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LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAEHLER MANIFOLDS WITH QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. In this paper, we study lightlike submanifolds of indefinite Kaehler manifolds. We introduce a class of lightlike submanifold called semi-invariant lightlike submanifold. We consider lightlike submanifold with respect to a quarter-symmetric non metric connection which is determined by the complex structure. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifold and the quarter symmetric non metric connection and some results.

1. Introduction

The geometry of lightlike submanifolds of a semi-Riemannian manifold was presented in [1] (see also [2]) by K.L. Duggal and A. Bejancu. In [3], [4], [5], [6], K. L. Duggal and B. Sahin introduced and studied geometry of classes of lightlike submanifolds in indefinite Kaehler and indefinite Sasakian manifolds which is an umbrella of CR-lightlike, SCR-lightlike, Screen real GCR-lightlike submanifolds. In [7], M. Atçeken and E. Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product. In [8], E. Kılıç and B. Şahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product and gave some examples and results for lightlike submanifolds. In [9], E. Kılıç and O. Bahadır studied lightlike hypersurfaces of a semi-Riemannian product manifold with respect to quarter symmetric non-metric connection.

In [10], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors [11], [12], [13], [14], [15], [16]. The idea of quarter-symmetric linear connections on a differential manifold was introduced by S.Golab [12]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y, \tag{1.1}$$

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for any vector fields X, Y on a manifold, where u is a $1-$form and φ is a tensor of type $(1,1)$.

In this paper, we study lightlike submanifolds of an indefinite Kaehler manifold. First, we introduce semi-invariant lightlike submanifolds of an indefinite Kaehler manifold. We define some special distribution of semi-invariant lightlike submanifolds. Then we give some examples and find their geometric properties. Finally, by considering the quarter-symmetric non-metric connection, we study lightlike submanifolds of an indefinite Kaehler manifold. Then we obtain some results on lightlike submanifolds of an indefinite Kaehler manifold admitting quarter-symmetric non-metric connections. We introduce semi-invariant lightlike submanifold of an indefinite Kaehler manifold.

2. Preliminaries

Let $(\widetilde{M}, \widetilde{g})$ be a real $(m+n)-$ dimensional semi-Riemannian manifold of constant index ν, $1 \leq \nu \leq m+n-1$ and (M, g) be an $m-$ dimensional submanifold of $\widetilde{M}$. If $\widetilde{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\widetilde{M}$. Denote by g the induced tensor field of $\widetilde{g}$ on M and suppose g is degenerate. Then, for each tangent space T_xM we consider

$$T_xM^{\perp} = \left\{ Y_x \in T_x\widetilde{M} \mid \widetilde{g}_x\left(Y_x, X_x\right) = 0, \forall X_x \in T_xM \right\}$$

which is a degenerate $n-$dimensional subspace of $T_x\widetilde{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary subspaces. For this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ called *radical (null) subspace*. If the mapping

$$Rad\,TM : x \in M \longrightarrow RadT_xM$$

defines a smooth distribution on M of *rank* $r > 0$, the submanifold M of $\widetilde{M}$ is called $r-$*lightlike* ($r-$*degenerate*) *submanifold* and $RadTM$ is called the *radical* (*lightlike*) *distribution* on M. In the following , there are four possible cases:

Case 1. M is called a $r-$lightlike submanifold if $1 \leq r < \min\{m, n\}$.
Case 2. M is called a coisotropic submanifold if $1 < r = n < m$.
Case 3. M is called an isotropic submanifold if $1 < r = m < n$.
Case 4. M is called a totally lightlike submanifold if $1 < r = m = n$ [2].

In this paper, we have considered case 1, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $RadTM$ in TM. Therefore,

$$TM = Rad\,TM \perp S\left(TM\right), \tag{2.1}$$

in which $\perp$ denotes orthogonal direct sum. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/Rad\ TM$. Denote an $r-$lightlike submanifold by $(M, g, S(TM), S(TM^{\perp}))$, where $S(TM^{\perp})$ is a complementary vector bundle of $Rad\,TM$ in $TM^{\perp}$ and $S(TM^{\perp})$ is non-degenerate with respect to $\widetilde{g}$. Let us define that $tr(TM)$ is a complementary (but never orthogonal) vectors bundle to TM in $T\widetilde{M}_{|M}$ and

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}), \tag{2.2}$$

where $ltr(TM)$ is an arbitrary lightlike transversal vector bundle of M. Then we have

$$\begin{aligned} T\widetilde{M}_{|M} &= TM \oplus tr(TM) \\ &= (Rad\, TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}) \end{aligned} \tag{2.3}$$

where $\oplus$ denotes direct sum, but it is not orthogonal [2].

The Gauss and Weingarten formulas given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y),\ \forall X, Y \in \Gamma(TM), \tag{2.4}$$

$$\widetilde{\nabla}_X V = -A_V X + \nabla^t_X V,\ \forall V \in \Gamma(trTM), \tag{2.5}$$

for any $X, Y \in \Gamma(TM)$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ while $\{h(X,Y), \nabla^t_X V\}$ belong to $\Gamma(ltr(TM))$.

