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SHADOW LEMMA ON FINSLER MANIFOLDS OF HYPERBOLIC TYPE

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Abstract. Let (M, F) be a compact Finsler manifold of hyperbolic type, $\tilde{M}_F$ be its universal Finslerian covering and α^F the critical exponent of the group of the deck transformations of $\tilde{M}_F$. In this paper we prove the existence of an α^F-Busemann quasi-density on the Gromov boundary $\tilde{M}_F^G(\infty)$ of $\tilde{M}_F$. Furthermore, we generalize the Shadow lemma to the compact Finsler manifolds of hyperbolic type.

1. Introduction and main results

Let (M, g) be a complete Riemannian manifold with negative sectional curvature. Then the universal cover X of M is diffeomorphic to the Euclidean space and can be compactified by adding a topological sphere ∂X. The fundamental group Γ of M acts by isometries on $X \cup \partial X$.

In [10], when X is a real hyperbolic space, S.-J. Patterson constructed a familly of measures $\{\mu_x\}_{x\in X}$ as follows :

$$\mu_{s,x_0,x} = \frac{\sum_{\gamma\in\Gamma} e^{-sd_g(x,\gamma x_0)}}{e^{-sd_g(x_0,\gamma x_0)}}\delta_{\gamma x_0}, \quad s > \alpha^g, \quad x \text{ and } x_0 \in X,$$

where d_g is the distance function induced on X by g, $\delta_{\gamma x_0}$ the Dirac point mass of weight one at γx_0 and α^g the critical exponent of Γ. Let $(s_n)_n$ be a sequence with $s_n > \alpha^g$ and $s_n \longrightarrow \alpha^g$ such that $\mu_{s_n,x_0,x}$ converges weakly, as well to the measure μ_x. The measure μ_x is Γ-quasi-invariant.

Let now (M, F) be a compact Finsler manifold of hyperbolic type (cf. Definition 2.3) and $\tilde{M}_F$ the universal Finslerian cover of (M, F) (cf. Definition 2.2). Let denote by $\tilde{M}_F^G(\infty)$ the Gromov boundary of $\tilde{M}_F$, $\Gamma \subset Iso(\tilde{M}_F)$ the group of the deck transformations of $\tilde{M}_F$ and α^F the critical exponent of Γ.

Following the original idea of Patterson (see [10]), we define on $\tilde{M}_F \cup \tilde{M}_F^G(\infty)$, a familly of measures $\{\mu_x\}_{x\in\tilde{M}_F}$. Using this construction, we prove the existence of a Busemann quasi-density of dimension α^F (cf. Definition 4.2) on $\tilde{M}_F \cup \tilde{M}_F^G(\infty)$.

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For any point $x \in \tilde{M}_F$ and $\rho > 0$, let denote by $\mathcal{O}^F_{x_0}(x, \rho)$ (cf. Definition 4.3) the shadow on $\tilde{M}^G_F(\infty)$ viewed from x_0 of the ball $B^+_F(x, \rho)$.

The main result of this paper is the following :

Theorem 1.1. *Let (M, F) be a compact Finsler manifold of hyperbolic type and $\tilde{M}_F$ be its universal Finslerian covering. Let Γ be the group of deck transformations of $\tilde{M}_F$, α^F its critical exponent and $\{\mu_x\}_{x\in\tilde{M}_F}$ be a Patterson-Sullivan density associated to Γ on $\tilde{M}_F \cup \tilde{M}^G_F(\infty)$. Then there exist a constant $R > 0$ and a function $b \geq 1$ such that for all $\rho \geq R$ and all $x \in \tilde{M}_F$,*

$$\frac{1}{b(\rho)r^{\alpha^F}} \leq \mu_{x_0}(\mathcal{O}^F_{x_0}(x, \rho)) \leq b(\rho)r^{\alpha^F},$$

where $r = e^{-d_F(x,x_0)}$ and $\mathcal{O}^F_{x_0}(x, \rho)$ is the shadow on $\tilde{M}^G_F(\infty)$ viewed from x_0 of the ball $B^+_F(x, \rho)$.

The Shadow lemma allows to estimate the measure of certain subsets of the boundary with respect to Busemann quasidensities. It was proved by D. Sullivan ([13]) in 1979 to all hyperbolic spaces. In 1997, G. Knieper ([8]) used the Shadow lemma to estimate the volume of geodesic spheres of the universal Riemannian covering of the Hadamard manifolds of rank 1,

From Theorem 1.1 since all compact orientable surfaces of genus greater than one admits a metric g_0 of strictly negative curvature, we deduce the following :

Corollary 1.1. *Let M be a compact orientable surface of genus greater than one, F a Finsler metric on M and $\tilde{M}_F$ be its universal Finslerian covering. Let Γ be the group of deck transformations of $\tilde{M}_F$, α^F its critical exponent and $\{\mu_x\}_{x\in\tilde{M}_F}$ be a Patterson-Sullivan density associated to Γ on $\tilde{M}_F \cup \tilde{M}^G_F(\infty)$. Then there exist a constant $R > 0$ and a function $b \geq 1$ such that for all $\rho \geq R$ and all $x \in \tilde{M}_F$,*

$$\frac{1}{b(\rho)r^{\alpha^F}} \leq \mu_{x_0}(\mathcal{O}^F_{x_0}(x, \rho)) \leq b(\rho)r^{\alpha^F},$$

where $r = e^{-d_F(x,x_0)}$ and $\mathcal{O}^F_{x_0}(x, \rho)$ is the shadow on $\tilde{M}^G_F(\infty)$ viewed from x_0 of the ball $B^+_F(x, \rho)$.

The paper is organized as follows : in section 2, we recall some basic facts about a Finsler manifold. Section 3 is devoted to the ideal boundary and the Gromov boundary of a Finsler manifold of hyperbolic type. In section 4, we construct a Busemann quasidensity and we provide the proof of the theorem 1.1.

2. Generality on Finsler manifold of hyperbolic type

In this section, we briefly recall some notions from Finsler geometry; see [2] or [11] and the references therein for more details.

