ON ELLIPTIC LINEAR TIMELIKE PARALLEL WEINGARTEN SURFACES SATISFYING THE CONDITION \[ 2a^r H^r + b^r K^r = c^r \]

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Abstract. In this study, firstly, we obtain the parallel Weingarten surfaces which satisfy the condition \( 2a^r H^r + b^r K^r = c^r \) in Minkowski 3-space. Then we give some geometric properties of these kind of surfaces, such as their Gauss map \( N^r \) and conformal structures. By using the conformal structures induced by \( a^r \psi' - b^r N^r \), we derive two fundamental elliptic partial differential equations which involve the immersion and the Gauss map.

1. Introduction

Parallel surfaces as a subject of differential geometry have been intriguing for mathematicians throughout history and so it has been a research field. In the theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which differential geometers are interested. Among these surfaces, parallel surfaces are also studied in many papers [5, 4, 12, 14]. A surface \( M' \) whose points are at a constant distance along the normal from another surface \( M \) is said to be parallel to \( M \). So, there are infinite number of surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of points which are on the normals to \( M \) at a non-zero constant distance \( r \) from \( M' \) [19]. Here Chern, Hopf studied the case of \( S \) being closed and Rosenberg-Earp considered the case of \( S \) to be immersed [3, 10, 15]. The surfaces are called \( H \)-surfaces, if they have constant mean curvature \( H \) and are called \( K \)-surfaces, if they have constant Gaussian curvature \( K \) in Minkowski space. In 1853, Bonnet remarked that the study of \( K \)-surfaces could be as difficult as the study of \( H \)-surfaces.

The two principal curvatures \( k_1, k_2 \) satisfy the following equations:

\[
K = k_1 k_2 \\
2H = k_1 + k_2
\]

and therefore \( k_i = H \mp \sqrt{H^2 - K} \), \( i = 1, 2 \) [10].

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Let $S$ be an orientable surface and $\psi : S \to \mathbb{R}^3$ be an immersion with Gauss map $\eta : S \to S^2$. It is said that $\psi$ is a linear Weingarten immersion if there exist three real numbers $a, b, c$, not all zero positive real numbers, such that
\begin{equation}
2aH + bK = c
\end{equation}
where $H$ and $K$ are the mean curvature and the Gaussian curvature, respectively. In such case we say that $S$ is an elliptic timelike parallel Weingarten surface where $a = aI + bII$ is a positive definite metric [8]. The equation above is elliptic only when $a^2 + bc > 0$ [10].

In this paper, we consider parallel Weingarten surfaces satisfying elliptic linearity condition such that $b^2 - 4ac > 0$ for timelike parallel surfaces. Then we give some differential-geometric properties of these kind of surfaces. By using the conformal structure induced by $a^2\eta^2 - b^2\eta^2$, we derive two fundamental elliptic partial differential equations which involve the immersion and the Gauss map in Theorem 3.1 for elliptic linear timelike parallel Weingarten surfaces. In Corollary 3.1, the result let us to recover the immersion from harmonic local diffeomorphism into the unit pseudosphere in terms of its Gauss map $\eta^2$ for an elliptic linear timelike parallel Weingarten surfaces. Through this study, the elliptic linear timelike parallel Weingarten surfaces will be abbreviated as ELTPW.

