LIGHTLIKE SUBMANIFOLDS OF INDEFINITE KAELHER MANIFOLDS WITH QUARTER SYMMETRIC NON-METRIC CONNECTION

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(Communicated by Cyriaque ATINDOGBE)

Abstract. In this paper, we study lightlike submanifolds of indefinite Kaehler manifolds. We introduce a class of lightlike submanifold called semi-invariant lightlike submanifold. We consider lightlike submanifold with respect to a quarter-symmetric non metric connection which is determined by the complex structure. We give some equivalent conditions for integrability of distributions with respect to the Levi-Civita connection of semi-Riemannian manifold and the quarter symmetric non metric connection and some results.

1. Introduction

The geometry of lightlike submanifolds of a semi-Riemannian manifold was presented in [1] (see also [2]) by K.L. Duggal and A. Bejancu. In [3], [4], [5], [6], K. L. Duggal and B. Sahin introduced and studied geometry of classes of lightlike submanifolds in indefinite Kaehler and indefinite Sasakian manifolds which is an umbrella of CR-lightlike, SCR-lightlike, Screen real GCR-lightlike submanifolds. In [7], M. Atçeken and E. Kılıç introduced semi-invariant lightlike submanifolds of a semi-Riemannian product. In [8], E. Kılıç and B. Şahin introduced radical anti-invariant lightlike submanifolds of a semi-Riemannian product and gave some examples and results for lightlike submanifolds. In [9], E. Kılıç and O. Bahadır studied lightlike hypersurfaces of a semi-Riemannian product manifold with respect to quarter symmetric non-metric connection.

In [10], H. A. Hayden introduced a metric connection with non-zero torsion on a Riemannian manifold. The properties of Riemannian manifolds with semi-symmetric (symmetric) and non-metric connection have been studied by many authors [11], [12], [13], [14], [15], [16]. The idea of quarter-symmetric linear connections on a differential manifold was introduced by S.Golab [12]. A linear connection is said to be a quarter-symmetric connection if its torsion tensor $\overline{T}$ is of the form

$$\overline{T}(X,Y) = u(Y)\varphi X - u(X)\varphi Y,$$

where $\varphi$ is the complex structure.

Date: Received: October 30, 2013; Revised: May 15, 2014; Accepted: August 21, 2014.

2010 Mathematics Subject Classification. 53C15, 53C25, 53C40.

Key words and phrases. Lightlike submanifolds, Indefinite Kaehler manifolds, Quarter-symmetric connections.
for any vector fields $X, Y$ on a manifold, where $u$ is a 1–form and $\varphi$ is a tensor of type $(1, 1)$.

In this paper, we study lightlike submanifolds of an indefinite Kaehler manifold. First, we introduce semi-invariant lightlike submanifolds of an indefinite Kaehler manifold. We define some special distribution of semi-invariant lightlike submanifolds. Then we give some examples and find their geometric properties. Finally, by considering the quarter-symmetric non-metric connection, we study lightlike submanifolds of an indefinite Kaehler manifold. Then we obtain some results on lightlike submanifolds of an indefinite Kaehler manifold admitting quarter-symmetric non-metric connections. We introduce semi-invariant lightlike submanifold of an indefinite Kaehler manifold.

2. Preliminaries

Let $(\tilde{M}, \tilde{g})$ be a real $(m+n)$–dimensional semi-Riemannian manifold of constant index $\nu$, $1 \leq \nu \leq m+n-1$ and $(M,g)$ be an $m$–dimensional submanifold of $\tilde{M}$. If $\tilde{g}$ is degenerate on the tangent bundle $TM$ of $M$ then $M$ is called a lightlike submanifold of $\tilde{M}$. Denote by $g$ the induced tensor field of $\tilde{g}$ on $M$ and suppose $g$ is degenerate. Then, for each tangent space $T_xM$ we consider

$$T_xM^\perp = \{ Y_x \in T_x\tilde{M} \mid \tilde{g}_x(Y_x, X_x) = 0, \forall X_x \in T_xM \}$$

which is a degenerate $n$–dimensional subspace of $T_x\tilde{M}$. Thus, both $T_xM$ and $T_xM^\perp$ are degenerate orthogonal subspaces but no longer complementary subspaces. For this case, there exists a subspace $RadTM = T_xM \cap T_xM^\perp$ called radical (null) subspace. If the mapping $RadTM : x \in M \rightarrow RadTM_x$ defines a smooth distribution on $M$ of rank $r > 0$, the submanifold $M$ of $\tilde{M}$ is called a $r$–lightlike ($r$–degenerate) submanifold and $RadTM$ is called the radical (lightlike) distribution on $M$. In the following, there are four possible cases:

Case 1. $M$ is called a $r$–lightlike submanifold if $1 \leq r < \min\{m, n\}$.

Case 2. $M$ is called a coisotropic submanifold if $1 < r = n < m$.

Case 3. $M$ is called an isotropic submanifold if $1 < r = m < n$.

Case 4. $M$ is called a totally lightlike submanifold if $1 < r = m = n$ [2].

In this paper, we have considered case 1, there exists a non-degenerate screen distribution $S(TM)$ which is a complementary vector subbundle to $RadTM$ in $TM$. Therefore,

$$TM = RadTM \perp S(TM),$$

in which $\perp$ denotes orthogonal direct sum. Although $S(TM)$ is not unique, it is canonically isomorphic to the factor vector bundle $TM/RadTM$. Denote an $r$–lightlike submanifold by $(M, g, S(TM), S(TM^\perp))$, where $S(TM^\perp)$ is a complementary vector bundle of $RadTM$ in $TM^\perp$ and $S(TM^\perp)$ is non-degenerate with respect to $\tilde{g}$. Let us define that $tr(TM)$ is a complementary (but never orthogonal) vectors bundle to $TM$ in $TM_{|M}$ and

$$tr(TM) = ltr(TM) \perp S(TM^\perp),$$

where $ltr(TM)$ is a complementary (but never orthogonal) vectors bundle to $TM$ in $TM$. Therefore,
where $ltr(TM)$ is an arbitrary lightlike transversal vector bundle of $M$. Then we have
\[ T\tilde{M}_{lTM} = TM \oplus tr(TM) \]
(2.3)
\[ = (Rad TM \oplus ltr(TM)) \perp S(TM) \perp S(TM^\perp) \]
where $\oplus$ denotes direct sum, but it is not orthogonal [2].