Let M be an n-dimensional C^∞ manifold. Denote by T_xM the tangent space at $x \in M$, and by $TM := \cup_{x\in M} T_xM$ the tangent bundle of M. Each element of TM has the form (x, y), where $x \in M$ and $y \in T_xM$. The natural projection $\pi : TM \longrightarrow M$ is given by $\pi(x, y) := x$. Let $(x^1, x^2, \cdots, x^n) = (x^i) : U \longrightarrow \mathbb{R}^n$ be a local coordinate system on an open subset $U \subset M$. As usual, $\{\frac{\partial}{\partial x^i}\}$ is the induced basis for T_xM. Any $y \in T_xM$ is expressed as $y = y^i \frac{\partial}{\partial x^i}$.

A Finsler structure on M is a function

$$F : TM \longrightarrow [0; +\infty)$$

with the following properties :

(1) *Regularity*: F is C^∞ on the slit tangent bundle $TM \setminus \{0\}$;
(2) *Positive homogeneity*: $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;
(3) *Strong convexity*: The $n \times n$ Hessian matrix

$$(g_{ij}) := \left(\left[\frac{1}{2} F^2 \right]_{y^i y^j} \right)$$

is positive definite at every point of $TM \setminus \{0\}$.

Note that the Finsler structure F is locally expressed as a function of (x_i, y_i) and the partial derivatives of $\frac{1}{2}F^2$ are taken with respect to y^i. It is easy to check that the positive-definiteness of $\frac{1}{2}F^2$ is independent of the choice of any basis of $T_x M$.

Let M be an n-dimensional C^∞ manifold. A smooth Riemannian metric g on M is a family $\{g_x\}_{x \in M}$ of inner products, one for each tangent space $T_x M$, such that the functions $g_{ij}(x) := g_x\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$ are C^∞. Since each g_x is an inner product, the matrix $(g_{ij}(x))$ is positive-definite at every $x \in M$. Then g defines a symmetric Finsler structure F on TM by :

$$F(x, y) := \sqrt{g_x(y, y)}.$$

Therefore, every Riemannian manifold is a Finsler Manifold.

Let $\sigma : [a, b] \longrightarrow M$ be a piecewise C^∞ curve with velocity $\frac{d\sigma}{dt} = \frac{d\sigma^i}{dt} \frac{\partial}{\partial x^i} \in T_{\sigma(t)} M$. Its length $l_F(\sigma) = \int_a^b F(\sigma, \frac{d\sigma}{dt}) dt$. For p and $x \in M$, denote by $C^\infty(p, x)$ the collection of all piecewise C^∞ curves $\sigma : [a, b] \longrightarrow M$ with $\sigma(a) = p$ and $\sigma(b) = x$. Define the metric distance from p to x by

$$d_F(p, x) = \inf_{\sigma \in C^\infty(p,x)} l_F(\sigma).$$

Note that if F is typically positively homogeneous (of degre 1) the distance d_F is non-symmetric.

We say that the Finsler structure F is absolute homogeneous if

$$F(x, \lambda y) = \mid \lambda \mid F(x, y) \quad \text{for all } \lambda \in \mathbb{R}.$$

In this case, the distance d_F is symmetric.

Let denote by $B_F^+(p, r) = \{x \in M : d_F(p, x) < r\}$ and $B_F^-(p, r) = \{x \in M : d_F(x, p) < r\}$.

Definition 2.1. Let (M, F) be a Finsler manifold.

(1) A piecewise C^∞ curve $c : [a, b] \longrightarrow \tilde{M}$ satisfying $F(\dot{c}) = 1$ is said to be minimal if $l_F(c) = d_F(c(a), c(b))$.
(2) A curve $c : [0, \infty) \longrightarrow \tilde{M}$ is called a forward ray if $c \mid_{[a,b]}$ is minimal for all $[a, b] \subset [0, \infty)$.
(3) A curve $c : (-\infty, 0] \longrightarrow \tilde{M}$ is called a backward ray if $c \mid_{[a,b]}$ is minimal for all $[a, b] \subset (-\infty, 0]$.
(4) A curve $c : \mathbb{R} \longrightarrow \tilde{M}$ is called a minimal geodesic if $c \mid_{[a,b]}$ is minimal for all $[a, b] \subset \mathbb{R}$.

Definition 2.2. Let (M, F) be a Finsler manifold M. We say that F is uniformly equivalent to a Riemannian metric g, if there is a constant c_F such that

$$\frac{1}{c_F} \cdot F \leq \|\cdot\|_g \leq c_F \cdot F,$$

where $\|v\|_g = \sqrt{g_x(y,y)}$ for all $v = (x,y) \in TM$.

Let $p : \tilde{M} \longrightarrow M$ be the universal Riemannian covering of M. Using the map p, we pull the Finsler structure F back to $\tilde{M}$. The resulting $\tilde{F}$ defines on $T\tilde{M} \setminus \{0\}$ a Finsler structure. We denote by $\tilde{M}_F$ the Finsler manifold $(\tilde{M}, \tilde{F})$; $\tilde{M}_F$ is the universal Finslerian covering of the Finsler manifold (M, F).

Let $\Gamma \subset Iso(\tilde{M}_F)$ be the group of deck transformations. We say that F is invariant under Γ if

$$F(d\tau(pv)v) = F(v) \ \ \forall v \in T\tilde{M}, \tau \in \Gamma.$$

Remark 2.1. Note that if M is a compact manifold and F is invariant under the group of deck transformations Γ then F and g are uniform equivalence.

Definition 2.3. A Finsler manifold (M, F) is called of hyperbolic type, if there exists on the manifold M a Riemannian metric g_0 of strictly negative curvature such that F and g_0 are uniformly equivalent (cf. Definition 2.2).

3. Ideal and Gromov Boundaries of Finsler manifolds of hyperbolic type

In this section, we study the ideal Boundary and the Gromov hyperbolic boundary of the universal Finslerian covering of a compact Finsler Manifold of hyperbolic type.