2. Preliminaries

Let $E_3^1$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric
\[ \langle dx, dx \rangle = dx_1^2 + dx_2^2 - dx_3^2 \]
where $\langle x_1, x_2, x_3 \rangle$ denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $x$ of $E_3^1$ is said to be spacelike if $\langle x, x \rangle > 0$ or $x = 0$, timelike if $\langle x, x \rangle < 0$ and lightlike or null if $\langle x, x \rangle = 0$ and $x \neq 0$. For $x \in E_3^1$, the norm is defined by $\|x\| = \sqrt{\langle x, x \rangle}$. Then the vector $x$ is called a spacelike unit vector if $\langle x, x \rangle = 1$ and a timelike unit vector if $\langle x, x \rangle = -1$. Similarly, a regular curve in $E_3^1$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [13]. For any two vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ of $E_3^1$, the inner product is the real number $\langle x, y \rangle = x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $x \times y = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [11]. Let $X = X(u, v)$ be a local parametrization and $\{X_u, X_v\}$ be a local base of the tangent plane at each point. The first fundamental form defined by
\[ I = Edu^2 + 2Fdudv + Gdv^2. \]
where the differentiable functions
\[ E = \langle X_u, X_u \rangle, \quad F = \langle X_u, X_v \rangle, \quad G = \langle X_v, X_v \rangle, \]
are called the coefficients of the first fundamental form $I$ [11].

The second fundamental form defined by
\[ II = edu^2 + 2fdudv + gdv^2. \]
where the differentiable functions
\[ e = -\langle N_u, N_u \rangle = \langle N, X_{uu} \rangle, \]
\[ f = -\langle N_u, N_v \rangle = -\langle N_v, N_u \rangle = \langle N, X_{uv} \rangle, \]
\[ g = -\langle N_v, N_v \rangle = \langle N, X_{vv} \rangle. \]
are called the coefficients of the second fundamental form $II$ \[11\]. Isothermic parameters $u,v$ satisfy the equation below
\[ ds^2 = E(du^2 + dv^2) \]

**Definition 2.1.** \([13]\) An immersion $\phi : M \to N$ is a smooth mapping such that $d\phi_P$ is one to one for all $P \in M$. An immersion $\psi : S \to \mathbb{E}_1^3$ of a surface $M$ is called timelike if the induced metric on $M$ is a Lorentzian metric, i.e., the normal on the surface is a spacelike vector \([1]\).

**Theorem 2.1.** \([7]\) A regular surface $M \subset \mathbb{R}^3$ is orientable if and only if there is a continuous map $P \to U(P)$ that assigns to each $P \in M$ a unit normal vector $U(P) \in T_P^\perp(M)$. The map $\eta : S \to \mathbb{S}^2 \subset \mathbb{R}^3$ is called the Gauss map of $S$. Theorem 2.2. \([16]\) Let $P,Q,\partial P/\partial y,\partial Q/\partial x$ be single-valued and continuous in a simply-connected region $R$ bounded by a simple closed curve $C$. Then
\[ \oint_C Pdx + Qdy = \iint_R \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy \]
where $\oint$ is used to emphasize that $C$ is closed and that it is described in the positive direction.

**Definition 2.3.** \([13]\) Let $n \geq 2$ and $0 \leq \nu \leq n$. Then the psedosphere of radius $r > 0$ in $\mathbb{R}^{\nu+1}$ is the hyperquadric $S_n^\nu(r) = q^{-1}(r^2) = \{ p \in \mathbb{R}^{\nu+1} : \langle p, p \rangle = r^2 \}$ with dimension $n$ and index $\nu$.

**Definition 2.4.** \([13]\) Let $M$ be a surface in Minkowski 3-space and $D$ be the Levi-Civita connection on Minkowski 3-space. Then,
\[ S : \chi(M) \to \chi(M), \quad S(X) = DXN \]
is called the shape operator, where $N$ is the unit normal vector on $M$. A semi-Riemannian submanifold $M$ of $\bar{M}$ is totally umbilic provided every point of $M$ is umbilic.
Definition 2.8. ([9]) Let $M$ and $M^r$ be two surfaces in Minkowski 3-space. The function
\[ f : M \rightarrow M^r \\
\quad P \rightarrow f(P) = P + rN_P \]
is called the parallelization function between $S_C$ and $S'_C$ and furthermore $S'_C$ is called the parallel surface to $S_C$ in $E^3_1$ where $r$ is a given positive real number and $N$ is the unit normal vector field on $S_C$.