The Gauss and Weingarten formulas given by
\[ \nabla_X Y = \nabla_X Y + h(X,Y), \forall X,Y \in \Gamma(TM), \]
(2.4)
\[ \nabla_X V = -\nabla_V X + \nabla_X V, \forall V \in \Gamma(tr(TM)), \]
(2.5)
for any $X, Y \in \Gamma(TM)$, where $\{\nabla_X Y, A_V X\}$ belong to $\Gamma(TM)$ while $\{h(X,Y), \nabla_X^l V\}$ belong to $\Gamma(ltr(TM))$.

Suppose $S(TM^\perp) \neq 0$, that is, $M$ is either in Case 1 or in Case 3. According to the decomposition (2.3) we consider the projection morphisms $L$ and $S$ of $tr(TM)$ on $ltr(TM)$ and $S(TM^\perp)$, respectively. Then (2.4)-(2.5) become
\[ \tilde{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y), \forall X,Y \in \Gamma(TM), \]
(2.6)
\[ \tilde{\nabla}_X N = -A_N X + \nabla_X N + D^s(X,N), \forall N \in \Gamma(ltr(TM)), \]
(2.7)
\[ \tilde{\nabla}_X W = -A_W X + \nabla_X W + D^l(X,W), \forall W \in \Gamma(s(TM^\perp)). \]
(2.8)
where $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $\{\nabla_X^l N, D^l(X,W)\}$ $\in \Gamma(ltr(TM))$, $\{\nabla_X^s W, D^s(X,N)\}$ $\in \Gamma(s(TM^\perp))$ and $\{\nabla_X Y, A_N X, A_W X\} \in \Gamma(TM)$ [2]. Then, taking account of (2.6)-(2.8) and the Levi-Civita connection $\tilde{\nabla}$ is a metric, we obtain
\[ \tilde{g}(h^s(X,Y), W) + g(Y, D^l(X,W)) = g(A_W X, Y), \]
(2.9)
\[ \tilde{g}(D^s(X,N), W) = \tilde{g}(A_W X, N). \]
(2.10)
Let $P$ be the projection of $S(TM)$ on $M$. Then according to (2.1) and (2.3) we have
\[ \nabla_X PY = \nabla_X^p PY + h^s(X,Y), \]
(2.11)
\[ \nabla_X \xi = -A^l_X \xi + \nabla_X^l \xi, \]
(2.12)
for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(Rad TM)$. By using above equations we obtain
\[ g(h^l(X,Y), \xi) + g(Y, h^l(X,\xi)) + g(Y, \nabla_X \xi) = 0, \]
(2.13)
\[ g(h^s(X,Y), N) = g(A_N X, PY), \]
(2.14)
\[ g(h^l(X,Y), \xi) = g(A^l_X Y, PY), \]
(2.15)
\[ g(A_N X, PY) = g(N, \tilde{\nabla}_X PY), \]
(2.16)
\[ g(h^l(X,\xi), \xi) = 0, A^l_X \xi = 0, \]
(2.17)
Taking into account that $\tilde{\nabla}$ is metric connection and by using (2.6), we get
\[ (\nabla_X g)(Y, Z) = \tilde{g}(h^l(X,Y), Z) + \tilde{g}(h^l(X,Z), Y), \]
(2.18)
However, from (2.11), it is easy to show that $\nabla^*\perp$ is a metric connection on $S(TM)$ [2].
3. Semi-invariant Lightlike Submanifolds of Indefinite Kaehler Manifolds

In this section, we introduce semi-invariant lightlike submanifolds of indefinite Kaehler manifolds.

Let \( (\tilde{\mathcal{M}}, \tilde{g}) \) be an indefinite Kaehler manifold \([17]\). This means that \( \tilde{\mathcal{M}} \) admits a tensor field \( J \) of type \((1,1)\) on \( \mathcal{M} \) such that, \( \forall X, Y \in \Gamma(T\tilde{\mathcal{M}}) \), we have

\[
J^2 = -I, \quad \tilde{g}(JX, JY) = \tilde{g}(X, Y), \quad (\tilde{\nabla}_X J)Y = 0.
\]

Let \( (M, g, S(TM), S(TM^\perp)) \) be a lightlike submanifold of an indefinite Kaehler manifold \( (\tilde{\mathcal{M}}, \tilde{g}, J) \). For each \( X \) tangent to \( M \), \( JX \) can be written as follows:

\[
JX = fX + wX = fX + w_tX + w_sX,
\]
where \( fX \) and \( wX \) are the tangential and the transversal parts of \( JX \), \( w_t \) and \( w_s \) are projections on \( ltrTM \) and \( S(TM^\perp) \), respectively. In addition, for any \( V \in \Gamma(tr(TM)) \), \( JV \) can be written as:

\[
JV = BV + CV,
\]
where \( BV \) and \( CV \) are the tangential and the transversal parts of \( JV \), respectively.

**Definition 3.1.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{\mathcal{M}}, \tilde{g}) \). If \( JRad TM \subset S(TM^\perp) \), \( JltrTM \subset S(TM^\perp) \) and \( J(S(TM^\perp)) \subset S(TM) \) then we say that \( M \) is a semi-invariant lightlike submanifold of indefinite Kaehler manifold.

