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ON ELLIPTIC LINEAR TIMELIKE PARALLEL WEINGARTEN SURFACES SATISFYING THE CONDITION $2a^rH^r + b^rK^r = c^r$

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Abstract. In this study, firstly, we obtain the parallel Weingarten surfaces which satisfy the condition $2a^rH^r + b^rK^r = c^r$ in Minkowski 3-space. Then we give some geometric properties of these kind of surfaces, such as their Gauss map N^r and conformal structures. By using the conformal structures induced by $a^r\psi^r - b^rN^r$, we derive two fundamental elliptic partial differential equations which involve the immersion and the Gauss map.

1. Introduction

Parallel surfaces as a subject of differential geometry have been intriguing for mathematicians throughout history and so it has been a research field. In the theory of surfaces, there are some special surfaces such as ruled surfaces, minimal surfaces and surfaces of constant curvature in which differential geometers are interested. Among these surfaces, parallel surfaces are also studied in many papers [5, 4, 12, 14]. A surface M^r whose points are at a constant distance along the normal from another surface M is said to be parallel to M. So, there are infinite number of surfaces because we choose the constant distance along the normal arbitrarily. From the definition it follows that a parallel surface can be regarded as the locus of points which are on the normals to M at a non-zero constant distance r from M [19]. Here Chern, Hopf studied the case of S being closed and Rosenberg-Earp considered the case of S to be immersed [3, 10, 15]. The surfaces are called $H-surfaces$, if they have constant mean curvature H and are called $K-surfaces$, if they have constant Gaussian curvature K in Minkowski space. In 1853, Bonnet remarked that the study of $K-surfaces$ could be as difficult as the study of $H-surfaces$.

The two principal curvatures k_1, k_2 satisfy the following equations:

$$\begin{aligned} K &= k_1k_2 \\ 2H &= k_1 + k_2 \end{aligned}$$

and therefore $k_i = H \mp \sqrt{H^2 - K}$, $(i = 1, 2)$ [10].

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Let S be an orientable surface and $\psi : S \to \mathbb{R}^3$ be an immersion with Gauss map $\eta : S \to S^2$. It is said that ψ is a linear Weingarten immersion if there exist three real numbers a, b, c, not all zero positive real numbers, such that

$$2aH + bK = c \tag{1.1}$$

where H and K are the mean curvature and the Gaussian curvature, respectively. In such case we say that S is a linear Weingarten surface where $\sigma = aI + bII$ is a positive definite metric [8]. The equation above is elliptic only when $a^2 + bc > 0$ [10].

In this paper, we consider parallel Weingarten surfaces satisfying elliptic linearity condition such that $b^2 - 4ac > 0$ for timelike parallel surfaces. Then we give some differential-geometric properties of these kind of surfaces. By using the conformal structure induced by $a^r\psi^r - b^r\eta^r$, we derive two fundamental elliptic partial differential equations which involve the immersion and the Gauss map in Theorem 3.1 for elliptic linear timelike parallel Weingarten surfaces. In Corollary 3.1, the result let us to recover the immersion from harmonic local diffeomorphism into the unit pseudosphere in terms of its Gauss map η^r for an elliptic linear timelike parallel Weingarten surfaces. Throught this study, the elliptic linear timelike parallel Weingarten surfaces will be abbreviated as ELTPW.

2. Preliminaries

Let $\mathbb{E}_1^3$ be the three-dimensional Minkowski space, that is, the three-dimensional real vector space $\mathbb{R}^3$ with the metric