The following theorem is fundamental for the study of the ideal boundary of Finsler manifolds of hyperbolic type. It was proved by Morse in dimension 2 and by Klingenberg in arbitrary dimensions. The fact that the Morse Lemma also holds in Finsler case was first observed by E. M. Zaustinsky (see [14]). Due to Klingenberg (see [7]), the Morse Lemma holds in any dimension.

Theorem 3.1 (Morse Lemma, cf. [12])**.** *Let (M, F) be a Finsler manifold of hyperbolic type and g_0 be a metric of strictly negative curvature on M such that F and g_0 are uniformly equivalent and $\tilde{M}$ be the universal covering of M.*

Then there is a constant $r_0 = r_0(F, g_0) > 0$ with the following properties.

(i) *for any two points x and $y \in \tilde{M}$, the g_0-geodesic-segment $\gamma : [0, d_{g_0}(x,y)] \to \tilde{M}$ from x to y and any F-minimal segment $c : [0, d_F(x,y)] \to \tilde{M}$ from x to y we have*

$$\max_{t \in [0, d_F(x,y)]} d_{g_0}(\gamma([0, d_{g_0}(x,y)], c(t))) \leq r_0.$$

(ii) *If $c : [0, \infty) \longrightarrow \tilde{M}$ is a F-forward ray, then there exists a g_0-ray $\gamma : [0, \infty) \longrightarrow \tilde{M}$ and conversely, if $\gamma : [0, \infty) \longrightarrow \tilde{M}$ is a g_0-ray, then there exists a F-forward ray $c : [0, \infty) \longrightarrow \tilde{M}$, such that*

$$\sup_{t \in [0, \infty)} d_{g_0}(\gamma([0, \infty), c(t))) \leq r_0.$$

This properties stay hold for backward rays and minimal geodesics.

Now let (M, F) be a compact Finsler manifold of hyperbolic type and $\tilde{M}_F$ be its universal Finslerian covering. Let g_0 denotes an associated metric of strictly negative curvature on M. Note that the universal Riemannian covering $\tilde{M}_0$ of (M, g_0) is a Hadamard manifold and let denote by $\tilde{M}_0(\infty)$ its ideal boundary. Two F-forward rays c and c' are said to be asymptotic if there exists a constant $D_0 \geq 0$ such that $d_H(c(\mathbb{R}_+), c'(\mathbb{R}_+)) \leq D_0$, where d_H is the Hausdorff distance with respect to the distance d_F. This defines an equivalence relation on the set of F-forward rays of $\tilde{M}_F$. Let $\tilde{M}_F(\infty)$ be the coset of asymptotic F-forward rays c of $\tilde{M}_F$. For each F-forward ray c of $\tilde{M}_F$, it follows from Morse lemma that there exists a g_0-geodesic ray γ such that $d_H(c(\mathbb{R}_+), \gamma(\mathbb{R}_+)) \leq D$, where D is the constant in Morse lemma. Let $[c]$ be the equivalence class of a F-forward ray c and let $[\gamma]$ the equivalence class of the g_0-geodesic γ. The map f defined by

$$f: \begin{array}{ccc} \tilde{M}_F(\infty) & \to & \tilde{M}_0(\infty) \\ [c] & \mapsto & [\gamma] \end{array}$$

is bijective. Then f defines on $\tilde{M}_F(\infty)$ a natural topology with respect to which $\tilde{M}_F(\infty)$ and $\tilde{M}_0(\infty)$ are homeomorphic ($\tilde{M}_F(\infty) \simeq \tilde{M}_0(\infty)$).

Let recall now some basic facts about Gromov hyperbolic spaces. Let (X, d) be a metric space with a reference point x_0. The Gromov product of the points x and y of X with respect to x_0 is the nonnegative real number $(x.y)_{x_0}$ defined by:

$$(x.y)_{x_0} = \frac{1}{2}\{d(x, x_0) + d(y, x_0) - d(x, y)\}$$

Let $\delta \geq 0$. A metric space (X, d) is said to be a δ-hyperbolic space if

$$(x.y)_{x_0} \geq \min\{(x.z)_{x_0}; (y.z)_{x_0}\} - \delta$$

for all x, y, z and every choice of reference point x_0. We call X a *Gromov hyperbolic space* if it is a δ-hyperbolic space for some $\delta \geq 0$. The usual hyperbolic space $\mathbb{H}^n$ is a δ-hyperbolic space, where $\delta = \log 3$. More generally, every Hadamard manifold with sectional curvature $\leq -k^2$ for some constant $k > 0$ is a δ-hyperbolic space, where $\delta = k^{-1} \log 3$ (see [1] or [4]).

Lemma 3.1. *(See* [4] *or* [6]*) Let (X, d) be a complete geodesic δ-hyperbolic space, x_0 a reference point in X, x and y two points of X. Then*

$$d(x_0, \gamma_{xy}) - 4\delta \leq (x.y)_{x_0} \leq d(x_0, \gamma_{xy})$$

for every geodesic segment γ_{xy} joining x and y.

Definition 3.1. A function $f : \mathbb{R} \to \mathbb{R}$ is called k-convex if for all $x, y \in \mathbb{R}$, and $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + k.$$

Proposition 3.1. *(See* [4] *or* [6]*) Let (X, d) be a δ-hyperbolic geodesic space and $c_1, c_2 : \mathbb{R} \to X$ two minimizing geodesics. The function*

$$f: \begin{array}{ccc} \mathbb{R} & \longrightarrow & \mathbb{R} \\ t & \longmapsto & d(c_1(t), c_2(t)) \end{array}$$

is 4δ-convex.

Definition 3.2. Let (X_1, d_1) and (X_2, d_2) be two metric spaces. A map $\Phi : X_1 \longrightarrow X_2$ is called a quasi-isometric map, if there exist constants $A > 1$ and $\alpha > 0$ with:

$$\frac{1}{A} d_1(x, y) - \alpha \leq d_2(\Phi(x), \Phi(y)) \leq A d_1(x, y) + \alpha \quad \forall x, y \in X_1.$$

In a metric space X, a quasi-geodesic (resp. quasi-geodesic ray) is a quasi-isometric map $\Phi : \mathbb{R} \longrightarrow X$ (resp. $\Phi : \mathbb{R}^+ \longrightarrow X$).