Suppose $S(TM^{\perp}) \neq 0$, that is, M is either in *Case 1* or in *Case 3*. According to the decomposition (2.3) we consider the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively. Then (2.4)- (2.5) become

$$\widetilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y),\ \forall X, Y \in \Gamma(TM), \tag{2.6}$$

$$\widetilde{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N),\ \forall N \in \Gamma(ltr(TM)), \tag{2.7}$$

$$\widetilde{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W)\ \forall W \in \Gamma(s(TM^{\perp})). \tag{2.8}$$

where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\{\nabla^l_X N,\ D^l(X,W)\} \in \Gamma(ltrTM)$, $\{\nabla^s_X W,\ D^s(X,N)\} \in \Gamma(s(TM^{\perp})$ and $\{\nabla_X Y,\ A_N X,\ A_W X\} \in \Gamma(TM)$ [2]. Then, taking account of (2.6)-(2.8) and the Levi-Civita connection $\widetilde{\nabla}$ is a metric, we obtain

$$\widetilde{g}(h^s(X,Y),W) + g(Y, D^l(X,W)) = g(A_W X, Y), \tag{2.9}$$

$$\widetilde{g}(D^s(X,N),W) = \widetilde{g}(A_W X, N). \tag{2.10}$$

Let P be the projection of $S(TM)$ on M. Then according to (2.1) and (2.3) we have

$$\nabla_X PY = \nabla^*_X PY + h^*(X,PY), \tag{2.11}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad\, TM)$. By using above equations we obtain

$$g(h^l(X,Y),\xi) + g(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.13}$$

$$g(h^*(X,PY),N) = g(A_N X, PY), \tag{2.14}$$

$$g(h^l(X,PY),\xi) = g(A^*_\xi X, PY), \tag{2.15}$$

$$g(A_N X, PY) = g(N, \widetilde{\nabla}_X PY), \tag{2.16}$$

$$g(h^l(X,\xi),\xi) = 0,\ A^*_\xi \xi = 0, \tag{2.17}$$

Taking into account that $\widetilde{\nabla}$ is metric connection and by using (2.6), we get

$$(\nabla_X g)(Y,Z) = \widetilde{g}(h^l(X,Y),Z) + \widetilde{g}(h^l(X,Z),Y), \tag{2.18}$$

However, from (2.11), it is easy to show that ∇^* is a metric connection on $S(TM)$ [2].

3. Semi-invariant Lightlike Submanifolds of Indefinite Kaehler Manifolds

In this section, we introduce semi-invariant lightlike submanifolds of indefinite Kaehler manifolds.

Let $(\widetilde{M}, \widetilde{g})$ be an indefinite Kaehler manifold [17]. This means that $\widetilde{M}$ admits a tensor field J of type $(1,1)$ on M such that, $\forall X, Y \in \Gamma(T\widetilde{M})$, we have

$$J^2 = -I, \qquad \widetilde{g}(JX, JY) = \widetilde{g}(X, Y), \qquad (\widetilde{\nabla}_X J)Y = 0. \tag{3.1}$$

Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of an indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g}, J)$. For each X tangent to M, JX can be written as follows:

$$JX = fX + wX = fX + w_l X + w_s X, \tag{3.2}$$

where fX and wX are the tangential and the transversal parts of JX, w_l and w_s are projections on $ltrTM$ and $S(TM^{\perp})$, respectively. In addition, for any $V \in \Gamma(tr(TM))$, JV can be written as;

$$JV = BV + CV, \tag{3.3}$$

where BV and CV are the tangential and the transversal parts of JV, respectively.

Definition 3.1. Let $(M, g, S(TM), S(TM^{\perp}))$ be a lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. If $JRad\,TM \subset S(TM)$, $JltrTM \subset S(TM)$ and $J(S(TM^{\perp})) \subset S(TM)$ then we say that M is a semi-invariant lightlike submanifold of indefinite Kaehler manifold.

If we set $L_1 = JRad\,TM$, $L_2 = JltrTM$ and $L_3 = J(S(TM^{\perp}))$ then we can write

$$S(TM) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3. \tag{3.4}$$

where L_0 is a $(m-4)-$ dimensional distribution. Hence we have the following decomposition

$$TM = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp Rad\,TM, \tag{3.5}$$

$$T\widetilde{M} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp S(TM^{\perp}) \perp \{Rad\,TM \oplus ltrTM\}. \tag{3.6}$$

If we set

$$L = L_0 \perp L_1 \perp Rad\,TM \qquad and \qquad L' = L_2 \perp L_3, \tag{3.7}$$

then we can write

$$TM = L \oplus L'. \tag{3.8}$$

In where L and L' is invariant and anti-invariant distributions with respect to J, respectively.

Proposition 3.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then M is invariant lightlike submanifold if and only if $L' = \{0\}$.*

Proposition 3.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. The distribution L_0 is invariant with respect to J.*

Example 3.1. Let $\widetilde{M} = R^8_4$ be a 8− dimensional manifold with signature $(-,-,+,+,-,-,+,+)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ be the standart coordinate system of R^8_4. If we set

$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),$$

then $J^2 = -I$ and J is a complex structure on R^8_4. Consider the submanifold M in $\widetilde{M}$ defined by the equations:

$$\begin{aligned} x_1 &= -\sqrt{2}t_2 + \sqrt{2}t_4 + \sqrt{2}t_5 - \sqrt{2}t_6, \\ x_2 &= \sqrt{2}t_1 - \sqrt{2}t_3 - \sqrt{2}t_4 + \sqrt{2}t_5 + 3\sqrt{2}t_6, \\ x_3 &= t_1 - t_2 + t_3 - t_5 - t_6, \\ x_4 &= t_1 + t_2 - t_3 + t_4 + t_6, \\ x_5 &= -\sqrt{2}t_2 + \sqrt{2}t_4 - \sqrt{2}t_5 - 3\sqrt{2}t_6, \\ x_6 &= \sqrt{2}t_1 + \sqrt{2}t_3 + \sqrt{2}t_4 + \sqrt{2}t_5 - \sqrt{2}t_6, \\ x_7 &= -t_1 - t_2 + t_3 + 3t_4 - 2t_5 - 5t_6, \\ x_8 &= t_1 - t_2 + t_3 + 2t_4 + 3t_5 - t_6, \end{aligned}$$