$$\langle d\mathbf{x}, d\mathbf{x}\rangle = dx_1^2 + dx_2^2 - dx_3^2$$

where (x_1, x_2, x_3) denotes the canonical coordinates in $\mathbb{R}^3$. An arbitrary vector $\mathbf{x}$ of $\mathbb{E}_1^3$ is said to be spacelike if $\langle \mathbf{x}, \mathbf{x}\rangle > 0$ or $\mathbf{x} = \mathbf{0}$, timelike if $\langle \mathbf{x}, \mathbf{x}\rangle < 0$ and lightlike or null if $\langle \mathbf{x}, \mathbf{x}\rangle = 0$ and $\mathbf{x} \neq \mathbf{0}$. For $\mathbf{x} \in \mathbb{E}_1^3$, the norm is defined by $\|\mathbf{x}\| = \sqrt{|\langle \mathbf{x}, \mathbf{x}\rangle|}$. Then the vector $\mathbf{x}$ is called a spacelike unit vector if $\langle \mathbf{x}, \mathbf{x}\rangle = 1$ and a timelike unit vector if $\langle \mathbf{x}, \mathbf{x}\rangle = -1$. Similarly, a regular curve in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors are spacelike, timelike or null (lightlike), respectively [13]. For any two vectors $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ of $\mathbb{E}_1^3$, the inner product is the real number $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1 + x_2y_2 - x_3y_3$ and the vector product is defined by $\mathbf{x} \times \mathbf{y} = ((x_2y_3 - x_3y_2), (x_3y_1 - x_1y_3), -(x_1y_2 - x_2y_1))$ [11]. Let $X = X(u, v)$ be a local parametrization and $\{X_u, X_v\}$ be a local base of the tangent plane at each point. The first fundamental form defined by

$$I = Edu^2 + 2Fdudv + Gdv^2.$$

where the differentiable functions

$$E = \langle X_u, X_u\rangle, \qquad F = \langle X_u, X_v\rangle, \qquad G = \langle X_v, X_v\rangle,$$

are called the coefficients of the first fundamental form I [11].

The second fundamental form defined by

$$II = edu^2 + 2fdudv + gdv^2.$$

where the differentiable functions

$$\begin{aligned} e &= -\langle X_u, N_u\rangle = \langle N, X_{uu}\rangle \\ f &= -\langle X_u, N_v\rangle = -\langle X_v, N_u\rangle = \langle N, X_{uv}\rangle \\ g &= -\langle X_v, N_v\rangle = \langle N, X_{vv}\rangle \end{aligned}$$

are called the coefficients of the second fundamental form II [11]. Isothermic parameters u, v satisfy the equation below

$$ds^2 = E(du^2 + dv^2)$$

[10].

Definition 2.1. ([13]) An immersion $\phi : M \to N$ is a smooth mapping such that $d\phi_P$ is one to one for all $P \in M$. An immersion $\psi : S \to \mathbb{E}_1^3$ of a surface M is called timelike if the induced metric on M is a Lorentzian metric, i.e., the normal on the surface is a spacelike vector ([1]).

Theorem 2.1. ([7]) A regular surface $M \subset \mathbb{R}^3$ is orientable if and only if there is a continuous map $P \to U(P)$ that assigns to each $P \in M$ a unit normal vector $U(P) \in M_P^{\perp}$.

Definition 2.2. ([6]) Let $S \subset \mathbb{R}^3$ be a surface with an orientation η. The map $\eta : S \to \mathbb{R}^3$ takes its values in the unit sphere

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3; x^2 + y^2 + z^2 = 1\right\}$$

The map $\eta : S \to S^2$, thus defined, is called the Gauss map of S.

Theorem 2.2. ([16]) Let $P, Q, \frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ be single-valued and continuous in a simply-connected region R bounded by a simple closed curve C. Then

$$\oint_C P dx + Q dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx dy$$

where $\oint_C$ is used to emphasize that C is closed and that it is described in the positive direction.

Definition 2.3. ([13]) Let $n \geq 2$ and $0 \leq \nu \leq n$. Then the psedosphere of radius $r > 0$ in R_ν^{n+1} is the hyperquadric

$$S_\nu^n(r) = q^{-1}(r^2) = \left\{p \in R_\nu^{n+1} : \langle p, p\rangle = r^2\right\}$$

with dimension n and index ν.

Definition 2.4. ([13]) Let M be a surface in Minkowski 3-space and D be the Levi-Civita connection on Minkowski 3-space. Then,

$$\begin{array}{rll} S: & \chi(M) \longrightarrow & \chi(M) \\ & X \longrightarrow & S(X) = D_X N \end{array}$$

is called the shape operator, where N is the unit normal vector on M.

Definition 2.5. ([13]) A smooth function $\psi : M \to N$ of semi-Riemannian manifolds is conformal provided $\psi^*(g_N) = h g_M$, for some function $h \in \mathfrak{F}(M)$ such that $h > 0$ or $h < 0$.

Definition 2.6. ([13]) The Laplacian Δf of a function $f \in \mathfrak{F}(M)$ is the divergence of its gradient: $\Delta f = \operatorname{div}(\operatorname{grad} f) \in \mathfrak{F}(M)$. A form w is said to be harmonic if $\Delta w = 0$ ([17]).