Lemma 3.2. *(see* [4]*) Let X_1 be a metric space and (X_2, d_2) be a geodesic Gromov hyperbolic space. If there exists a quasi-isometric map $\Phi : X_1 \longrightarrow X_2$, then X_1 is also a Gromov hyperbolic space.*

Now let X be a Gromov hyperbolic manifold, x_0 a reference point in X. We say that the sequence $(x_i)_{i\in\mathbb{N}}$ of points in X converges at infinity if

$$\lim_{i,j\to\infty}(x_i \cdot x_j)_{x_0} = \infty.$$

If x_1 is another reference point in X,

$$(x \cdot y)_{x_0} - d(x_0, x_1) \leq (x \cdot y)_{x_1} \leq (x \cdot y)_{x_0} + d(x_0, x_1).$$

Then the definition of the sequence that converges at infinity does not depend on the choice of the reference point. Let us recall the following equivalence relation $\mathcal{R}$ on the set of sequences of points in X that converge at infinity :

$$(x_i)\mathcal{R}(y_j) \Longleftrightarrow \lim_{i,j\to\infty}(x_i \cdot y_j)_{x_0} = \infty.$$

The Gromov boundary $X^G(\infty)$ of X is the coset of sequences that converge at infinity.

Let X be a simply connected manifold which is a Gromov hyperbolic space. One defines on the set $X \cup X^G(\infty)$ a topology as follows (see [4] page 22) :

(1) if $x \in X$, a sequence $(x_i)_{i\in\mathbb{N}}$ converges to x with respect to the topology of X.
(2) if $(x_i)_{i\in\mathbb{N}}$ defines a point $\xi \in X^G(\infty)$, $(x_i)_{i\in\mathbb{N}}$ converges to ξ.
(3) For $\eta \in X^G(\infty)$ and $k > 0$, let

$$V_k(\eta) := \left\{ y \in X \cup X^G(\infty) \ / \ (y \cdot \eta)_{x_0} > k \right\},$$

where

$$(x \cdot y)_{x_0} = \inf \left\{ \liminf_{i\to\infty}(x_i \cdot y_i)_{x_0} \ / \ x_i \to x, \ y_i \to y \right\}$$

for x and y elements of $X \cup X^G(\infty)$.

The set of all $V_k(\eta)$ and the open metric balls of X generate a topology on $X \cup X^G(\infty)$. With respect to this topology, X is dense in $X \cup X^G(\infty)$ and $X \cup X^G(\infty)$ is compact.

Lemma 3.3. *(see* [5]*) Let X be a δ-hyperbolic space. Then*

(1) *Each geodesic $\gamma : \mathbb{R} \longrightarrow X$ defines two distinct points at infinity $\gamma(+\infty)$ and $\gamma(-\infty)$.*
(2) *For each $(\eta, x) \in X^G(\infty) \times X$, there exists a geodesic ray γ such that $\gamma(0) = x$ and $\gamma(+\infty) = \eta$. For any other geodesic ray γ' with $\gamma'(0) = x$ and $\gamma'(+\infty) = \eta$ we have $d(\gamma'(t), \gamma(t)) \leq 4\delta$ for all $t \geq 0$.*

Definition 3.3. Let $\xi \in X^G(\infty)$ and $c : \mathbb{R}_+ \longrightarrow X$ be a minimal geodesic ray satisfying $c(+\infty) = \xi$. The function

$$b_c(x) := \lim_{t\to\infty}(d(x, c(t)) - t)$$

is well-defined on X and is called the Busemann function for the geodesic c.

Lemma 3.4. *(see [5]) Let X be a $\delta-$hyperbolic space, $\xi \in X^G(\infty)$, $x,\ y \in X$ and c a geodesic ray with $c(0) = x$ and $c(+\infty) = \xi$. Then there exists a neighbourhood $\mathcal{V}$ of ξ in $X \cup X^G(\infty)$ such that*

$$|b_c(y) - (d(z, y) - d(z, x))| \leq K \text{ for all } z \in \mathcal{V} \cap X,$$

where b_c is the busemann function for the geodesic c and K is a constant depending only on δ.

Lemma 3.5. *(see [4]) Let X_1 be a metric space and (X_2, d_2) be a geodesic Gromov hyperbolic space. If there exists a quasi-isometric map $\phi : X_1 \longrightarrow X_2$, then X_1 is also a Gromov hyperbolic space. Moreover, if the map*

$$x \longmapsto d_2(x, \phi(X_1))$$

is bounded above, $X_1^G(\infty) \simeq X_2^G(\infty)$ ie $X_1^G(\infty)$ is homeomorphic to $X_2^G(\infty)$.

The following lemma give an homeomorphism between the ideal boundary and the Gromov hyperbolic boundary of Hadamard manifolds :

Lemma 3.6. *(see [3]) Let X_0 be a Hadamard manifold with sectional curvature $K_{X_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. There exists a natural homeomorphism*

$$\phi : X_0 \cup X_0^G(\infty) \longrightarrow X_0 \cup X_0(\infty).$$

In particular, $X_0^G(\infty) \simeq X_0(\infty)$.

Using Morse lemma, lemma 3.6 and the properties of the ideal boundaries, we obtain the following lemma :

Lemma 3.7. *Let (M, F) be a compact Finsler manifold of hyperbolic type and $\tilde{M}_F$ be its universal Finslerian covering. Let g_0 be an associated metric of strictly negative curvature on M and $\tilde{M}_0$ be the universal Riemannian covering of (M, g_0). We have*

$$\tilde{M}_F(\infty) \simeq \tilde{M}_0(\infty) \simeq \tilde{M}_0^G(\infty) \simeq \tilde{M}_F^G(\infty).$$

4. Shadow lemma

Definition 4.1. Let X be a Gromov hyperbolic manifold with reference point x_0 and Γ be a discrete and infinite subgroup of the isometry group $Iso(X)$ of X. For a given point $x \in X$, the limit set $\Lambda(\Gamma, x)$ is the set of the accumulation points of the orbit Γx in $X^G(\infty)$.