where t_i, $1 \le i \le 6$, are real parameters. Then

$$TM = Span\{U_1, U_2, U_3, U_4, U_5, U_6\},$$

where

$$\begin{aligned} U_1 &= \sqrt{2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}, \\ U_2 &= -\sqrt{2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - \sqrt{2}\frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}, \\ U_3 &= -\sqrt{2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_6} + \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8}, \\ U_4 &= \sqrt{2}\frac{\partial}{\partial x_1} - \sqrt{2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_6} + 3\frac{\partial}{\partial x_7} + 2\frac{\partial}{\partial x_8}, \\ U_5 &= \sqrt{2}\frac{\partial}{\partial x_1} + \sqrt{2}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \sqrt{2}\frac{\partial}{\partial x_5} + \sqrt{2}\frac{\partial}{\partial x_6} - 2\frac{\partial}{\partial x_7} + 3\frac{\partial}{\partial x_8}, \\ U_6 &= -\sqrt{2}\frac{\partial}{\partial x_1} + 3\sqrt{2}\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - 3\sqrt{2}\frac{\partial}{\partial x_5} - \sqrt{2}\frac{\partial}{\partial x_6} - 5\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}. \end{aligned}$$

it is easy to check that M is a lightlike submanifold and U_1 is a degenerate vector. Then we have $Rad\,TM = Span\{U_1\}$ and $S(TM) = Span\{U_2, U_3, U_4, U_5, U_6\}$. By direct calculations we obtain

$$ltrTM = Span\{N = -\sqrt{2}\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} + \sqrt{2}\frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}\},$$

and

$$S(TM^{\perp}) = Span\{W = 3\sqrt{2}\frac{\partial}{\partial x_1} + \sqrt{2}\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - 3\sqrt{2}\frac{\partial}{\partial x_5} + 3\sqrt{2}\frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} + 5\frac{\partial}{\partial x_8}\}.$$

Thus M is a 6− dimensional 1− lightlike submanifold. Moreover we get

$$J\xi = U_2 \in \Gamma(S(TM)),\ JN = U_3 \in \Gamma(S(TM)),$$
$$JW = U_6 \in \Gamma(S(TM)),\ JU_4 = U_5.$$

Therefore we obtain $L_0 = Span\{U_4, U_5\}$, $L_1 = Span\{J\xi\}$, $L_2 = Span\{JN\}$ and $L_3 = Span\{JW\}$. Thus M is a semi-invariant lightlike submanifold of $\widetilde{M}$.

Now, let M be a semi-invariant lightlike submanifold of an indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$. Since J is parallel on M, From (3.2) and (3.3), we have

$$\nabla_X JY + h(X,JY) = f\nabla_X Y + w\nabla_X Y + Bh(X,Y) + Ch(X,Y), \tag{3.9}$$

for any $X,Y \in \Gamma(TM)$. If we take tangential and transversal parts of the equation (3.9) we have

$$\nabla_X JY = f\nabla_X Y + Bh(X,Y) \tag{3.10}$$

and

$$h(X,JY) = w\nabla_X Y + Ch(X,Y). \tag{3.11}$$

Similarly since J is parallel, from (3.2) and (3.3) we get

$$\begin{aligned} \widetilde{\nabla}_X JY &= \nabla_X fY + h^l(X,fY) + h^s(X,fY) - A_{w_lY}X + \nabla^l_X w_lY \\ &\quad + D^s(X,w_lY) - A_{w_sY}X + \nabla^s_X w_sY + D^l(X,w_sY) \end{aligned} \tag{3.12}$$

and

$$J\widetilde{\nabla}_X Y = f\nabla_X Y + w_l\nabla_X Y + w_s\nabla_X Y + Bh(X,Y) + Ch(X,Y). \tag{3.13}$$

Moreover if we take tangential and transversal parts of the equations (3.12) and (3.13), we have

$$(\nabla_X f)Y = A_{w_lY}X + A_{w_sY}X + Bh(X,Y) \tag{3.14}$$

$$D^s(X,w_lY) = -\nabla^s_X w_sY + w_s\nabla_X Y - h^s(X,fY) + Ch^s(X,Y) \tag{3.15}$$

$$D^l(X,w_sY) = -\nabla^l_X w_lY + w_l\nabla_X Y - h^l(X,fY) + Ch^l(X,Y) \tag{3.16}$$

for any $X,Y \in \Gamma(TM)$.

From the Gauss equation we have the following lemma

Lemma 3.1. *Let $(M,g,S(TM),S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$. Then*

$$g(h(X,Y),W) = g(A_W X,Y),$$

for any $X \in \Gamma(L)$, $Y \in \Gamma(L')$ and $W \in \Gamma(S(TM^\perp))$.

Theorem 3.1. *Let $(M,g,S(TM),S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$. Then M is mixed geodesic if and only if*

$$A^*_\xi X \in \Gamma(L_0 \perp L_2)$$

and

$$A_W X \in \Gamma(L_0 \perp L_2 \perp Rad\, TM)$$

for any $X \in \Gamma(L)$, $Y \in \Gamma(L')$ and $W \in \Gamma(S(TM^\perp))$.

Proof. For any $X \in \Gamma(L)$, $Y \in \Gamma(L')$ and $W \in \Gamma(S(TM^\perp))$, M is mixed geodesic if and only if $g(h(X,Y),\xi) = 0$ and $g(h(X,Y),W) = 0$. From the (2.12), Gauss and Weingarten formulas we get $g(h(X,Y),\xi) = g(Y,A^*_\xi X)$. By using Lemma 3.1 and (3.7) proof is completed. $\square$

Theorem 3.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then M is $L-$ geodesic if and only if*

$$\nabla^*_X J\xi \in \Gamma(L_1 \perp L_3)$$

and

$$g(A_W X, Y) = g(D^l(X, W), Y)$$

for any $X, Y \in \Gamma(L)$ and $W \in \Gamma(S(TM^{\perp}))$.