Definition 2.7. ([13]) A point P of $M \subset \bar{M}$ is umbilic provided there is a normal vector $z \in T_P(M)^{\perp}$ such that

$$II(v, w) \quad = \quad \langle v, w\rangle\, z$$

for all $v, w \in T_P(M)$. Then z is called the normal curvature vector of M at P. A semi-Riemannian submanifold M of $\bar{M}$ is totally umbilic provided every point of M is umbilic.

Definition 2.8. ([9]) Let M and M^r be two surfaces in Minkowski 3-space. The function

$$\begin{array}{rll} f: & M \longrightarrow & M^r \\ & P \longrightarrow & f(P) = P + r\mathbf{N}_P \end{array}$$

is called the parallelization function between S_C and S_C^r and furthermore S_C^r is called the parallel surface to S_C in $\mathbb{E}_1^3$ where r is a given positive real number and $\mathbf{N}$ is the unit normal vector field on S_C.

Theorem 2.3. ([9]) Let M be a surface and M^r be a parallel surface of M in Minkowski 3-space. Let $f : M \to M^r$ be the parallelization function. Then for $X \in \chi(M)$,

1. $f_*(X) = X + rS(X)$
2. $S^r(f_*(X)) = S(X)$
3. f preserves the principal directions of the curvature, that is

$$S^r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where S^r is the shape operator on M^r, and k is a principal curvature of M at P in the direction of the vector X.

Definition 2.9 ([1]) A surface in a 3-dimensional Lorentz space is called a spacelike surface if the induced metric on the surface is a Lorentzian metric, i.e., the normal on the surface is a timelike vector.

Definition 2.10. ([18]) Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Let N^r and S^r be the unit normal vector field and the shape operator of M^r, respectively. The Gaussian and the mean curvature functions are defined, respectively,

$$K^r(f(P)) \;=\; \det S^r_{f(P)} \quad \text{and} \quad H^r(f(P)) \;=\; \frac{1}{2}\, iz S^r_{f(P)},$$

where $P \in M$, $f(P) \in M^r$ and $\langle N, N\rangle = 1$.

Theorem 2.4. ([18]) Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Let N^r and S^r be a unit normal vector field and the shape operator of M^r, respectively. The Gaussian and the mean curvatures are given, respectively, in terms of the coefficients of the fundamental forms I^r and II^r as:

$$K^r = \frac{e^r g^r - f^{r2}}{E^r G^r - F^{r2}} \quad \text{and} \quad H^r = \frac{e^r G^r - 2f^r F^r + g^r E^r}{2(E^r G^r - F^{r2})} \tag{2.1}$$

Lemma 2.1. ([18]) Let M be a timelike surface and M^r be its parallel surface in $\mathbb{E}_1^3$. The surface M is spacelike one if and only if the surface M^r is timelike parallel surface.

Corollary 2.1. ([18]) Let M be a timelike surface and M^r be a parallel surface of M in $\mathbb{E}_1^3$. Then we have

$$K = \frac{K^r}{1 - 2rH^r + r^2K^r} \quad \text{and} \quad H = \frac{H^r - rK^r}{1 - 2rH^r + r^2K^r}$$

where the Gaussian and the mean curvatures of M and M^r be denoted by K, H and K^r, H^r, respectively.

3. On elliptic linear timelike parallel Weingarten surfaces satisfying the condition $2a^r H^r + b^r K^r = c^r$

Let S be an orientable timelike parallel surface and $\psi^r : S \to \mathbb{R}^3_1$ be an immersion with the Gauss map $\eta^r : S \to S^2$ for S surface. Using Corollary 2.1 in the equation (1.1), we get

$$2aH^r + bK^r = c. \tag{3.1}$$

This equation is elliptic if and only if $a^2 + bc > 0$ where H^r and K^r are the mean curvature and the Gaussian curvature for timelike parallel surfaces. In that case we say that the immersion ψ^r is ELTPW if there exist three real numbers a, b, c, not all zero, positive real numbers.

Some interesting examples of ELTPW immersions are given by the timelike parallel surfaces with constant mean curvature, that is $b = 0$, and the timelike parallel surfaces with positive constant Gaussian curvature, that is $a = 0$.