If we set \( L_1 = JRad TM, \quad L_2 = JltrTM \) and \( L_3 = J(S(TM^\perp)) \) then we can write

\[
S(TM) = L_0 \perp \{L_1 \oplus L_2\} \perp L_3.
\]

where \( L_0 \) is a \((m - 4)\)-dimensional distribution. Hence we have the following decomposition

\[
TM = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp Rad TM,
\]

\[
\tilde{TM} = L_0 \perp \{L_1 \oplus L_2\} \perp L_3 \perp S(TM^\perp) \perp \{Rad TM \oplus ltrTM\}.
\]

If we set \( L = L_0 \perp L_1 \perp Rad TM \) and \( L' = L_2 \perp L_3 \),

then we can write

\[
TM = L \oplus L'.
\]

In where \( L \) and \( L' \) is invariant and anti-invariant distributions with respect to \( J \), respectively.

**Proposition 3.1.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{\mathcal{M}}, \tilde{g}) \). Then \( M \) is invariant lightlike submanifold if and only if \( L' = \{0\} \).

**Proposition 3.2.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{\mathcal{M}}, \tilde{g}) \). The distribution \( L_0 \) is invariant with respect to \( J \).
Example 3.1. Let $\tilde{M} = R^8_t$ be a 8–dimensional manifold with signature $(-, -, +, +, -, -, +, +)$ and $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8)$ be the standard coordinate system of $R^8_t$. If we set

$$J(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) = (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7),$$

then $J^2 = -I$ and $J$ is a complex structure on $R^8_t$. Consider the submanifold $M$ in $\tilde{M}$ defined by the equations:

$$x_1 = -\sqrt{2}t_2 + \sqrt{2}t_4 + \sqrt{2}t_5 - \sqrt{2}t_6,$$

$$x_2 = \sqrt{2}t_1 - \sqrt{2}t_3 - \sqrt{2}t_4 + \sqrt{2}t_5 + 3\sqrt{2}t_6,$$

$$x_3 = t_1 - t_2 + t_3 - t_5 - t_6,$$

$$x_4 = t_1 + t_2 - t_3 + t_4 + t_6,$$

$$x_5 = -\sqrt{2}t_2 + \sqrt{2}t_4 - \sqrt{2}t_5 - 3\sqrt{2}t_6,$$

$$x_6 = \sqrt{2}t_1 + \sqrt{2}t_3 + \sqrt{2}t_4 + \sqrt{2}t_5 - \sqrt{2}t_6,$$

$$x_7 = -t_1 - t_2 + t_3 + 3t_4 - 2t_5 - 5t_6,$$

$$x_8 = t_1 - t_2 + t_3 + 2t_4 + 3t_5 - t_6,$$

where $t_i$, $1 \leq i \leq 6$, are real parameters. Then

$$TM = \text{Span}\{U_1, U_2, U_3, U_4, U_5, U_6\},$$

where

$$U_1 = \sqrt{2} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} + \sqrt{2} \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8},$$

$$U_2 = -\sqrt{2} \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - \sqrt{2} \frac{\partial}{\partial x_5} - \frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8},$$

$$U_3 = -\sqrt{2} \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} + \sqrt{2} \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7} + \frac{\partial}{\partial x_8},$$

$$U_4 = \sqrt{2} \frac{\partial}{\partial x_1} - \sqrt{2} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \sqrt{2} \frac{\partial}{\partial x_4} + \sqrt{2} \frac{\partial}{\partial x_6} + 3\frac{\partial}{\partial x_7} + 2\frac{\partial}{\partial x_8},$$

$$U_5 = \sqrt{2} \frac{\partial}{\partial x_1} + \sqrt{2} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} - \sqrt{2} \frac{\partial}{\partial x_5} + \sqrt{2} \frac{\partial}{\partial x_6} - 2\frac{\partial}{\partial x_7} + 3\frac{\partial}{\partial x_8},$$

$$U_6 = -\sqrt{2} \frac{\partial}{\partial x_1} + 3\sqrt{2} \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - 3\sqrt{2} \frac{\partial}{\partial x_5} - \sqrt{2} \frac{\partial}{\partial x_6} - 5\frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}.$$

It is easy to check that $M$ is a lightlike submanifold and $U_1$ is a degenerate vector. Then we have $\text{Rad} TM = \text{Span}\{U_1\}$ and $S(TM) = \text{Span}\{U_2, U_3, U_4, U_5, U_6\}$. By direct calculations we obtain

$$ltr TM = \text{Span}\{N = -\sqrt{2} \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_4} + \sqrt{2} \frac{\partial}{\partial x_5} + \frac{\partial}{\partial x_7} - \frac{\partial}{\partial x_8}\},$$

and

$$S(TM^\perp) = \text{Span}\{W = 3\sqrt{2} \frac{\partial}{\partial x_1} + \sqrt{2} \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_4} - 3\sqrt{2} \frac{\partial}{\partial x_5} + 3\sqrt{2} \frac{\partial}{\partial x_6} - \frac{\partial}{\partial x_7} + 5\frac{\partial}{\partial x_8}\}.$$

Thus $M$ is a 6–dimensional 1–lightlike submanifold. Moreover we get

$$J\xi = U_2 \in \Gamma(S(TM)), \quad JN = U_3 \in \Gamma(S(TM)),$$

$$JW = U_6 \in \Gamma(S(TM)), \quad JU_4 = U_5.$$
Now, let $M$ be a semi-invariant lightlike submanifold of an indefinite Kaehler manifold $(\tilde{M}, \tilde{g})$. Since $J$ is parallel on $M$, From (3.2) and (3.3), we have

\[ \nabla_X JY + h(X, JY) = f\nabla_X Y + w\nabla_X Y + Bh(X, Y) + Ch(X, Y), \]

for any $X, Y \in \Gamma(TM)$. If we take tangential and transversal parts of the equation (3.9) we have

\[ \nabla_X JY = f\nabla_X Y + Bh(X, Y) \] \hspace{1cm} (3.10)

and

\[ h(X, JY) = w\nabla_X Y + Ch(X, Y). \] \hspace{1cm} (3.11)