Theorem 2.3. ([9]) Let $M$ be a surface and $M^r$ be a parallel surface of $M$ in Minkowski 3-space. Let $f : M \rightarrow M^r$ be the parallelization function. Then for $X \in \chi(M)$,
1. $f^*(X) = X + rS(X)$
2. $S'(f_*(X)) = S(X)$
3. $f$ preserves the principal directions of the curvature, that is $S'(f_*(X)) = k_1 + rk_1$ where $S'$ is the shape operator on $M^r$ and $k_1$ is a principal curvature of $M$ at $P$ in the direction of the vector $X$.

Definition 2.9 ([1]) A surface in a 3-dimensional Lorentz space is called a spacelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal on the surface is a timelike vector.

Definition 2.10. ([18]) Let $M$ be a timelike surface and $M^r$ be a parallel surface of $M$ in $E^3_1$. Let $N^r$ and $S^r$ be the unit normal vector field and the shape operator of $M^r$, respectively. The Gaussian and the mean curvature functions are defined, respectively,
\[ K^r(f(P)) = \det S^r_{f(P)} \quad \text{and} \quad H^r(f(P)) = \frac{1}{2} \langle N, S^r_{f(P)} \rangle, \]
where $P \in M$, $f(P) \in M^r$ and $\langle N, N \rangle = 1$.

Theorem 2.4. ([18]) Let $M$ be a timelike surface and $M^r$ be a parallel surface of $M$ in $E^3_1$. Let $N^r$ and $S^r$ be a unit normal vector field and the shape operator of $M^r$, respectively. The Gaussian and the mean curvatures are given, respectively, in terms of the coefficients of the fundamental forms $I^r$ and $II^r$ as:
\[ K^r = \frac{e^r g^r - f^r}{E^r G^r - F^r} \quad \text{and} \quad H^r = \frac{e^r G^r - 2f^r F^r + g^r E^r}{2(E^r G^r - F^r)} \]

Lemma 2.1. ([18]) Let $M$ be a timelike surface and $M^r$ be its parallel surface in $E^3_1$. The surface $M$ is spacelike one if and only if the surface $M^r$ is timelike parallel surface.

Corollary 2.1. ([18]) Let $M$ be a timelike surface and $M^r$ be a parallel surface of $M$ in $E^3_1$. Then we have
\[ K = \frac{K^r}{1 - 2rH^r + r^2K^r} \quad \text{and} \quad H = \frac{H^r - rK^r}{1 - 2rH^r + r^2K^r} \]
where the Gaussian and the mean curvatures of $M$ and $M^r$ be denoted by $K$, $H$ and $K^r$, $H^r$, respectively.
3. On elliptic linear timelike parallel Weingarten surfaces satisfying the condition $2aH^r + bK^r = c$

Let $S$ be an orientable timelike parallel surface and $\psi^r : S \to \mathbb{R}^3_1$ be an immersion with the Gauss map $\eta^r : S \to S^2$ for $S$ surface. Using Corollary 2.1 in the equation (1.1), we get

$$2aH^r + bK^r = c.$$  

(3.1)

This equation is elliptic if and only if $a^2 + bc > 0$ where $H^r$ and $K^r$ are the mean curvature and the Gaussian curvature for timelike parallel surfaces. In that case we say that the immersion $\psi^r$ is ELTPW if there exist three real numbers $a, b, c$, not all zero, positive real numbers.

Some interesting examples of ELTPW immersions are given by the timelike parallel surfaces with constant mean curvature, that is $b = 0$, and the timelike parallel surfaces with positive constant Gaussian curvature, that is $a = 0$.