Lemma 3.1. *Let $\psi^r : S \to \mathbb{R}^3_1$ be an ELTPW immersion satisfying (3.1). Then there exists a Gauss map $\eta^r : S \to S^2$ and two real numbers α, β such that*

$$2\alpha H^r + \beta K^r = \gamma^r \geqslant 0 \tag{3.2}$$

and $\alpha I^r + \beta II^r$ is a positive definite metric, where $I^r = \langle d\psi^r, d\psi^r\rangle$ and $II^r = \langle d\psi^r, -d\eta^r\rangle$ are the first and second fundamental forms of the immersion, respectively.

Proof. Let $\{f_*(X_1), f_*(X_2)\}$ be an orthonormal basis consisting of timelike and spacelike vectors at a point P respectively, which diagonalizes $d\eta^r$. That is, for $d\eta^r(f_*(X_i)) = \dfrac{k_i}{1+rk_i}(f_*(X_i))$, where $i = 1, 2$, we have

$$\begin{aligned} \sigma^r(f_*(X_1) \wedge f_*(X_2), f_*(X_1) \wedge f_*(X_2)) &= -(-(\alpha + \frac{\beta k_1}{1+rk_1}))(\alpha + \frac{\beta k_2}{1+rk_2}) \\ &= (\alpha^2 + \beta(2\alpha H^r + \beta K^r)) \end{aligned}$$

and then

$$\sigma^r(f_*(X_1) \wedge f_*(X_2), f_*(X_1) \wedge f_*(X_2)) = (\alpha^2 + \beta\gamma^r) > 0$$

with $\sigma^r = \alpha I^r + \beta II^r$. So, σ^r is positive definite. □

Thus, we will assume that every ELTPW immersion satisfies the above result. Moreover, Gauss map η^r given by Lemma 3.1 will be called its associated Gauss map.

Now, we tried to obtain a condition in order to define associated Gauss map of ELTPW immersion in Lemma 3.2.

Lemma 3.2. *Let $\psi^r : S \to \mathbb{R}^3_1$ be an ELTPW immersion satisfying (3.2) with associated Gauss map $\eta^r : S \to S^2$ and the Gaussian curvature $K(P) > 0$. Then $\eta^r(P)$ is the inner normal vector at a point P if and only if $\alpha \geq 0$ or $\beta \geq 0$.*

Proof. If η^r is not the inner normal at P then the principal curvatures $\dfrac{k_1}{1+rk_1}(P)$, $\dfrac{k_2}{1+rk_2}(P)$ are both negative. Using (3.2), we get □

$$\frac{k_2}{1+rk_2} = \frac{\gamma^r - \dfrac{\alpha k_1}{1+rk_1}}{\alpha + \dfrac{\beta k_1}{1+rk_1}} < 0 \tag{3.3}$$

when $\beta \neq 0$.Since $\sigma^r = \alpha I^r + \beta II^r$ is positive definite, we get

$$\alpha + \frac{\beta k_1}{1+rk_1} < 0. \tag{3.4}$$

By using (3.4) in (3.3), we have $0 \leq \gamma^r < \dfrac{\alpha k_1}{1+rk_1}$. Therefore $\dfrac{\alpha k_1}{1+rk_1} > 0$. Since $\dfrac{k_1}{1+rk_1} < 0$, we have $\alpha < 0$ and $\beta < 0$. If $\beta = 0$, the proof of the lemma is obvious. Namely, since η^r is the inner normal, $\alpha \geqslant 0$ or $\beta \geqslant 0$.

Conversely, let η^r is the inner normal and $\alpha < 0$ or $\beta < 0$. The following inequality

$$\alpha + \frac{\beta k_1}{1+rk_1} > 0 \tag{3.5}$$

is satisfied since σ^r is positive definite. From (3.5), we obtain

$$\alpha < -\frac{\beta k_1}{1+rk_1} < 0$$
$$\frac{\beta k_1}{1+rk_1} > 0.$$

As a result, $\dfrac{k_1}{1+rk_1} < 0$. Thus $\eta^r(P)$ is the outer normal. That is a contradiction.