Moreover if we take tangential and transversal parts of the equations (3.12) and (3.13), we have

\[ \tilde{\nabla}_X JY = \nabla_X fY + h^i(X, fY) - A_{w_i}Y + \nabla^i_X w_iY + D^i(X, w_iY) - A_{w_i}Y + \nabla^i_X w_iY + D^i(X, w_iY) \]

and

\[ J\tilde{\nabla}_X Y = f\nabla_X Y + w_l\nabla_X Y + w_s\nabla_X Y + Bh(X, Y) + Ch(X, Y). \]

Moreover if we take tangential and transversal parts of the equations (3.12) and (3.13), we have

\[ (\nabla_X f)Y = A_{w_i}Y + A_{w_i}Y + Bh(X, Y) \]

\[ D^i(X, w_iY) = -\nabla^i_X w_iY + w_s\nabla_X Y - h^s(X, fY) + Ch(X, Y) \]

\[ D^i(X, w_iY) = -\nabla^i_X w_iY + w_l\nabla_X Y - h^l(X, fY) + Ch(X, Y) \]

for any $X, Y \in \Gamma(TM)$.

From the Gauss equation we have the following lemma

**Lemma 3.1.** Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\tilde{M}, \tilde{g})$. Then

\[ g(h(X, Y), W) = g(A_W X, Y), \]

for any $X \in \Gamma(L), Y \in \Gamma(L^\perp)$ and $W \in \Gamma(S(TM^\perp))$.

**Theorem 3.1.** Let $(M, g, S(TM), S(TM^\perp))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\tilde{M}, \tilde{g})$. Then $M$ is mixed geodesic if and only if

\[ A^s_X Y \in \Gamma(L_0 \perp L_2) \]

and

\[ A_W X \in \Gamma(L_0 \perp L_2 \perp \text{Rad } TM) \]

for any $X \in \Gamma(L), Y \in \Gamma(L^\perp)$ and $W \in \Gamma(S(TM^\perp))$.

**Proof.** For any $X \in \Gamma(L), Y \in \Gamma(L^\perp)$ and $W \in \Gamma(S(TM^\perp))$, $M$ is mixed geodesic if and only if $g(h(X, Y), \xi) = 0$ and $g(h(X, Y), W) = 0$. From the (2.12), Gauss and Weingarten formulas we get $g(h(X, Y), \xi) = g(Y, A^s_X X)$. By using Lemma 3.1 and (3.7) proof is completed. $\square$
Theorem 3.2. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\). Then \(M\) is \(L\)–geodesic if and only if
\[
\nabla^X_J\xi \in \Gamma(L_1 \perp L_3)
\]
and
\[
g(A_W X, Y) = g(D^l(X, W), Y)
\]
for any \(X, Y \in \Gamma(L)\) and \(W \in \Gamma(S(TM^\perp))\).

Proof. For any \(X, Y \in \Gamma(L)\) and \(W \in \Gamma(S(TM^\perp))\), \(M\) is mixed geodesic if and only if \(g(h(X, Y), \xi) = 0\) and \(g(h(X, Y), W) = 0\). By using (2.11), Gauss and Weingarten formulas, we get
\[
g(h^s(X, Y), W) = g(Y, A_W X) - g(D^l(X, W), Y),
\]
(3.17) and
\[
g(h^l(X, Y), \xi) = -g(JY, \nabla^X_J\xi).
\]
(3.18)
From the (3.17) and (3.18) we have our assertion. \(\square\)

Theorem 3.3. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\). Then the following assertions are equivalent;
(i) There are no \(L_1\) and \(L_3\) component of \(A_W X\) and \(A^*_\xi X\), for any \(X \in \Gamma(L')\).
(ii) \(M\) is \(L'\)–geodesic.
(iii) There is no \(L'\)–component of \(A_{JY} X\), for any \(X \in \Gamma(L')\).

Proof. For any \(X, Y \in \Gamma(L')\) and \(W \in \Gamma(S(TM^\perp))\), by using (2.12) and Gauss-Weingarten formulas we have
\[
g(h(X, Y), \xi) = g(Y, A^*_\xi X)
\]
(3.19) and
\[
g(h(X, Y), W) = g(Y, A_W X).
\]
(3.20)
from (3.19) and (3.20) we get \((i) \Leftrightarrow (ii)\). Moreover from (3.1), Gauss and Weingarten formulas we get
\[
g(h(X, Y), W) = -g(A_{JY} X, JW),
\]
(3.21) and
\[
g(h(X, Y), \xi) = -g(A_{JY} X, JW).
\]
(3.22)
From (3.21) and (3.22) we have \((ii) \Leftrightarrow (iii)\). \(\square\)

Theorem 3.4. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\). Then the following assertions are equivalent;
(i) \(L\) is parallel distribution.
(ii) \(h(X, JY) = 0\), for any \(X, Y \in \Gamma(L)\).
(iii) \((\nabla^X f) Y = Jh(X, Y)\), for any \(X \in \Gamma(L')\).
Proof. For any \( X,Y \in \Gamma(L) \), by using Gauss formula and (3.1) we have
\[
\begin{align*}
\tag{3.23} g(\nabla_X Y, J\xi) &= -g(h(X, JY), \xi), \\
\tag{3.24} g(\nabla_X Y, Ju) &= -g(h^*(X, JY), u).
\end{align*}
\]
From (3.23) and (3.24) we get (i) \( \Leftrightarrow \) (ii). Moreover from (3.1), (3.10) and Gauss formula we obtain
\[
\tag{3.25} h(X, JY) = (\nabla_X f)Y + w\nabla_X Y + Jh(X, Y).
\]
If we take tangential and transversal parts of this last equation we have
\[
\tag{3.26} (\nabla_X f)Y = Jh(X, Y).
\]
This is (ii) \( \Leftrightarrow \) (iii). \( \square \)

**Proposition 3.3.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \). Then \( L' \) is parallel distribution if and only if \( A_{JYX} \) belongs to \( \Gamma(L') \), for any \( X,Y \in \Gamma(L) \).