Proof. For any $X, Y \in \Gamma(L)$ and $W \in \Gamma(S(TM^{\perp}))$, M is mixed geodesic if and only if $g(h(X, Y), \xi) = 0$ and $g(h(X, Y), W) = 0$. By using (2.11), Gauss and Weingarten formulas, we get

$$g(h^s(X, Y), W) = g(Y, A_W X) - g(D^l(X, W), Y), \tag{3.17}$$

and

$$g(h^l(X, Y), \xi) = -g(JY, \nabla^*_X J\xi). \tag{3.18}$$

From the (3.17) and (3.18) we have our assertion. □

Theorem 3.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent;*
(i) *There are no L_1 and L_3 component of $A_W X$ and $A^*_\xi X$, for any $X \in \Gamma(L^{'})$.*
(ii) *M is $L^{'}-$ geodesic.*
(iii) *There is no $L^{'}-$ component of $A_{JY} X$, for any $X \in \Gamma(L^{'})$.*

Proof. For any $X, Y \in \Gamma(L^{'})$ and $W \in \Gamma(S(TM^{\perp}))$, by using (2.12) and Gauss-Weingarten formulas we have

$$g(h(X, Y), \xi) = g(Y, A^*_\xi X) \tag{3.19}$$

and

$$g(h(X, Y), W) = g(Y, A_W X). \tag{3.20}$$

from (3.19) and (3.20) we get $(i) \Leftrightarrow (ii)$. Moreover from (3.1), Gauss and Weingarten formulas we get

$$g(h(X, Y), W) = -g(A_{JY} X, JW), \tag{3.21}$$

and

$$g(h(X, Y), \xi) = -g(A_{JY} X, JW). \tag{3.22}$$

From (3.21) and (3.22) we have $(ii) \Leftrightarrow (iii)$. □

Theorem 3.4. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent;*
(i) *L is parallel distribution.*
(ii) *$h(X, JY) = 0$, for any $X, Y \in \Gamma(L)$.*
(iii) *$(\nabla_X f)Y = Jh(X, Y)$, for any $X \in \Gamma(L^{'})$.*

Proof. For any $X, Y \in \Gamma(L)$, by using Gauss formula and (3.1) we have

$$g(\nabla_X Y, J\xi) = -g(h^l(X, JY), \xi), \tag{3.23}$$
$$g(\nabla_X Y, Ju) = -g(h^s(X, JY), u). \tag{3.24}$$

From (3.23) and (3.24) we get $(i) \Leftrightarrow (ii)$. Moreover from (3.1), (3.10) and Gauss formula we obtain

$$h(X, JY) = (\nabla_X f)Y + w\nabla_X Y + Jh(X, Y). \tag{3.25}$$

If we take tangential and transversal parts of this last equation we have

$$(\nabla_X f)Y = Jh(X, Y).$$

This is $(ii) \Leftrightarrow (iii)$. □

Proposition 3.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then $L^{'}$ is parallel distribution if and only if $A_{JY}X$ belongs to $\Gamma(L^{'})$, for any $X, Y \in \Gamma(L^{'})$.*

Proof. For any $X, Y \in \Gamma(L^{'})$ and $Z \in \Gamma(L_0)$, from (3.1), Gauss and Weingarten formulas we get

$$g(\nabla_X Y, N) = -g(A_{JY}X, JN), \tag{3.26}$$
$$g(\nabla_X Y, JN) = -g(A_{JY}X, N), \tag{3.27}$$
$$g(\nabla_X Y, Z) = g(A_{JY}X, JZ). \tag{3.28}$$

From (3.26), (3.27) and (3.28), the proof is completed. □

Theorem 3.5. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then the following assertions are equivalent;*
(i) *The distribution L is integrable.*
(ii) *$h(X, JY) = h(Y, JX)$, for any $X, Y \in \Gamma(L)$.*
(iii) *$(\nabla_X J)Y = (\nabla_Y J)X$, for any $X, Y \in \Gamma(L)$.*

Proof. For any $X, Y \in \Gamma(L)$, from (3.1) and Gauss formula we obtain

$$g([X, Y], J\xi) = g(h^l(Y, JX) - h^l(X, JY), \xi), \tag{3.29}$$

and

$$g([X, Y], Ju) = g(h^s(Y, JX) - h^s(X, JY), u). \tag{3.30}$$

Thus from (3.29) and (3.30) we get $(i) \Leftrightarrow (ii)$. Moreover by using Gauss formula, from (3.1) and (3.10) we obtain

$$h(X, JY) = -(\nabla_X f)Y + w\nabla_X Y, \tag{3.31}$$

and

$$h(Y, JX) = -(\nabla_Y f)X + +w\nabla_Y X. \tag{3.32}$$

Comparing the tangential and normal parts (3.31) and (3.32), we have $(ii) \Leftrightarrow (iii)$. □

Proposition 3.4. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then the distribution $L^{'}$ is integrable if and only if*

$$A_{JX}Y = A_{JY}X,$$

for any $X, Y \in \Gamma(L^{'})$.