Let (X, d) be a metric space and Γ be a discrete and infinite subgroup of the isometry group $Iso(X)$ of X. For $x_0,\ x \in X$ and $s \in \mathbb{R}$,

$$P_s(x, x_0) := \sum_{\gamma \in \Gamma} e^{-sd(x, \gamma x_0)}$$

denotes the Poincaré series associated to Γ. The number

$$\alpha := \inf\{s \in \mathbb{R}; P_s(x, x_0) < \infty\}$$

is called the critical exponent of Γ and is independent of x and x_0. The subgroup Γ is called of divergence type if the Poincaré series diverges for $s = \alpha$. The following lemma introduces a usefull modification (due to Patterson) of the Poincaré series if Γ is not of divergence type.

Lemma 4.1. *(see [10]) Let Γ be a discrete group with critical exponent α. There exists a function $f : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ which is continuous, nondecreasing and such that*

$$\text{for all } a > 0, \quad \lim_{r \longrightarrow +\infty} \frac{f(r+a)}{f(r)} = 1$$

and the modified series

$$\tilde{P}_s(x, x_0) := \sum_{\gamma \in \Gamma} f(d(x, \gamma x_0)) e^{-d(x, \gamma x_0)}$$

converges for $s > \alpha$ and diverges for $s \leq \alpha$.

Now let (M, F) be a compact Finsler manifold of hyperbolic type, $\tilde{M}_F$ be its universal Finslerian covering. Let g_0 denote a metric of strictly negative curvature on M. The universal covering $\tilde{M}_0$ of (M, g_0) is a hadamard manifold satisfying $K_{\tilde{M}_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$.

Let Γ be the group of deck transformations of $\tilde{M}$ and α^{g_0} be its critical exponent with respect to the metric g_0. It follows from theorem 5.1 in [8] that :

$$\alpha^{g_0} = h(g_0) := \lim_{r \longrightarrow \infty} \frac{\log \mathrm{vol}_{g_0} B_{g_0}(x, r)}{r}.$$

The fact that M is compact implies the existence of a constant $\lambda \geq 1$ such that

$$\lambda^{-1} d_{g_0}(x, y) \leq d_F(x, y) \leq \lambda d_{g_0}(x, y) \text{ for all } x,\ y \in \tilde{M}_F.$$

Then, the critical exponent α^F of Γ with respect to the metric d_F belongs to $[\lambda^{-1} h(g_0), \lambda h(g_0)] \subset \mathbb{R}_+^*$.

Lemma 4.2. *Let (M, F) be a compact Finsler manifold of hyperbolic type, $\tilde{M}_F$ be its universal Finslerian covering and Γ be the group of deck transformations of $\tilde{M}_F$. Then*

1. $\Lambda^F(\Gamma, x) = \overline{\Gamma x} \cap \tilde{M}_F^G(\infty)$.
2. $\gamma(\Lambda^F(\Gamma, x)) = \Lambda^F(\Gamma, x)$ *for all* $\gamma \in \Gamma$ *and* $x \in \tilde{M}$.
3. $\Lambda^F(\Gamma, x)$ *is independent of* x.
4. $\Lambda^F(\Gamma, x) = \tilde{M}_F^G(\infty)$.

Proof of Lemma 4.2. (1) Direct because $\Lambda^F(\Gamma, x) = \overline{\Gamma x} \setminus \Gamma x$ and $\Gamma x \subset \tilde{M}_F$.

(2) Let $\xi \in \Lambda^F(\Gamma, x)$. There exists a sequence $\gamma_n \in \Gamma$ such that $\lim_{n \longrightarrow \infty} \gamma_n x = \xi$. Then $\lim_{n \longrightarrow \infty} \gamma \cdot \gamma_n x = \gamma \xi$.

(3) For all $\xi \in \Lambda^F(\Gamma, x)$, by the definition there is a sequence $(\gamma_n)_n$ of points of Γ such that $\lim_{n \longrightarrow \infty} \gamma_n x = \xi$. Then

$$\lim_{m,n \longrightarrow \infty} (\gamma_n x \cdot \gamma_m x)_{x_0} = \lim_{m,n \longrightarrow \infty} [d_F(\gamma_n x, x_0) + d_F(\gamma_m x, x_0) - d_F(\gamma_n x, \gamma_m x)] = +\infty.$$

For all $y \in \tilde{M}_F$, we have :

$$\begin{aligned}
2(\gamma_n x \cdot \gamma_n y)_{x_0} &= d_F(\gamma_n x, x_0) + d_F(\gamma_n y, x_0) - d_F(\gamma_n x, \gamma_n y) \\
&= d_F(\gamma_n x, x_0) + d_F(\gamma_n y, x_0) - d_F(x, y) \\
&\geq d_F(\gamma_n x, x_0) - d_F(x, y)
\end{aligned}$$

Hence,

$$\lim_{n \longrightarrow \infty} (\gamma_n x \cdot \gamma_n y)_{x_0} = +\infty \text{ and } \lim_{n \longrightarrow \infty} \gamma_n y = +\xi.$$

then $\xi \in \Lambda^F(\Gamma, y)$.