**Proof.** For any \( X,Y \in \Gamma(L') \) and \( Z \in \Gamma(L_0) \), from (3.1), Gauss and Weingarten formulas we get
\[
\begin{align*}
\tag{3.26} g(\nabla_X Y, N) &= -g(A_{JYX}, JN), \\
\tag{3.27} g(\nabla_X Y, JN) &= -g(A_{JYX}, N), \\
\tag{3.28} g(\nabla_X Y, Z) &= g(A_{JYX}, JZ).
\end{align*}
\]
From (3.26), (3.27) and (3.28), the proof is completed. \( \square \)

**Theorem 3.5.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \). Then the following assertions are equivalent;

(i) The distribution \( L \) is integrable.

(ii) \( h(X, JY) = h(Y, JX) \), for any \( X,Y \in \Gamma(L) \).

(iii) \( (\nabla_X J)Y = (\nabla_Y J)X \), for any \( X,Y \in \Gamma(L) \).

**Proof.** For any \( X,Y \in \Gamma(L) \), from (3.1) and Gauss formula we obtain
\[
\tag{3.29} g([X,Y], J\xi) = g(h^*(Y, JX) - h^*(X, JY), \xi),
\]
and
\[
\tag{3.30} g([X,Y], Ju) = g(h^*(Y, JX) - h^*(X, JY), u).
\]
Thus from (3.29) and (3.30) we get (i) \( \Leftrightarrow \) (ii). Moreover by using Gauss formula, from (3.1) and (3.10) we obtain
\[
\tag{3.31} h(X, JY) = -(\nabla_X f)Y + w\nabla_X Y,
\]
and
\[
\tag{3.32} h(Y, JX) = -(\nabla_Y f)X + w\nabla_Y X.
\]
Comparing the tangential and normal parts (3.31) and (3.32), we have (ii) \( \Leftrightarrow \) (iii). \( \square \)
Proposition 3.4. Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\). Then the distribution \(L'\) is integrable if and only if
\[ A_{JX}Y = A_{JY}X, \]
for any \(X, Y \in \Gamma(L')\).

Proof. For any \(X, Y \in \Gamma(L')\), \(Z \in \Gamma(L_0)\) and \(N \in \Gamma(ltrTM)\), from (3.1) and Weingarten formula we obtain
\[ g([X, Y], JN) = g(A_{JY}X - A_{JX}Y, N), \]
and
\[ g([X, Y], Z) = g(A_{JY}X - A_{JX}Y, JZ). \]
From (3.33) and (3.34) proof is completed. \(\square\)

Theorem 3.6. Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\). Then \(M\) is locally a product manifold according to the decomposition (3.25) if and only if \(f\) is parallel with respect to induced connection \(\nabla\), that is \(\nabla f = 0\).

Proof. Let \(M\) be locally a product manifold. Then the leaves of distributions \(L\) and \(L'\) are both total geodesic in \(M\). Since the distribution \(L\) is invariant with respect to \(J\) then, for any \(Y \in \Gamma(L)\), \(JY \in \Gamma(L)\). Thus \(\nabla_X Y\) and \(\nabla_X fY\) belong to \(\Gamma(L)\), for any \(X \in \Gamma(TM)\). From the Gauss formula, we obtain
\[ \nabla_X fY + h(X, fY) = f\nabla_X Y + \omega \nabla_X Y + Jh(X, Y). \]
Comparing the tangential and normal parts with respect to \(L\) of (3.35), we have
\[ (\nabla_X f)Y = 0. \]
For any \(X \in \Gamma(TM)\) and \(Z \in \Gamma(L')\), since \(fZ = 0\), we get \(\nabla_X fZ = 0\) and \(f\nabla_X Z = 0\), that is \((\nabla_X f)Z = 0\). Thus we have \(\nabla f = 0\) on \(M\).

Conversely, we assume that \(\nabla f = 0\) on \(M\). Then we have \(\nabla_X fY = f\nabla_X Y\), for any \(X, Y \in \Gamma(L)'\) and \(\nabla_W fZ = f\nabla_W Z\), for any \(W, Z \in \Gamma(L)\). Thus it follows that \(\nabla_X Y \in \Gamma(L)\) and \(\nabla_X Z \in \Gamma(L)\). Hence, the leaves of the distributions \(L\) and \(L'\) are total geodesic in \(M\). \(\square\)

Theorem 3.7. Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\). The radical distribution is integrable if and only if
\(\begin{align*}
(i) \quad h^*(\xi, J\xi') &= h^*(\xi', J\xi) \quad \xi, \xi' \in \Gamma(Rad TM) \\
(ii) \quad A^*_\xi \xi' &= A^*_{\xi'} \xi \quad \xi, \xi' \in \Gamma(Rad TM) \\
(iii) \quad h^!\xi(J\xi'; J\xi') &= h^!\xi(J\xi; J\xi) \quad \xi, \xi' \in \Gamma(Rad TM).
\end{align*}\)

Proof. For any \(\xi, \xi' \in \Gamma(Rad TM)\), from (2.11), (3.1) and the Gauss formula we obtain
\[ g([\xi, \xi'], JN) = g(-h^*(\xi, J\xi'), h^*(\xi', J\xi), N), \]
and
\[ g([\xi, \xi'], J\xi_1) = g(-h^!\xi(J\xi; J\xi') + h^!\xi(J\xi, J\xi), \xi_1). \]
Furthermore by using (3.1), Gauss formula and (2.12) we obtain
\[(3.39)\quad g([\xi,\xi'],Ju) = g(-h^s(\xi,J\xi') + h^s(\xi',J\xi),u),\]
and
\[(3.40)\quad g([\xi,\xi'],X) = g(-A^*_\xi\xi + A^*_\xi\xi',X),\]
for any \(X \in \Gamma(L_0)\). From (3.37), (3.38), (3.39) and (3.40) we have the our assertion. \(\square\)