Theorem 3.1. *Let $\psi^r : S \to \mathbb{R}^3_1$ be an ELTPW immersion satisfying (3.2) with associated Gauss map $\eta^r : S \to S^2$. Then*

$$\Delta^{\sigma^r}\psi^r = \frac{\gamma^r + \beta K^r}{\sqrt{\alpha^2 + \beta\gamma^r}}\eta^r \quad \textit{and} \quad \Delta^{\sigma^r}\eta^r = 2\frac{-\gamma^r(\alpha + H^r\beta)}{\beta\sqrt{\alpha^2+\beta\gamma^r}}\eta^r$$

where Δ^{σ^r} is the Laplacian operator.This is an example of a theorem.

Proof. Let (u, v) be isothermal parameters for σ^r, that is,

$$\begin{aligned} I^r &= E^r_1 du^2 + 2F^r_1 dudv + G^r_1 dv^2 \\ II^r &= E^r_2 du^2 + 2F^r_2 dudv + G^r_2 dv^2 \end{aligned}$$

These equations using in Lemma 3.1., we get

$$\sigma^r = \lambda^r(du^2 + dv^2)$$

where $\lambda^r = \alpha E^r_1 + \beta E^r_2$. □

We can write

$$\begin{aligned} \alpha\psi^r_u + \beta\eta^r_u &= \mu_{11}\eta^r \wedge \psi^r_u + \mu_{12}\eta^r \wedge \psi^r_v \\ \alpha\psi^r_v + \beta\eta^r_v &= \mu_{21}\eta^r \wedge \psi^r_u + \mu_{22}\eta^r \wedge \psi^r_v \end{aligned} \tag{3.6}$$

where $\eta^r \wedge \psi_u^r$ and $\eta^r \wedge \psi_v^r$ are on the basis of the tangent plane for timelike parallel surface for certain real numbers μ_{11}, μ_{12}, μ_{21}, μ_{22}. Now, inner product of this equations with ψ_u^r and ψ_v^r, gives

$$\mu_{12} = \frac{-\lambda^r}{\sqrt{|(F_1^r)^2 - E_1^r G_1^r|}} \quad \text{and} \quad \mu_{11} = 0 \tag{3.7}$$

also

$$\mu_{21} = \frac{\lambda^r}{\sqrt{|(F_1^r)^2 - E_1^r G_1^r|}} \quad \text{and} \quad \mu_{22} = 0 \tag{3.8}$$

Then using the equations (3.7)-(3.8) in (3.6), we get the following formula:

$$\begin{aligned} \alpha\psi_u^r + \beta\eta_u^r &= \frac{-\lambda^r}{\sqrt{|(F_1^r)^2 - E_1^r G_1^r|}}\eta^r \wedge \psi_v^r \\ \alpha\psi_v^r + \beta\eta_v^r &= \frac{\lambda^r}{\sqrt{|(F_1^r)^2 - E_1^r G_1^r|}}\eta^r \wedge \psi_u^r \end{aligned} \tag{3.9}$$

By using that

$$\begin{aligned} -(\lambda^r) &= (\alpha F_1^r + \beta F_2^r)^2 - (\alpha E_1^r + \beta E_2^r)(\alpha G_1^r + \beta G_2^r) \\ &= \alpha^2((F_1^r)^2 - E_1^r G_1^r) + \beta(2\alpha F_1^r F_2^r - \alpha E_2^r G_1^r - \alpha E_1^r G_2^r) \\ &\quad + \beta^2((F_2^r)^2 - E_2^r G_2^r). \end{aligned} \tag{3.10}$$

If we write equation (2.1) in (3.10), we get

$$(\lambda^r)^2 = (\alpha^2 + \beta\gamma^r)(E_1^r G_1^r - (F_1^r)^2). \tag{3.11}$$

Using the equations (3.11) in (3.9), we obtain

$$\begin{aligned} \alpha\psi_u^r + \beta\eta_u^r &= -\sqrt{\alpha^2 + \beta\gamma^r}\eta^r \wedge \psi_v^r \\ \alpha\psi_v^r + \beta\eta_v^r &= \sqrt{\alpha^2 + \beta\gamma^r}\eta^r \wedge \psi_u^r \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} \alpha\psi_u^r \wedge \eta^r + \beta\eta_u^r \wedge \eta^r &= \sqrt{\alpha^2 + \beta\gamma^r}\psi_v^r \\ \alpha\psi_v^r \wedge \eta^r + \beta\eta_v^r \wedge \eta^r &= -\sqrt{\alpha^2 + \beta\gamma^r}\psi_u^r \end{aligned} \tag{3.13}$$