(4) Let g_0 denotes a metric of strictly negative curvature on M. The universal Riemannian covering $\tilde{M}_0$ of (M, g_0) is a Hadamard manifold satisfying $K_{\tilde{M}_0} \leq -k_0^2 < 0$ for some constant $k_0 > 0$. Then $\Lambda^{g_0}(\Gamma, x) = \tilde{M}_0(\infty)$ (see [9]). Since Γ is cocompact, the identity map $I : \tilde{M}_0 \longrightarrow \tilde{M}_F$ defines a homeomorphism $I^* : \tilde{M}_0^G(\infty) \longrightarrow \tilde{M}_F^G(\infty)$ (see lemma lemma 3.7). Let $\xi \in \tilde{M}_F^G(\infty)$ and $\eta \in \tilde{M}_0^G(\infty)$ such that $\xi = I^*(\eta)$. The fact that $\tilde{M}_0^G(\infty) = \Lambda^{g_0}(\Gamma)$, there is a sequence $(\gamma_n)_n$ in Γ and $y \in \tilde{M}_F$ such that the sequence $(\gamma_n y)_n$ converges to η in $\tilde{M} \cup \tilde{M}_0^G(\infty)$. Then $I(\gamma_n y)_n = (\gamma_n y)_n$ converges to $I^*(\eta) = \xi$ in $\tilde{M}_F \cup \tilde{M}_F^G(\infty)$.

□

Definition 4.2. Let X be a Gromov hyperbolic manifold with reference point x_0, $\alpha \in \mathbb{R}_+$, and Γ be a discrete and infinite subgroup of $Iso(X)$. A family $\{\mu_x\}_x \in X$ of finite nontrivial Borel measures on $X \cup X^G(\infty)$ is an α-dimensional Busemann quasidensity if :

(1) $\sup \mu_x \subset \Lambda(\Gamma, x)$, where $\Lambda(\Gamma, x)$ is the limit set of the orbit Γx in $X^G(\infty)$.
(2) $\mu_{\gamma x}(\gamma A) = \mu_x(A)$ for all $\gamma \in \Gamma$, $A \subset X^G(\infty)$, A measurable, $x \in X$.
(3) There exists a constant $\lambda \geq 1$ sucvh that for all $x \in X$,

$$\frac{1}{\lambda} e^{-\alpha b_c(x_0)} \leq \frac{d\mu_{x_0}}{d\mu_x}(\xi) \leq \lambda e^{-alpha(x_0)}$$

for allmost all $\xi \in X^G(\infty)$, where c is a geodesic satisfying $c(0) = x$ and $c(\infty) = \xi$ and b_c is the Busemann function for the geodesic c.

The next lemma states the existence of a Busemann quasidensity on an universal Finslerian covering of compact Finsler manifolds of hyperbolic type.

Lemma 4.3. *Let (M, F) be a compact Finsler manifold of hyperbolic type and $\tilde{M}_F$ be its universal Finslerian covering. Let Γ be the group of deck transformations of $\tilde{M}_F$ and let α^F be its critical exponent. Then there exists an α^F-dimensional Busemann quasidensity $\{\mu_x\}_{x \in \tilde{M}_F}$ on $\tilde{M}_F \cup \tilde{M}_F^G(\infty)$.*

Proof of Lemma 4.3. We have to construct a family of measures $\{\mu_x\}_x \in \tilde{M}_F$ which satisfies the axiomatic definition 4.2
. A natural way to obtain Busemann quasidensity was given by Patterson (see [10]) in case of Fuchsian groups. Let x_0 be a reference point of the Gromov hyperbolic manifold $\tilde{M}_F$. For $s > \alpha^F$ and $x \in \tilde{M}_F$, we consider the measure

$$\mu_{s,x_0,x} := \frac{\sum_{\gamma \in \Gamma} f(d_F(x, \gamma x_0)) e^{-s d_F(x, \gamma x_0)}}{\tilde{P}_s^F(x_0, x_0)} \delta_{\gamma x_0},$$

where f is a usefull modification function (due to Patterson) of the Poincaré series if Γ is not of divergence type and

$$\tilde{P}_s^F(x_0, x_0) = \sum_{\gamma \in \Gamma} f(d_F(x_0, \gamma x_0)) e^{-s d_F(x_0, \gamma x_0)}.$$

Let $(s_n)_n$ be a sequence with $s_n > \alpha^F$ and $s_n \longrightarrow \alpha^F$ such that $\mu_{s_n, x_0, x}$ converges weakly, as well to the measure μ_x. For $x \notin \Gamma x_0$, we choose a subsequence of $(s_n)_n$, denoted by $(s_n^x)_n$ such that the measure $\mu_{s_n^x, x_0, x}$ is also weakly convergent. For all points of the same orbit Γx we can choose the same subsequence, that is, $s_n^{x'} = s_n^x$

if $x' \in \Gamma x$.
These choices yield a family $\{\mu_x\}_{x\in\tilde{M}_F}$ of measures.

(1) Using the triangle inequality of d_F and the properties of the usefull function f, we have
$$\frac{1}{2} \le \frac{f(d_F(x,\gamma x_0))}{f(d_F(x_0,\gamma x_0))} \le \frac{3}{2}$$
for allmost $\gamma \in \Gamma$.
Then, there exist constants a and b depending only on $d_F(x,x_0)$ such that
$$ae^{-sd_F(x,x_0)} \le \mu_{s,x_0,x}(\tilde{M}_F \cup \tilde{M}_F^G(\infty)) \le be^{sd_F(x,x_0)}.$$
This implies that $\{\mu_x\}_x \in \tilde{M}_F$ is a family of finite nontrivial Borel measures on $\tilde{M}_F \cup \tilde{M}_F^G(\infty))$.

(2) For all $z \in \tilde{M}_F \cup \tilde{M}_F^G(\infty)) \setminus \Lambda^F(\Gamma,x)$, there is an open neighbourhood $\mathcal{U}$ of z with
$$\Gamma x \cap \mathcal{U} \setminus \{z\} = \emptyset.$$
Then
$$\begin{array}{rcl}\mu_{s_n,x_0,x}(\mathcal{U}) & = & \frac{\sum_{\gamma\in\Gamma,\gamma x_0\in\mathcal{U}} f(d_F(x,\gamma x_0))e^{-s_n d_F(x,\gamma x_0)}}{\tilde{P}^F_{s_n}(x_0,x_0)} \\ & \le & \frac{f(d_F(x,z))e^{-s_n d_F(x,z)}}{\tilde{P}^F_{s_n}(x_0,x_0)}.\end{array}$$
The fact that $\tilde{P}^F_{s_n}(x_0,x_0)$ diverges for $s=\alpha^F$ implies that $\mu_x(\mathcal{U})=0$.