**Lemma 3.1.** Let $\psi^r : S \to \mathbb{R}^3_1$ be an ELTPW immersion satisfying (3.1). Then there exists a Gauss map $\eta^r : S \to S^2$ and two real numbers $\alpha, \beta$ such that

$$2\alpha H^r + \beta K^r = \gamma^r \geq 0$$

and $\alpha I^r + \beta II^r$ is a positive definite metric, where $I^r = \langle d\psi^r, d\psi^r \rangle$ and $II^r = \langle d\psi^r, -d\eta^r \rangle$ are the first and second fundamental forms of the immersion, respectively.

**Proof.** Let $\{f_*(X_1), f_*(X_2)\}$ be an orthonormal basis consisting of timelike and spacelike vectors at a point $P$ respectively, which diagonalizes $d\eta^r$. That is, for $d\eta^r(f_*(X_i)) = \frac{k_i}{1 + rk_i}(f_*(X_i))$, where $i = 1, 2$, we have

$$\sigma^r(f_*(X_1) \wedge f_*(X_2), f_*(X_1) \wedge f_*(X_2)) = -((\alpha + \beta k_1)(\alpha + \beta k_2))$$

$$= (\alpha^2 + \beta(2\alpha H^r + \beta K^r))$$

and then

$$\sigma^r(f_*(X_1) \wedge f_*(X_2), f_*(X_1) \wedge f_*(X_2)) = (\alpha^2 + \beta \gamma^r) > 0$$

with $\sigma^r = \alpha I^r + \beta II^r$. So, $\sigma^r$ is positive definite.

Thus, we will assume that every ELTPW immersion satisfies the above result. Moreover, Gauss map $\eta^r$ given by Lemma 3.1 will be called its associated Gauss map.

Now, we tried to obtain a condition in order to define associated Gauss map of ELTPW immersion in Lemma 3.2.

**Lemma 3.2.** Let $\psi^r : S \to \mathbb{R}^3_1$ be an ELTPW immersion satisfying (3.2) with associated Gauss map $\eta^r : S \to S^2$ and the Gaussian curvature $K(P) > 0$. Then $\eta^r(P)$ is the inner normal vector at a point $P$ if and only if $\alpha \geq 0$ or $\beta \geq 0$.

**Proof.** If $\eta^r$ is not the inner normal at $P$ then the principal curvatures $\frac{k_1}{1 + rk_1}(P)$, $\frac{k_2}{1 + rk_2}(P)$ are both negative. Using (3.2), we get

\[ \square \]
(3.3) \[
\frac{k_2}{1 + rk_2} = \frac{\gamma - \frac{\alpha k_1}{1 + rk_1}}{\alpha + \frac{\beta k_1}{1 + rk_1}} < 0
\]

when \( \beta \neq 0 \). Since \( \sigma^r = \alpha I^r + \beta II^r \) is positive definite, we get

(3.4) \[
\alpha + \frac{\beta k_1}{1 + rk_1} < 0.
\]

By using (3.4) in (3.3), we have

\[
0 \leq \frac{\gamma}{\alpha + \frac{\beta k_1}{1 + rk_1}} < \frac{\alpha k_1}{1 + rk_1}.
\]

Therefore \( \frac{\alpha k_1}{1 + rk_1} > 0 \).

As a result, \( \frac{k_1}{1 + rk_1} < 0 \). Thus \( \eta^r(P) \) is the outer normal. That is a contradiction.

**Theorem 3.1.** Let \( \psi^r : S \rightarrow \mathbb{R}^3 \) be an ELTPW immersion satisfying (3.2) with associated Gauss map \( \eta^r : S \rightarrow S^2 \). Then

\[
\Delta \sigma^r \psi^r = \frac{\gamma^r + \beta K^r}{\sqrt{\alpha^2 + \beta \gamma^r}} \eta^r \quad \text{and} \quad \Delta \sigma^r \eta^r = \frac{2}{\beta \sqrt{\alpha^2 + \beta \gamma^r}} \eta^r
\]

where \( \Delta \sigma^r \) is the Laplacian operator. This is an example of a theorem.