If we extract the derivative of the first equation with respect to v from the derivative of the second equation with respect to u in (3.13), we get

$$(2\alpha H^r + 2\beta K^r)(\psi_u^r \wedge \psi_v^r) = \sqrt{\alpha^2 + \beta\gamma^r}(\psi_{\mathrm{uu}}^r + \psi_{vv}^r) \tag{3.14}$$

Using (3.2) in (3.14), we obtain

$$(\psi_{\mathrm{uu}}^r + \psi_{vv}^r) = \frac{\gamma^r + \beta K^r}{\sqrt{\alpha^2 + \beta\gamma^r}}(\psi_u^r \wedge \psi_v^r) \tag{3.15}$$

On the other hand, if we add the derivative of the first equation with respect to u to the derivative of the second equation with respect to v in (3.13), we have

$$\alpha(\psi_{\mathrm{uu}}^r + \psi_{vv}^r) - \beta(\eta_{\mathrm{uu}}^r + \eta_{vv}^r) = -2H^r\sqrt{\alpha^2 + \beta\gamma^r}(\psi_u^r \wedge \psi_v^r) \tag{3.16}$$

If we write (3.15) in (3.16), we get

$$(\eta_{\mathrm{uu}}^r + \eta_{vv}^r) = 2\frac{-\gamma^r(\alpha + H^r\beta)}{\beta\sqrt{\alpha^2 + \beta\gamma^r}}(\psi_u^r \wedge \psi_v^r)$$

$(\beta \neq 0)$.

Remark 3.1. Since $(H^r)^2 \geqslant K^r$ on every timelike surface, given an ELTPW immersion satisfying (3.2) the equality in the above inequality occurs when

$$K^r = \begin{cases} \text{if } \beta, \gamma^r \neq 0, & \dfrac{(\gamma^r)^2}{(\alpha \mp \sqrt{\alpha^2 + \beta\gamma^r})^2} \\ \text{if } \beta = 0, & \dfrac{(\gamma^r)^2}{4\alpha^2} \\ \text{if } \beta \neq 0 \text{ and } \gamma^r = 0, & \dfrac{4\alpha^2}{(\beta^r)^2} \end{cases}$$

According to expression $\Delta^{\sigma^r}\eta^r$ verfied at Theorem 3.1 we can consider η^r as a harmonic map, that is,this is an example of a remark element.

Corollary 3.1. *Let $\psi^r : S \to \mathbb{R}^3_1$ be an ELTPW immersion satisfying (3.2) with associated Gauss map $\eta^r : S \to S^2$. If we consider the conformal structure induced by $\sigma^r = \alpha I^r + \beta II^r$ on S, then η^r is harmonic. Moreover, if $\gamma^r \neq 0$, ψ^r can be recovered as*

$$\psi^r = \frac{\alpha}{\gamma^r}\eta^r - \frac{\sqrt{\alpha^2 + \beta\gamma^r}}{\gamma^r} \int \eta^r \wedge \eta^r_v \, du - \eta^r \wedge \eta^r_u \, dv \tag{3.17}$$

for the isothermal parameters (u, v) on S.This is an example of a theorem.

Proof. From equation (3.12),

$$\begin{aligned} \alpha\psi^r_u &= -\beta\eta^r_u - \sqrt{\alpha^2 + \beta\gamma^r}\ \eta^r \wedge \psi^r_v \\ \alpha\psi^r_v &= -\beta\eta^r_v + \sqrt{\alpha^2 + \beta\gamma^r}\ \eta^r \wedge \psi^r_u \end{aligned} \tag{3.18}$$

There putting ψ^r_v into the first equation in (3.18) where the second equation in (3.18), we can write

$$-\beta\gamma^r\psi^r_u = -\alpha\beta\eta^r_u + \beta\sqrt{\alpha^2 + \beta\gamma^r}\ \eta^r \wedge \eta^r_v$$

Analogously, putting ψ^r_u into the second equation in (3.18) where the first equation in (3.18), we can write

$$-\beta\gamma^r\psi^r_v = -\alpha\beta\eta^r_v - \beta\sqrt{\alpha^2 + \beta\gamma^r}\ \eta^r \wedge \eta^r_u$$

Hence, if $\beta \neq 0$, the immersion is recovered using (3.17) □

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