(3) Let $\eta\in\Gamma$, and A be a measurable subset of $\tilde{M}_F \cup \tilde{M}_F^G(\infty))$. We have :
$$\begin{array}{rcl}\mu_{s,x_0,\eta x}(\eta A) & = & \frac{\sum_{\gamma\in\Gamma,\gamma x_0\in\eta A} f(d_F(\eta x,\gamma x_0))e^{-sd_F(\eta x,\gamma x_0)}}{\tilde{P}^F_s(x_0,x_0)} \\ & = & \frac{\sum_{\gamma'\in\Gamma,\gamma' x_0\in A} f(d_F(x,\gamma' x_0))e^{-sd_F(x,\gamma' x_0)}}{\tilde{P}^F_s(x_0,x_0)} \\ & = & \mu_{s,x_0,x}(A).\end{array}$$
Then $\mu_{\eta x}(\eta A) = \mu_x(A)$ for all $\eta\in\Gamma$.

(4) Let now $\xi \in \tilde{M}_F^G(\infty)$ and a sequence $(U_n)_n$ of open sets in $\tilde{M}_F \cup \tilde{M}_F^G(\infty))$ with $\lim_{n\longrightarrow\infty} = \{\xi\}$. By lemma 3.4, there $n_0\in\mathbb{N}$ such that
$$|b_c(x_0) - (d_F(\gamma x_0,x_0) - d(x,x_0))\,| \le K$$
for all $n\ge n_0$ and $\gamma x_0 \in U_n$, where c is the F-forward ray joining x and ξ, b_c the Busemann function for the geodesic c, and K a constant depending only on the metric g_0. Finally, using the lemma 4.1, we deduce the existence of a constant $\lambda\ge 1$ such that :
$$\frac{1}{\lambda}e^{-\alpha^F b_c(x_0)} \le \frac{d\mu_{x_0}}{d\mu_x}(\xi) \le \lambda e^{-\alpha^F b_c(x_0)}.$$

□

Definition 4.3. Let (M,F) be a compact Finsler manifold of hyperbolic type, $\tilde{M}_F$ its universal Finslerian covering and $\tilde{M}_F^G(\infty)$ the Gromov boundary of $\tilde{M}_F$. For $y\in\tilde{M}_F\cup\tilde{M}_F^G(\infty)$, $x\in\tilde{M}_F$ and $\rho>0$, the shadow $\mathcal{O}^F_y(x,\rho)$ of the ball $B^+_F(x,\rho)$ viewed from the point y is the set of alls points $\xi\in\tilde{M}_F^G(\infty)$ such that all F-forward rays $c_{y\xi}$ connecting y and ξ satisfy $c_{y\xi}\cap B^+_F(x,\rho)\neq\emptyset$.

Lemma 4.4. *Let (M, F) be a compact Finsler manifold of hyperbolic type, $\tilde{M}_F$ its universal Finslerian covering. Let Γ be the group of the deck transformations of $\tilde{M}_F$ and $\mathcal{F}$ be a fundamental domain of Γ in $\tilde{M}_F$. Then there exist constants $R > 0$ and $\epsilon > 0$ such that for all $y \in \tilde{M}_F$, $x \in \mathcal{F}$,*

$$C_\epsilon^{g_0}(v) \subset \mathcal{O}_y^F(x, R),$$

where $v \in S_{x_0}\tilde{M}_0$, $C_\epsilon^{g_0}(v) := \{c_w(\infty) : w \in S_{x_0}\tilde{M}_0 \text{ and } (v, w) < \epsilon\}$ and c_w is the g_0-geodesic satisfying $\dot{c}_w(0) = w$.

Proof of Lemma 4.4. Let g_0 be a metric of strictly negative curvature associated to F on M. The universal Riemanniann covering $\tilde{M}_0$ of (M, g_0) is a Hadamard manifold of strictly negative curvature. Then there exist constants $R_0 > 0$ and $\epsilon > 0$ such that for all $x \in \mathcal{F}$ and $y \in \tilde{M}_F$, $C_\epsilon^{g_0}(v) \subset \mathcal{O}_y^{g_0}(x, R_0)$, for some $v \in S_{x_0}\tilde{M}_0$. Let $\xi \in C_\epsilon^{g_0}(v)$, γ_0 the g_0-geodesic connecting y and ξ. There exists $t_0 \geq 0$ such that $d_{g_0}(\gamma_0(t_0), x) \leq R_0$. Then, there is $\lambda \geq 1$ such that $d_F(\gamma_0(t_0), x) \leq \lambda R_0$.
Let γ be a F-forward ray connecting y and ξ. By Morse Lemma (see 3.1) there is a constant $k_1 > 0$ such that $d_F(\gamma_0(t_0), \gamma(\mathbb{R}_+)) \leq k_1$. Let $t_1 \in \mathbb{R}_+$ such that $d_F(\gamma_0(t_0), \gamma(t_1)) \leq k_1$. Then, $d_F(x, \gamma(t_1) \leq \lambda R_0 + k_1$. This implies that

$$C_\epsilon^{g_0}(v) \subset \mathcal{O}_y^F(x, R),$$

where $R = \lambda R_0 + k_1$. □

Lemma 4.5. *Let (M, F) be a compact Finsler manifold of hyperbolic type, $\tilde{M}_F$ its universal Finslerian covering. Let Γ be the group of the deck transformations of $\tilde{M}_F$, $\mathcal{F}$ be a fundamental domain of Γ in $\tilde{M}_F$ and $\{\mu_x\}_{x\in\tilde{M}_F}$ an α^F-dimensional Busemann quasidensity on $\tilde{M}_F \cup \tilde{M}_F^G(\infty)$. Then, for all $\epsilon > 0$, there exists a constant $l > 0$*

$$\mu_x(C_\epsilon^{g_0}(v)) \geq l \text{ for all } v \in S_{x_0}\tilde{M}_F \text{ and } x \in \mathcal{F}.$$