**Proof.** Let \((u, v)\) be isothermal parameters for \( \sigma^r \), that is,

\[
I^r = E_1^r du^2 + 2F_1^r du dv + G_1^r dv^2
\]

\[
II^r = E_2^r du^2 + 2F_2^r du dv + G_2^r dv^2
\]

These equations using in Lemma 3.1., we get

\[
\sigma^r = \lambda^r (du^2 + dv^2)
\]

where \( \lambda^r = \alpha E_1^r + \beta E_2^r \).

We can write

\[
\alpha \psi_u^r + \beta \eta_u^r = \mu_{11} \eta^r \wedge \psi_u^r + \mu_{12} \eta^r \wedge \psi_v^r
\]

\[
\alpha \psi_v^r + \beta \eta_v^r = \mu_{21} \eta^r \wedge \psi_u^r + \mu_{22} \eta^r \wedge \psi_v^r
\]

(3.6)
where $\eta^r \wedge \psi^r_u$ and $\eta^r \wedge \psi^r_v$ are on the basis of the tangent plane for timelike parallel surface for certain real numbers $\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22}$. Now, inner product of this equations with $\psi^r_u$ and $\psi^r_v$, gives

$$\mu_{12} = \frac{-\lambda^r}{\sqrt{((F_1^r)^2 - E_1^r G_1^r)}} \quad \text{and} \quad \mu_{11} = 0$$

also

$$\mu_{21} = \frac{\lambda^r}{\sqrt{((F_1^r)^2 - E_1^r G_1^r)}} \quad \text{and} \quad \mu_{22} = 0$$

Then using the equations (3.7)-(3.8) in (3.6), we get the following formula:

$$\alpha \psi^r_u + \beta \eta^r_u = \frac{-\lambda^r}{\sqrt{((F_1^r)^2 - E_1^r G_1^r)}} \eta^r \wedge \psi^r_v$$

(3.9)

$$\alpha \psi^r_v + \beta \eta^r_v = \frac{\lambda^r}{\sqrt{((F_1^r)^2 - E_1^r G_1^r)}} \eta^r \wedge \psi^r_u$$

By using that

$$-(\lambda^r) = (\alpha F_1^r + \beta F_2^r)^2 - (\alpha E_1^r + \beta E_2^r)(\alpha G_1^r + \beta G_2^r)$$

(3.10)

$$= \alpha^2((F_1^r)^2 - E_1^r G_1^r) + \beta(2\alpha F_1^r F_2^r - \alpha E_2^r G_1^r - \alpha E_1^r G_2^r)$$

$$+ \beta^2((F_1^r)^2 - E_2^r G_2^r)$$

If we write equation (2.1) in (3.10), we get

$$\alpha \psi^r_u + \beta \eta^r_u = (\alpha^2 + \beta \gamma^r) (E_1^r G_1^r - (F_1^r)^2).$$

Using the equations (3.11) in (3.9), we obtain

$$\alpha \psi^r_u + \beta \eta^r_u = -\sqrt{\alpha^2 + \beta \gamma^r} \eta^r \wedge \psi^r_v$$

(3.12)

$$\alpha \psi^r_v + \beta \eta^r_v = \sqrt{\alpha^2 + \beta \gamma^r} \eta^r \wedge \psi^r_u$$

and

$$\alpha \psi^r_v \wedge \eta^r + \beta \eta^r_u \wedge \eta^r = \sqrt{\alpha^2 + \beta \gamma^r} \psi^r_v$$

(3.13)

$$\alpha \psi^r_u \wedge \eta^r + \beta \eta^r_v \wedge \eta^r = -\sqrt{\alpha^2 + \beta \gamma^r} \psi^r_u$$

If we extract the derivative of the first equation with respect to $v$ from the derivative of the second equation with respect to $u$ in (3.13), we get

$$2\alpha H^r + 2\beta K^r)(\psi^r_u \wedge \psi^r_v) = \sqrt{\alpha^2 + \beta \gamma^r} (\psi^r_u + \psi^r_v)$$