Proof. For any $X, Y \in \Gamma(L^{'})$, $Z \in \Gamma(L_0)$ and $N \in \Gamma(ltrTM)$, from (3.1) and Weingarten formula we obtain

$$g([X, Y], JN) = g(A_{JY}X - A_{JX}Y, N), \tag{3.33}$$

and

$$g([X, Y], Z) = g(A_{JY}X - A_{JX}Y, JZ). \tag{3.34}$$

From (3.33) and (3.34) proof is completed. □

Theorem 3.6. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. Then M is locally a product manifold according to the decomposition (3.25) if and only if f is parallel with respect to induced connection ∇, that is $\nabla f = 0$.*

Proof. Let M be locally a product manifold. Then the leaves of distributions L and $L^{'}$ are both total geodesic in M. Since the distribution L is invariant with respect to J then, for any $Y \in \Gamma(L)$, $JY \in \Gamma(L)$. Thus $\nabla_X Y$ and $\nabla_X fY$ belong to $\Gamma(L)$, for any $X \in \Gamma(TM)$. From the Gauss formula, we obtain

$$\nabla_X fY + h(X, fY) = f\nabla_X Y + w\nabla_X Y + Jh(X, Y). \tag{3.35}$$

Comparing the tangential and normal parts with respect to L of (3.35), we have

$$(\nabla_X f)Y = 0. \tag{3.36}$$

For any $X \in \Gamma(TM)$ and $Z \in \Gamma(L^{'})$, Since $fZ = 0$, we get $\nabla_X fZ = 0$ and $f\nabla_X Z = 0$, that is $(\nabla_X f)Z = 0$. Thus we have $\nabla f = 0$ on M.

Conversely, we assume that $\nabla f = 0$ on M. Then we have $\nabla_X fY = f\nabla_X Y$, for any $X, Y \in \Gamma(L)$ and $\nabla_W fZ = f\nabla_W Z$, for any $W, Z \in \Gamma(L^{'})$. Thus it follows that $\nabla_X Y \in \Gamma(L)$ and $\nabla_W Z \in \Gamma(L^{'})$. Hence, the leaves of the distributions L and $L^{'}$ are total geodesic in M. □

Theorem 3.7. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$. The radical distribution is integrable if and only if*

(i) $h^*(\xi, J\xi^{'}) = h^*(\xi^{'}, J\xi)$ $\xi, \xi^{'} \in \Gamma(Rad\,TM)$

(ii) $A^*_{\xi^{'}}\xi = A^*_{\xi}\xi^{'}$ $\xi, \xi^{'} \in \Gamma(Rad\,TM)$

(ii) $h^l(\xi, J\xi^{'}) = h^l(\xi^{'}, J\xi)$ $\xi, \xi^{'} \in \Gamma(Rad\,TM)$.

Proof. For any $\xi, \xi^{'} \in \Gamma(Rad\ TM)$, from (2.11), (3.1) and the Gauss formula we obtain

$$g([\xi, \xi^{'}], JN) = g(-h^*(\xi, J\xi^{'}) + h^*(\xi^{'}, J\xi), N), \tag{3.37}$$

and

$$g([\xi, \xi^{'}], J\xi_1) = g(-h^l(\xi, J\xi^{'}) + h^l(\xi^{'}, J\xi), \xi_1). \tag{3.38}$$

Furthermore by using (3.1), Gauss formula and (2.12) we obtain

$$g([\xi, \xi'], Ju) = g(-h^s(\xi, J\xi') + h^s(\xi', J\xi), u), \tag{3.39}$$

and

$$g([\xi, \xi'], X) = g(-A^*_{\xi'}\xi + A^*_{\xi}\xi', X), \tag{3.40}$$

for any $X \in \Gamma(L_0)$. From (3.37), (3.38), (3.39) and (3.40) we have the our assertion. □

4. Lightlike Submanifolds of indefinite Kaehler manifolds with quarter symmetric non-metric connection

Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ and $\widetilde{\nabla}$ be the Levi-Civita connection on $\widetilde{M}$. If we set

$$\widetilde{D}_X Y = \widetilde{\nabla}_X Y + \pi(Y)JX \tag{4.1}$$

for any $X, Y \in \Gamma(T\widetilde{M})$, then $\widetilde{D}$ is linear connection on $\widetilde{M}$, where π is a $1-$ form on $\widetilde{M}$ with U as associated vector field, that is

$$\pi(X) = \widetilde{g}(X, U). \tag{4.2}$$

Let the torsion tensor of $\widetilde{D}$ on $\widetilde{M}$ be denoted by $\widetilde{T}$.

$$\widetilde{T}(X, Y) = \pi(Y)JX - \pi(X)JY, \tag{4.3}$$

and

$$(\widetilde{D}_X\widetilde{g})(Y, Z) = \pi(Y)\widetilde{g}(JX, Z) + \pi(Z)\widetilde{g}(JX, Y) \tag{4.4}$$

for any $X, Y \in \Gamma(TM)$. Thus $\widetilde{D}$ is a quarter-symmetric non-metric connection on $\widetilde{M}$.

Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then the Gauss and Weingarten formulas with respect to $\widetilde{D}$ are given by,

$$\widetilde{D}_X Y = D_X Y + \widetilde{h}^l(X, Y) + \widetilde{h}^s(X, Y), \tag{4.5}$$

$$\widetilde{D}_X N = -\widetilde{A}_N X + \widetilde{\nabla}^l_X N + \widetilde{D}^s(X, N), \tag{4.6}$$

$$\widetilde{D}_X W = -\widetilde{A}_W X + \widetilde{\nabla}^s_X W + \widetilde{D}^l(X, W), \tag{4.7}$$

for any $X, Y \in \Gamma(TM)$, $N \in \Gamma(ltrTM)$ *and* $W \in \Gamma(S(TM^{\perp}))$, where $D_X Y, \widetilde{A}_N X, \widetilde{A}_W X \in \Gamma(TM)$ and $\widetilde{\nabla}^l$ and $\widetilde{\nabla}^s$ are linear connections on $ltrTM$ and $S(TM^{\perp})$, respectively. Both $\widetilde{A}_N$ and $\widetilde{A}_W$ are linear operators on $\Gamma(TM)$. From (3.2), (4.5), (4.6) and (4.7) we obtain