Proof of the Lemma 4.5. If it was not true, it would exist sequence $\{x_n\}_n$ of points of $\mathcal{F}$ and $\{v_n\}_n \in S_{x_0}\tilde{M}_F$ such that $\mu_{x_n}(C_\epsilon^{g_0}(v_n))$ converges to 0. One can suppose that $x_n \longrightarrow x \in \mathcal{F}$ and $v_n \longrightarrow v \in S_x\tilde{M}_F$. Then, there exists n_0 such that $n \geq n_0$ implies that $C_{\frac{\epsilon}{2}}^{g_0}(v) \subset C_\epsilon^{g_0}(v_n)$. Then $\mu_x(C_{\frac{\epsilon}{2}}^{g_0}(v)) = 0$. This is absurd because $C_{\frac{\epsilon}{2}}^{g_0}(v)$ is an open set of $\tilde{M}_F^G(\infty)$ and $supp\mu_x = \tilde{M}_F^G(\infty)$. □

Corollary 4.1. *Let (M, F), $\tilde{M}_F$, Γ and $\{\mu_x\}_{x\in\tilde{M}_F}$ be as in lemma 4.5. Then, there exist constants $l > 0$ and $R_1 > 0$ such that for all $\rho \geq R_1$*

$$\mu_x(\mathcal{O}_y^F(x, \rho) \geq l \text{ for all } x \text{ and } y \in \tilde{M}_F.$$

Proof of the corollary 4.1. Using the lemmas 4.4 and 4.5, there exists constants $l > 0$ and $R > 0$ such that for all $\rho \geq R$

$$\mu_x(\mathcal{O}_y^F(x, \rho) \geq l \text{ for all } y \in \tilde{M}_F \text{ and } x \in \mathcal{F}.$$

The fact that $\mathcal{F}$ is a fundamental domain of Γ in $\tilde{M}_F$ and $\gamma\left(\mathcal{O}_y^F(x, \rho)\right) = \mathcal{O}_{\gamma y}^F(\gamma x, \rho)$ implies that :

$$\begin{aligned}
\mu_x\left(\mathcal{O}_y^F(x, \rho)\right) &= \mu_{\gamma x'}\left(\mathcal{O}_{\gamma(\gamma^{-1}y)}^F(\gamma x', \rho)\right) \\
&= \mu_{x'}\left(\gamma\left(\mathcal{O}_{\gamma^{-1}y}^F(x', \rho)\right)\right) \\
&= \mu_{x'}\left(\mathcal{O}_{\gamma^{-1}y}^F(x', \rho)\right).
\end{aligned}$$

□

Now we can give a proof of Theorem 1.1

Proof of the teorem 1.1. By lemma 4.3, there exists a constant $\lambda > 1$ such that for all $\xi \in \tilde{M}_F^G(\infty)$,

$$\lambda^{-1} e^{-\alpha^F b_c(x_0)} \leq \frac{d\mu_{x_0}}{d\mu_x}(\xi) \leq \lambda e^{-\alpha^F b_c(x_0)}$$

for all $x \in \tilde{M}_F$, where c is a F-forward ray connecting x and ξ and

$$b_c(y) = \lim_{t \longrightarrow \infty} \left[d_F(y, c(t)) - t \right].$$

Then

$$\lambda^{-1} \int_{\mathcal{O}_{x_0}^F(x,\rho)} e^{-\alpha^F b_c(x_0)} d\mu_x(\xi) \leq \mu_{x_0}\left(\mathcal{O}_{x_0}^F(x,\rho)\right) \leq \lambda \int_{\mathcal{O}_{x_0}^F(x,\rho)} e^{-\alpha^F b_c(x_0)} d\mu_x(\xi)$$

for all $\xi \in \tilde{M}_F^G(\infty)$ and $x \in \tilde{M}_F$, where c is a F-forward ray connecting x and ξ, b_c the Busemann function for the geodesic c.

Morse lemma and the defintion of the shadow $\mathcal{O}_{x_0}^F(x,\rho)$ imply the existence of a constant $D > 0$ such that

$$d_F(x, x_0) - D \leq b_c(x_0) \leq d_F(x, x_0) + D \quad \text{for all } x \in \tilde{M}_F.$$

Then,

$$\begin{array}{rcl} \mu_{x_0}\left(\mathcal{O}_{x_0}^F(x,\rho)\right) & \leq & \lambda \int_{\mathcal{O}_{x_0}^F(x,\rho)} e^{-\alpha^F d_F(x,x_0) - D} d\mu_x(\xi) \\ & \leq & \lambda e^{-\alpha^F d_F(x,x_0) - D} \mu_x\left(\mathcal{O}_{x_0}^F(x,\rho)\right) \\ & \leq & \lambda e^{\alpha^F D} r^{\alpha^F} \mu_x\left(\mathcal{O}_{x_0}^F(x,\rho)\right) \\ & \leq & b' e^{\alpha^F D} r^{\alpha^F}, \end{array}$$

where $b' = \sup_{x \in \tilde{M}_F} \mu_x(\tilde{M}_F^G(\infty))$. Moreover,

$$\mu_{x_0}\left(\mathcal{O}_{x_0}^F(x,\rho)\right) \geq \lambda^{-1} e^{-\alpha^F D} r^{\alpha^F} \mu_x\left(\mathcal{O}_{x_0}^F(x,\rho\right).$$

Using corollary 4.1, we have

$$\mu_x\left(\mathcal{O}_{x_0}^F(x,\rho\right) \geq l \quad \text{since } \rho \geq R.$$

Finally, putting $b(\rho) = \max\{\frac{e^{\alpha^F D}\lambda}{l}, e^{\alpha^F D b'\lambda}, \frac{3}{2}\}$ we obtain :

$$\frac{1}{b(\rho) r^{\alpha^F}} \leq \mu_{x_0}(\mathcal{O}_{x_0}^F(x,\rho)) \leq b(\rho) r^{\alpha^F},$$

for all $x \in \tilde{M}_F$. ☐

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