4. Lightlike Submanifolds of indefinite Kaehler manifolds with quarter symmetric non-metric connection

Let \((M,g,S(TM),S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M},\tilde{g})\) and \(\tilde{\nabla}\) be the Levi-Civita connection on \(\tilde{M}\). If we set
\[(4.1)\quad \tilde{D}_XY = \tilde{\nabla}_XY + \pi(Y)JX,\]
for any \(X,Y \in \Gamma(T\tilde{M})\), then \(\tilde{D}\) is linear connection on \(\tilde{M}\), where \(\pi\) is a 1-form on \(\tilde{M}\) with \(U\) as associated vector field, that is
\[(4.2)\quad \pi(X) = \tilde{g}(X,U).\]
Let the torsion tensor of \(\tilde{D}\) on \(\tilde{M}\) be denoted by \(\tilde{T}\).
\[(4.3)\quad \tilde{T}(X,Y) = \pi(Y)JX - \pi(X)JY,\]
and
\[(4.4)\quad (\tilde{D}_X\tilde{g})(Y,Z) = \pi(Y)\tilde{g}(JX,Z) + \pi(Z)\tilde{g}(JX,Y)\]
for any \(X,Y \in \Gamma(T\tilde{M})\). Thus \(\tilde{D}\) is a quarter-symmetric non-metric connection on \(\tilde{M}\).

Let \((M,g,S(TM),S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M},\tilde{g})\) with quarter symmetric non-metric connection \(\tilde{D}\). Then the Gauss and Weingarten formulas with respect to \(\tilde{D}\) are given by,
\[(4.5)\quad \tilde{D}_XY = D_XY + \tilde{h}(X,Y) + \tilde{h}^s(X,Y),\]
\[(4.6)\quad \tilde{D}_XN = -\tilde{A}_NX + \tilde{\nabla}_XN + \tilde{D}^s(X,N),\]
\[(4.7)\quad \tilde{D}_XW = -\tilde{A}_WX + \tilde{\nabla}_WX + \tilde{D}^l(X,W),\]
for any \(X,Y \in \Gamma(TM)\), \(N \in \Gamma(\text{ltr}TM)\) and \(W \in \Gamma(S(TM^\perp))\), where \(D_XY, \tilde{A}_NX, \tilde{A}_WX \in \Gamma(TM)\) and \(\tilde{\nabla}^l\) and \(\tilde{\nabla}^s\) are linear connections on \(\text{ltr}TM\) and \(S(TM^\perp)\), respectively. Both \(A_N\) and \(A_W\) are linear operators on \(\Gamma(TM)\). From (3.2), (4.5), (4.6) and (4.7)
we obtain

\begin{align}
(4.8) & \quad D_X Y = \nabla_X Y + \pi(Y)f X, \\
(4.9) & \quad \tilde{h}^i(X, Y) = h^i(X, Y) + \pi(Y)w_i X, \\
(4.10) & \quad \tilde{h}^s(X, Y) = h^s(X, Y) + \pi(Y)w_s X, \\
(4.11) & \quad \tilde{A}_N X = A_N X - \pi(N)f X, \\
(4.12) & \quad \hat{\nabla}^l_X N = \nabla^l_X N + \pi(N)w_i X, \\
(4.13) & \quad \tilde{D}^s(X, N) = D^s(X, N) + \pi(N)w_s X, \\
(4.14) & \quad \tilde{A}_W X = A_W X - \pi(W)f X, \\
(4.15) & \quad \hat{\nabla}^s_X W = \nabla^s_X W + \pi(N)w_s X, \\
(4.16) & \quad \tilde{D}^l(X, N) = D^l(X, N) + \pi(N)w_i X,
\end{align}

From (4.8) we get

\begin{align}
(4.17) & \quad (D_X g)(Y, Z) = g(h(X, Y), Z) + g(h(X, Z), Y) - \pi(Y)g(f X, Z) \\
& \quad - \pi(Z)g(f X, Y)
\end{align}

On the other hand, the torsion tensor of the induced connection $D$ is

\begin{align}
(4.18) & \quad T^D(X, Y) = \pi(Y)f X - \pi(X)f Y
\end{align}

From the last two equations we have the following proposition.

**Proposition 4.1.** Let $(M, g, S(TM), S(TM^⊥))$ be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold $(\bar{M}, \bar{g})$ with quarter symmetric non-metric connection $\bar{D}$. Then the induced connection $D$ is a quarter-symmetric non-metric connection on the lightlike submanifold $M$.

For any $X, Y \in \Gamma(TM)$ we can write

\begin{align}
(4.19) & \quad D_X PY = D^*_X PY + \tilde{h}^*(X, PY) \\
\intertext{and}
(4.20) & \quad D_X \xi = -\tilde{A}^*_\xi X + \tilde{\nabla}^*_X \xi
\end{align}

for any $X, Y \in \Gamma(TM)$, where $D^*_X PY, \tilde{A}^*_X \xi \in \Gamma(S(TM))$ and $\tilde{h}^*(X, PY) \in \Gamma(Rad TM)$. From (4.8), (4.19), (4.20) we obtain

\begin{align}
(4.21) & \quad D^*_X PY = \nabla^*_X PY + \pi(PY)f X, \\
(4.22) & \quad \tilde{h}^*(X, PY) = h^*(X, PY) + \pi(PY)\sum_{i} \eta_i(f X)\xi_i,
\end{align}

and

\begin{align}
(4.23) & \quad \tilde{A}^*_\xi X = A^*_\xi X - \pi(\xi)f X, \\
(4.24) & \quad \tilde{\nabla}^*_X \xi = \nabla^*_X \xi + \pi(\xi)\eta(f X)\xi,
\end{align}