$$D_X Y = \nabla_X Y + \pi(Y) f X, \tag{4.8}$$

$$\widetilde{h}^l(X,Y) = h^l(X,Y) + \pi(Y) w_l X, \tag{4.9}$$

$$\widetilde{h}^s(X,Y) = h^s(X,Y) + \pi(Y) w_s X, \tag{4.10}$$

$$\widetilde{A}_N X = A_N X - \pi(N) f X, \tag{4.11}$$

$$\widetilde{\nabla}^l_X N = \nabla^l_X N + \pi(N) w_l X, \tag{4.12}$$

$$\widetilde{D}^s(X,N) = D^s(X,N) + \pi(N) w_s X, \tag{4.13}$$

$$\widetilde{A}_W X = A_W X - \pi(W) f X, \tag{4.14}$$

$$\widetilde{\nabla}^s_X W = \nabla^s_X W + \pi(N) w_s X, \tag{4.15}$$

$$\widetilde{D}^l(X,N) = D^l(X,N) + \pi(N) w_l X, \tag{4.16}$$

From (4.8) we get

$$(D_X g)(Y,Z) = g(h(X,Y),Z) + g(h(X,Z),Y) - \pi(Y) g(fX,Z) - \pi(Z) g(fX,Y) \tag{4.17}$$

On the other hand, the torsion tensor of the induced connection D is

$$T^D(X,Y) = \pi(Y) f X - \pi(X) f Y \tag{4.18}$$

From the last two equations we have the following proposition.

Proposition 4.1. *Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then the induced connection D is a quarter-symmetric non-metric connection on the lightlike submanifold M.*

For any $X, Y \in \Gamma(TM)$ we can write

$$D_X PY = D^*_X PY + \widetilde{h}^*(X, PY) \tag{4.19}$$

and

$$D_X \xi = -\widetilde{A}^*_\xi X + \widetilde{\nabla}^{*t}_X \xi \tag{4.20}$$

for any $X, Y \in \Gamma(TM)$, where $D^*_X PY$, $\widetilde{A}^*_\xi X \in \Gamma(S(TM))$ and $\widetilde{h}^*(X, PY) \in \Gamma(Rad\,TM)$. From (4.8), (4.19), (4.20) we obtain

$$D^*_X PY = \nabla^*_X PY + \pi(PY) P f X, \tag{4.21}$$

$$\widetilde{h}^*(X, PY) = h^*(X, PY) + \pi(PY) \sum_i \eta_i(fX) \xi_i, \tag{4.22}$$

and

$$\widetilde{A}^*_\xi X = A^*_\xi X - \pi(\xi) P f X, \tag{4.23}$$

$$\widetilde{\nabla}^{*t}_X \xi = \nabla^{*t}_X \xi + \pi(\xi) \eta(fX) \xi, \tag{4.24}$$

where $D^*_X PY$, $\widetilde{A}^*_\xi X \in \Gamma(S(TM))$, $\eta_i(X) = g(X, N_i)$, and $\{\xi_1, ..., \xi_r\}$ is basis of $\Gamma(Rad\,TM)$ for $i \in \{1, ..., r\}$. From (4.9), (4.11), (4.22) and (4.23)) we have

$$g(\widetilde{h}^l(X, PY), \xi) = g(\widetilde{A}^*_\xi X, PY) + \pi(\xi) g(PfX, PY) + \pi(Y) g(w_l X, \xi) \tag{4.25}$$

and

$$g(\widetilde{h}^*(X,PY),N) = g(\widetilde{A}_N X,PY)+\pi(N)g(fX,PY) +\pi(PY)\eta(fX). \tag{4.26}$$

Also from (4.23) we obtain

$$g(\widetilde{A}^*_\xi PX,PY)=g(A^*_\xi PY,PX)-\pi(\xi)g(fPX,PY),$$

and

$$g(\widetilde{A}^*_\xi PY,PX)=g(A^*_\xi PY,PX)-\pi(\xi)g(fPY,PX).$$

Thus, from the last two equations and from (4.23) we get

$$g(\widetilde{A}^*_\xi PX,PY)=g(\widetilde{A}^*_\xi PY,PX), \tag{4.27}$$

and

$$\widetilde{A}^*_\xi \xi = -\pi(\xi)J\xi. \tag{4.28}$$

From (4.9), (4.10) and (4.14) we have the following lemma.

Lemma 4.1. *Let $(M,g,S(TM),S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then we have*

$$g(\widetilde{h}(X,Y),W)=g(A_W X,Y)=g(\widetilde{A}_W X,Y)+\pi(N)g(fX,Y),$$

for any $W\in\Gamma(S(TM^\perp))$, $X\in\Gamma(L)$ and $Y\in\Gamma(L^{'})$.

From (4.9), (4.10) and (4.23) we get the following lemma.

Lemma 4.2. *Let $(M,g,S(TM),S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then*

$$g(\widetilde{h}(X,Y),\xi)=g(A^*_\xi,Y)=g(\widetilde{A}^*_\xi,Y)+\pi(\xi)g(PfX,Y),$$

for any $\xi\in\Gamma(Rad\,TM)$, $X\in\Gamma(L)$ and $Y\in\Gamma(L^{'})$.

Remark 4.1. Since $\widetilde{h}$ is not symmetric, for any $X\in\Gamma(L)$ and $Y\in\Gamma(L^{'})$ if we take $\widetilde{h}(X,Y)=0$ then $\widetilde{h}(Y,X)$ may not be zero. Thus we need new definition.