Using (3.2) in (3.14), we obtain

$$\alpha \psi^r_u \wedge \psi^r_v + \psi^r_v \wedge \psi^r_v = \frac{\gamma^r + \beta K^r}{\sqrt{\alpha^2 + \beta \gamma^r}} (\psi^r_u \wedge \psi^r_v)$$

(3.15)

On the other hand, if we add the derivative of the first equation with respect to $u$ to the derivative of the second equation with respect to $v$ in (3.13), we have

$$\alpha (\psi^r_u + \psi^r_v) - \beta (\eta^r_u + \eta^r_v) = -2H^r \sqrt{\alpha^2 + \beta \gamma^r} (\psi^r_u \wedge \psi^r_v)$$

If we write (3.15) in (3.16), we get

$$(\eta^r_u + \eta^r_v) = \frac{-\gamma^r (\alpha + H^r \beta)}{\beta \sqrt{\alpha^2 + \beta \gamma^r}} (\psi^r_u \wedge \psi^r_v)$$
Remark 3.1. Since \((H^r)^2 \geq K^r\) on every timelike surface, given an ELTPW immersion satisfying (3.2) the equality in the above inequality occurs when

\[
K^r = \begin{cases} 
\frac{(\gamma^r)^2}{\alpha + \sqrt{\alpha^2 + \beta \gamma^r}}^2 & \text{if } \beta, \gamma^r \neq 0, \\
\frac{(\gamma^r)^2}{4\alpha^2} & \text{if } \beta = 0, \\
\frac{4\alpha^2}{(\beta^r)^2} & \text{if } \beta \neq 0 \text{ and } \gamma^r = 0.
\end{cases}
\]

According to expression \(\Delta^r \eta^r\) verified at Theorem 3.1 we can consider \(\eta^r\) as a harmonic map, that is, this is an example of a remark element.

Corollary 3.1. Let \(\psi^r : S \rightarrow \mathbb{R}^3\) be an ELTPW immersion satisfying (3.2) with associated Gauss map \(\eta^r : S \rightarrow S^2\). If we consider the conformal structure induced by \(\sigma^r = \alpha I^r + \beta II^r\) on \(S\), then \(\eta^r\) is harmonic. Moreover, if \(\gamma^r \neq 0\), \(\psi^r\) can be recovered as

\[
\psi^r = \frac{\alpha}{\gamma^r} \eta^r - \frac{\sqrt{\alpha^2 + \beta \gamma^r}}{\gamma^r} \int \eta^r \wedge \eta^r_u \, du - \eta^r \wedge \eta^r_v \, dv
\]

for the isothermal parameters \((u, v)\) on \(S\). This is an example of a theorem.

Proof. From equation (3.12),

\[
\begin{align*}
\alpha \psi_u^r &= -\beta \eta_u^r - \sqrt{\alpha^2 + \beta \gamma^r} \eta^r \wedge \psi_u^r \\
\alpha \psi_v^r &= -\beta \eta_v^r + \sqrt{\alpha^2 + \beta \gamma^r} \eta^r \wedge \psi_v^r
\end{align*}
\]

There putting \(\psi_v^r\) into the first equation in (3.18) where the second equation in (3.18), we can write

\[-\beta \gamma^r \psi_u^r = -\alpha \beta \eta_v^r + \beta \sqrt{\alpha^2 + \beta \gamma^r} \eta^r \wedge \eta_v^r\]

Analogously, putting \(\psi_u^r\) into the second equation in (3.18) where the first equation in (3.18), we can write

\[-\beta \gamma^r \psi_v^r = -\alpha \beta \eta_u^r - \beta \sqrt{\alpha^2 + \beta \gamma^r} \eta^r \wedge \eta_u^r\]

Hence, if \(\beta \neq 0\), the immersion is recovered using (3.17) \(\square\)

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