where $D^*_X PY, \tilde{A}^*_X \xi \in \Gamma(S(TM))$, $\eta_i(X) = g(X, N_i)$, and $\{\xi_1, ..., \xi_r\}$ is basis of $\Gamma(Rad TM)$ for $i \in \{1, ..., r\}$. From (4.9), (4.11), (4.22) and (4.23) we have

\begin{align}
(4.25) & \quad g(\tilde{h}^i(X, PY), \xi) = g(\tilde{A}^*_\xi X, PY) + \pi(\xi)g(Pf X, PY) \\
& \quad + \pi(Y)g(w_i X, \xi)
\end{align}
and
\[
(4.26) \quad g(\tilde{h}^*(X, PY), N) = g(\tilde{A}_N X, PY) + \pi(N)g(fX, PY) + \pi(PY)\eta(fX).
\]

Also from (4.23) we obtain
\[
g(\tilde{A}_Y^* PX, PY) = g(A_Y^* PY, PX) - \pi(\xi)g(fPX, PY),
\]
and
\[
g(\tilde{A}_X^* PY, PX) = g(A_X^* PY, PX) - \pi(\xi)g(fPY, PX).
\]
Thus, from the last two equations and from (4.23) we get
\[
(4.27) \quad g(\tilde{A}_Y^* PX, PY) = g(\tilde{A}_Y^* PY, PX),
\]
and
\[
(4.28) \quad \tilde{A}_Y^* \xi = -\pi(\xi)J\xi.
\]

From (4.9), (4.10) and (4.14) we have the following lemma.

**Lemma 4.1.** Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\bar{M}, \bar{g})\) with quarter symmetric non-metric connection \(\bar{D}\). Then we have
\[
g(\bar{h}(X, Y), W) = g(\bar{A}_W X, Y) = g(\bar{A}_W X, Y) + \pi(N)g(fX, Y),
\]
for any \(W \in \Gamma(S(TM^⊥))\), \(X \in \Gamma(L)\) and \(Y \in \Gamma(L^′)\).

From (4.9), (4.10) and (4.23) we get the following lemma.

**Lemma 4.2.** Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\bar{M}, \bar{g})\) with quarter symmetric non-metric connection \(\bar{D}\). Then
\[
g(\tilde{h}(X, Y), \xi) = g(A_X^* Y, \xi) = g(\tilde{A}_X^* Y, \xi) + \pi(\xi)g(\pi fX, Y),
\]
for any \(\xi \in \Gamma(Rad TM)\), \(X \in \Gamma(L)\) and \(Y \in \Gamma(L^′)\).

**Remark 4.1.** Since \(\tilde{h}\) is not symmetric, for any \(X \in \Gamma(L)\) and \(Y \in \Gamma(L^′)\) if we take \(\tilde{h}(X, Y) = 0\) then \(\tilde{h}(Y, X)\) may not be zero. Thus we need new definition.

**Definition 4.1.** Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\bar{M}, \bar{g})\) with quarter symmetric non-metric connection \(\bar{D}\). If \(\tilde{h}(X, Y) = 0\), for any \(\forall X \in \Gamma(L)\) and \(\forall Y \in \Gamma(L^′)\), then \(M\) is called \(L^∠\) mixed geodesic submanifold.

From the last two lemma, we have the following theorem:

**Theorem 4.1.** Let \((M, g, S(TM), S(TM^⊥))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\bar{M}, \bar{g})\) with quarter symmetric non-metric connection \(\bar{D}\). Then the following assertions are equivalent:
\[(i)\] \(M\) is mixed geodesic.
\[(ii)\] \(M\) is \(L^∠\) mixed geodesic with respect to quarter symmetric non-metric connection.
\[(iii)\] \(g(\tilde{A}_W X, Y) = -\pi(N)g(fX, Y)\) and \(g(\tilde{A}_X^* Y, \xi) = -\pi(\xi)g(\pi fX, Y)\), for any \(X \in \Gamma(L)\) and \(Y \in \Gamma(L^′)\).
Theorem 4.2. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\) with quarter symmetric non-metric connection \(\tilde{D}\). \(M\) is \(L\)– geodesic with respect to quarter symmetric non-metric connection if and only if

(i) \(\nabla^*_X J\xi \in \Gamma(L_1 \perp L_3)\) and \(\nabla^*_X JY \in \Gamma(L_0 \perp L_1 \perp L_3)\).

(ii) \(\tilde{A}_W X \in \Gamma(L_1 \perp L_3)\).

Proof. \(M\) is \(L\)– geodesic if and only if \(g(\tilde{h}(X, Y), W) = 0\) and \(g(\tilde{h}(X, Y), \xi) = 0\), for any \(X, Y \in \Gamma(L)\). From (4.9), (4.10) and Theorem 3.2 we have

\[
g(\tilde{h}(X, Y), W) = g(A_W X, Y) = 0,
\]

and

\[
g(\tilde{h}(X, Y), \xi) = g(h^I(X, Y), \xi) = 0.
\]

Thus proof is completed. \(\square\)

Theorem 4.3. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\) with quarter symmetric non-metric connection \(\tilde{D}\). \(M\) is \(L'\)– geodesic with respect to quarter symmetric non-metric connection if and only if

\[
A_J Y X = -\pi(Y) X,
\]

for any \(X, Y \in \Gamma(L')\).

Proof. For any \(X, Y \in \Gamma(L')\) and \(W \in \Gamma(S(TM^\perp))\), from (4.5), (4.1), (3.1) and Weingarten formula we obtain

\[
g(\tilde{h}(X, Y), W) = -g(A_J Y X + \pi(Y) X, JW),
\]

and

\[
g(\tilde{h}(X, Y), \xi) = -g(A_J Y X + \pi(Y) X, J\xi).
\]

From the last two equations proof is completed. \(\square\)

From (4.9), (4.10) and Theorem 3.4 we have the following Corollary.