Definition 4.1. Let $(M,g,S(TM),S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. If $\widetilde{h}(X,Y)=0$, for any $\forall X\in\Gamma(L)$ and $\forall Y\in\Gamma(L^{'})$, then M is called $L-$ mixed geodesic submanifold.

From the last two lemma, we have the following theorem:

Theorem 4.1. *Let $(M,g,S(TM),S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M},\widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then the following assertions are equivalent:*
(i) *M is mixed geodesic.*
(ii) *M is $L-$ mixed geodesic with respect to quarter symmetric non-metric connection.*
(iii) *$g(\widetilde{A}_W X,Y)=-\pi(N)g(fX,Y)$ and $g(\widetilde{A}^*_\xi,Y)=-\pi(\xi)g(PfX,Y)$, for any $X\in\Gamma(L)$ and $Y\in\Gamma(L^{'})$.*

Theorem 4.2. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. M is $L-$ geodesic with respect to quarter symmetric non-metric connection if and only if*
(i) $\nabla^{*}_{X} J\xi \in \Gamma(L_1 \perp L_3)$ *and* $\nabla^{*}_{X} JY \in \Gamma(L_0 \perp L_1 \perp L_3)$.
(ii) $A_W X \in \Gamma(L_1 \perp L_3)$.

Proof. M is $L-$ geodesic if and only if $g(\widetilde{h}(X,Y), W) = 0$ and $g(\widetilde{h}(X,Y), \xi) = 0$, for any $X, Y \in \Gamma(L)$. From (4.9), (4.10) and Theorem 3.2 we have

$$g(\widetilde{h}(X,Y), W) = g(A_W X, Y) = 0,$$

and

$$g(\widetilde{h}(X,Y), \xi) = g(h^l(X,Y), \xi) = 0.$$

Thus proof is completed. □

Theorem 4.3. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. M is $L^{'}-$ geodesic with respect to quarter symmetric non-metric connection if and only if*

$$A_{JY} X = -\pi(Y) X,$$

for any $X, Y \in \Gamma(L^{'})$.

Proof. For any $X, Y \in \Gamma(L^{'})$ and $W \in \Gamma(S(TM^{\perp}))$, from (4.5), (4.1), (3.1) and Weingarten formula we obtain

$$g(\widetilde{h}(X,Y), W) = -g(A_{JY} X + \pi(Y) X, JW),$$

and

$$g(\widetilde{h}(X,Y), \xi) = -g(A_{JY} X + \pi(Y) X, J\xi).$$

From the last two equations proof is completed. □

From (4.9), (4.10) and Theorem 3.4 we have the following Corollary.

Corollary 4.1. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then the following assertions are equivalent:*
(i) *L is parallel distribution with respect to quarter symmetric non-metrik connection D.*
(ii) *L is parallel distribution with respect to induced connection ∇.*
(iii) $h(X, JY) = 0,\ X, Y \in \Gamma(L)$.
(iv) $\widetilde{h}(X, JY) = 0,\ X, Y \in \Gamma(L)$.
(v) $(\nabla_X f) Y = J h(X,Y),\ X, Y \in \Gamma(L)$.

Theorem 4.4. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\widetilde{M}, \widetilde{g})$ with quarter symmetric non-metric connection $\widetilde{D}$. Then the distribution $L^{'}$ is parallel with respect to quarter symmetric non-metric connection if and only if*

$$\widetilde{A}_{JY} X \in \Gamma(L^{'}),$$

for any $X, Y \in \Gamma(L')$.

Proof. For any $X, Y \in \Gamma(L')$ we know that $fX = 0$. Thus from (4.11) and (4.14) we get $\widetilde{A}_{JY}X = A_{JY}X$. Therefore by using Proposition(3.3) we have our assertion. □

From (4.9), (4.10) and Theorem 3.5 we have the following theorem

Theorem 4.5. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold* $(\widetilde{M}, \widetilde{g})$ *with quarter symmetric non-metric connection* $\widetilde{D}$. *The distribution* L *is integrable with respect to quarter symmetric non-metric connection if and only if*

$$\widetilde{h}(X, JY) = \widetilde{h}(Y, JX),$$

for any $X, Y \in \Gamma(L)$.

Therefore from Theorem 3.5 we have the following corollary

Corollary 4.2. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold* $(\widetilde{M}, \widetilde{g})$ *with quarter symmetric non-metric connection* $\widetilde{D}$. *Then the following assertions are equivalent:*
(i) L *is integrable distribution with respect to quarter symmetric non-metric connection .*
(ii) L *is integrable distribution with respect to* ∇ *connection .*
(iii) $\widetilde{h}(X, JY) = \widetilde{h}(Y, JX),\ X, Y \in \Gamma(L)$.
(iv) $h(X, JY) = h(Y, JX),\ X, Y \in \Gamma(L)$.
(v) $(\nabla_X f)Y = (\nabla_Y f)X,\ X, Y \in \Gamma(L)$.

Proposition 4.2. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold* $(\widetilde{M}, \widetilde{g})$ *with quarter symmetric non-metric connection* $\widetilde{D}$. *Then* L' *is integrable distribution with respect to quarter symmetric non-metric connection if and only if*

$$\widetilde{A}_{JY}X = \widetilde{A}_{JX}Y,$$

for any $X, Y \in \Gamma(L')$.

Proof. For any $X, Y \in \Gamma(L')$ we know that $fX = 0$. Thus from (4.11) and (4.14) we get $\widetilde{A}_{JY}X = A_{JY}X$. Therefore from Proposition 3.4 proof is completed. □

From Proposition 3.4 and Proposition 4.2 we have the following corollary.

Corollary 4.3. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold* $(\widetilde{M}, \widetilde{g})$ *with quarter symmetric non-metric connection* $\widetilde{D}$. *Then the following assertions are equivalent:*
(i) L' *is integrable distribution with respect to quarter symmetric non-metric connection.*
(ii) L' *is integrable distribution with respect to* ∇ *connection.*
(iii) $\widetilde{A}_{JY}X = \widetilde{A}_{JX}Y,\ X, Y \in \Gamma(L')$.
(iv) $A_{JY}X = A_{JX}Y,\ X, Y \in \Gamma(L')$.

Theorem 4.6. *Let* $(M, g, S(TM), S(TM^{\perp}))$ *be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold* $(\widetilde{M}, \widetilde{g})$ *with quarter symmetric non-metric*

connection $\widetilde{D}$. The radical distribution $Rad\,TM$ is integrable with respect to quarter symmetric non-metric connection if and only if
(i) $\widetilde{h}^*(\xi, J\xi^{'}) = \widetilde{h}^*(\xi^{'}, J\xi),\ for\ any\ \xi, \xi^{'} \in \Gamma(Rad\,TM),$
(ii) $\widetilde{h}^l(\xi, J\xi^{'}) = \widetilde{h}^l(\xi^{'}, J\xi)\ for\ any\ \xi, \xi^{'} \in \Gamma(Rad\,TM),$
(iii) $\widetilde{A}^*_{\xi}\xi^{'} - \widetilde{A}^*_{\xi^{'}}\xi = -\pi(\xi)f\xi^{'} + \pi(\xi^{'})f\xi\ for\ any\ \xi, \xi^{'} \in \Gamma(Rad\,TM).$

Proof. For any $\xi, \xi^{'} \in \Gamma(Rad\,TM)$ we know that $\eta(f\xi) = 0$, thus from (4.22) we have

$$\widetilde{h}^*(\xi, J\xi^{'}) = h^*(\xi, J\xi^{'}), \tag{4.29}$$

$$\widetilde{h}^*(\xi^{'}, J\xi) = h^*(\xi^{'}, J\xi). \tag{4.30}$$

Since $w\xi = 0$, from (4.9) we get

$$\widetilde{h}^l(\xi, J\xi^{'}) = h^l(\xi, J\xi^{'}), \tag{4.31}$$

$$\widetilde{h}^l(\xi^{'}, J\xi) = h^l(\xi^{'}, J\xi). \tag{4.32}$$

From (4.23) we obtain

$$\widetilde{A}^*_{\xi}\xi^{'} - \widetilde{A}^*_{\xi^{'}}\xi = A^*_{\xi}\xi^{'} - A^*_{\xi^{'}}\xi - \pi(\xi)f\xi^{'} + \pi(\xi^{'})f\xi. \tag{4.33}$$

From (4.29), (4.30), (4.31), (4.32), (4.33) and Theorem(3.7) proof is completed. □

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References

[1] Duggal, K. L. and Bejancu, A., Lightlike Submanifolds of Semi-Riemannian Manifolds and Applications, Kluwer, Dordrecht, 1996.
[2] Duggal, K. L. and Sahin, B., Differential Geometry of Lightlike Submanifolds, Birkhauser Verlag AG, Basel-Boston-Berlin, 2010.
[3] Duggal, K. L. and Sahin, B. Screen Cauchy Riemann lightlike submanifolds, Acta Math. Hungar., 106(1-2) (2005), 137165
[4] Duggal, K. L. and Sahin, B. Generalized Cauchy Riemann lightlike submanifolds, Acta Math. Hungar., 112(1-2), (2006), 113136.
[5] Duggal, K. L. and Sahin, B. Lightlike submanifolds of indefinite Sasakian manifolds, Int. J. Math. Math. Sci., 2007, Art ID 57585, 121.[162]
[6] Duggal, K. L. and Sahin, B. Contact generalized CR-lightlike submanifolds of Sasakian submanifolds. Acta Math. Hungar., 122, No. 1-2, (2009), 4558.
[7] Atçen, M. and Kılıç, E., Semi-Invariant lightlike submanifolds of a semi-Riemannian product manifold, Kodai Math. J., Vol. 30, No. 3, (2007), pp. 361-378.
[8] Kılıç, E. and Şahin, B., Radical anti-invariant lightlike submanifolds of a semi-Riemannian product manifold, Turkish J. Math., 32, (2008), 429-449.
[9] Kılıc, E. and Bahadir, O., Lightlike Hypersurfaces of a Semi-Riemannian Product Manifold and Quarter-Symmetric Nonmetric Connections, Hindawi Publishing Corporation International Journal of Mathematics and Mathematical Science, 17 pages, 2012
[10] H. A. Hayden, Sub-spaces of a space with torsion, Proceedings of the London Mathematical Society, vol. 34, 1932, 27-50.
[11] B. G. Schmidt, Conditions on a connection to be a metric connection, Commun. Math. Phys. 29 (1973), 55-59.
[12] S. Golab, On semi-symmetric and quarter-symmetric linear connections, Tensor 29 (1975), 249-254.

[13] N, S. Agashe and M, R, Chafle, A semi symetric non-metric connection in a Riemannian manifold, Indian J. Pure Appl. Math. 23 (1992), 399-409.
[14] Y. Liang. On semi-symmetric recurrent-metric connection, Tensor 55 (1994), 107-112.
[15] M.M Tripathi, A new connection in a Riemannian manifold, International Journal of Geometry. 1 (2006), 15-24.
[16] K.Yano, On semi-symmetric metric connections, Rev. Roumania Math. Pures Appl. 15, 1970 , 1579-1586.
[17] Barros M, Romero A. Indefinite Kaehler manifolds. Math Ann 1982;261:5562.

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