Corollary 4.1. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\) with quarter symmetric non-metric connection \(\tilde{D}\). Then the following assertions are equivalent:

(i) \(L\) is parallel distribution with respect to quarter symmetric non-metric connection \(D\).

(ii) \(L\) is parallel distribution with respect to induced connection \(\nabla\).

(iii) \(h(X, JY) = 0, X, Y \in \Gamma(L)\).

(iv) \(\tilde{h}(X, JY) = 0, X, Y \in \Gamma(L)\).

(v) \((\nabla_X f)Y = Jh(X, Y), X, Y \in \Gamma(L)\).

Theorem 4.4. Let \((M, g, S(TM), S(TM^\perp))\) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \((\tilde{M}, \tilde{g})\) with quarter symmetric non-metric connection \(\tilde{D}\). Then the distribution \(L'\) is parallel with respect to quarter symmetric non-metric connection if and only if

\[
\tilde{A}_J Y X \in \Gamma(L'),
\]
for any \( X, Y \in \Gamma(L') \).

**Proof.** For any \( X, Y \in \Gamma(L') \) we know that \( fX = 0 \). Thus from (4.11) and (4.14) we get \( \tilde{A}_{JY} X = A_{JY} X \). Therefore by using Proposition (3.3) we have our assertion. \( \square \)

From (4.9), (4.10) and Theorem 3.5 we have the following theorem

**Theorem 4.5.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \) with quarter symmetric non-metric connection \( \tilde{D} \). The distribution \( L \) is integrable with respect to quarter symmetric non-metric connection if and only if

\[
\tilde{h}(X, JY) = \tilde{h}(Y, JX),
\]

for any \( X, Y \in \Gamma(L) \).

Therefore from Theorem 3.5 we have the following corollary

**Corollary 4.2.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \) with quarter symmetric non-metric connection \( \tilde{D} \). Then the following assertions are equivalent:

(i) \( L \) is integrable distribution with respect to quarter symmetric non-metric connection .

(ii) \( L \) is integrable distribution with respect to \( \nabla \) connection .

(iii) \( \tilde{h}(X, JY) = \tilde{h}(Y, JX), \ X, Y \in \Gamma(L) \).

(iv) \( h(X, JY) = h(Y, JX), \ X, Y \in \Gamma(L) \).

(v) \( (\nabla_X f)Y = (\nabla_Y f)X, \ X, Y \in \Gamma(L) \).

**Proposition 4.2.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \) with quarter symmetric non-metric connection \( \tilde{D} \). Then \( L' \) is integrable distribution with respect to quarter symmetric non-metric connection if and only if

\[
\tilde{A}_{JY} X = \tilde{A}_{JX} Y,
\]

for any \( X, Y \in \Gamma(L') \).

**Proof.** For any \( X, Y \in \Gamma(L') \) we know that \( fX = 0 \). Thus from (4.11) and (4.14) we get \( \tilde{A}_{JY} X = A_{JY} X \). Therefore from Proposition 3.4 proof is completed. \( \square \)

From Proposition 3.4 and Proposition 4.2 we have the following corollary.

**Corollary 4.3.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \) with quarter symmetric non-metric connection \( \tilde{D} \). Then the following assertions are equivalent:

(i) \( L' \) is integrable distribution with respect to quarter symmetric non-metric connection.

(ii) \( L' \) is integrable distribution with respect to \( \nabla \) connection.

(iii) \( \tilde{A}_{JY} X = \tilde{A}_{JX} Y, \ X, Y \in \Gamma(L') \).

(iv) \( A_{JY} X = A_{JX} Y, \ X, Y \in \Gamma(L') \).

**Theorem 4.6.** Let \( (M, g, S(TM), S(TM^\perp)) \) be a semi-invariant lightlike submanifold of the indefinite Kaehler manifold \( (\tilde{M}, \tilde{g}) \) with quarter symmetric non-metric
connection $\tilde{\nabla}$. The radical distribution $\text{Rad} TM$ is integrable with respect to quarter symmetric non-metric connection if and only if

(i) $\tilde{h}^*(\xi, J\xi') = h^*(\xi, J\xi')$, for any $\xi, \xi' \in \Gamma(\text{Rad} TM)$,

(ii) $\tilde{h}^l(\xi, J\xi') = h^l(\xi, J\xi')$ for any $\xi, \xi' \in \Gamma(\text{Rad} TM)$,

(iii) $\tilde{A}^*_\xi \xi' - A^*_\xi \xi = -\pi(\xi)f\xi' + \pi(\xi')f\xi$ for any $\xi, \xi' \in \Gamma(\text{Rad} TM)$.

Proof. For any $\xi, \xi' \in \Gamma(\text{Rad} TM)$ we know that $\eta(f\xi) = 0$, thus from (4.22) we have

(4.29) $\tilde{h}^*(\xi, J\xi') = h^*(\xi, J\xi')$,

(4.30) $\tilde{h}^*(\xi', J\xi) = h^*(\xi', J\xi)$.

Since $w\xi = 0$, from (4.9) we get

(4.31) $\tilde{h}^l(\xi, J\xi') = h^l(\xi, J\xi')$,

(4.32) $\tilde{h}^l(\xi', J\xi) = h^l(\xi', J\xi)$.

From (4.23) we obtain

(4.33) $\tilde{A}^*_\xi \xi' - A^*_\xi \xi = A^*_\xi \xi' - A^*_\xi \xi - \pi(\xi)f\xi' + \pi(\xi')f\xi$.

From (4.29), (4.30), (4.31), (4.32), (4.33) and Theorem (3.7) proof is completed. □

Acknowledgement

We have greatly benefited from the referee’s report. So we wish to express our gratitude to the reviewer for his/her valuable suggestions which improved the content and presentation of the paper.

References


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