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APPLICATION OF THE EXTENDED TRIAL EQUATION METHOD TO THE NONLINEAR EVOLUTION EQUATIONS

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Abstract. In this paper we investigate the exact solutions of the nonlinear partial differential equations. We have applied the extended trial equation method to nonlinear partial differential equations. By using this method we have successfully obtained analytical solutions of the two-dimensional Bratu type equation. We think that this method can also be applied to other nonlinear evolution equations.

1. Introduction

The investigation of exact solutions for nonlinear partial differential equations plays an important role in applied mathematics and physical science. To find exact solutions of partial differential equations, several methods such as Tanh-method [1-4], (G'/G)- expansion method [5- 8], homogeneous balance method [9,10], first integral method [11], sine-cosine method [12,13], Hirotas bilinear transformation [14,15], trial equation method [16-18], and so on, have been proposed. Recently, Pandir et al. proposed a new version of the trial equation method, called as extended trial equation method to solve nonlinear partial differential equations [19]. In this study, we apply the extended trial equation method to the nonlinear partial differential equations. By the use of the complete discrimination system for polynomial method, we obtain the classification of exact solutions, including rational, soliton and elliptic solutions, to the two dimensional Bratu type equation [20,21]:

$$u_{xx} + u_{yy} + \lambda exp(su) = 0. \tag{1.1}$$

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2. The Extended Trial Equation method

The main steps of the extended trial equaiton method are outlines as follows [19]:

Step1. Consider the nonlinear partial differential equation for a function of u:

$$P(u, u_t, u_x, u_{xx}, u_{tt}, u_{xt}, \ldots) = 0. \tag{2.1}$$

Take the wave transformation:

$$u(x,t) = u(\eta), \quad \eta = kx + \omega t, \tag{2.2}$$

where k and ω are arbitrary constants. Under the wave transformation (2.2), Eq.(2.1) reduces to the nonlinear ordinary differential equation of the form:

$$N(u, u', u'', \ldots) = 0. \tag{2.3}$$

Step 2. Take the finite series and the trial equation as follows:

$$u = \sum_{i=0}^{\delta} \tau_i \Gamma^i, \tag{2.4}$$

where,

$$(\Gamma')^2 = \Lambda(\Gamma) = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} = \frac{\xi_\theta \Gamma^\theta + \ldots + \xi_1 \Gamma + \xi_0}{\zeta_\epsilon \Gamma^\epsilon + \ldots + \zeta_1 \Gamma + \zeta_0}. \tag{2.5}$$

By equations Eq.(2.4) and Eq.(2.5), we can write

$$(u')^2 = \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i \tau_i \Gamma^{i-1} \right)^2, \tag{2.6}$$

$$u'' = \frac{\Phi'(\Gamma)\Psi(\Gamma) - \Phi(\Gamma)\Psi'(\Gamma)}{2\Psi^2(\Gamma)} \left(\sum_{i=0}^{\delta} i \tau_i \Gamma^{i-1} \right) + \frac{\Phi(\Gamma)}{\Psi(\Gamma)} \left(\sum_{i=0}^{\delta} i(i-1) \tau_i \Gamma^{i-2} \right), \tag{2.7}$$

where $\Phi(\Gamma)$ and $\Psi(\Gamma)$ are polynomials. Substituting Eq.(2.6) and Eq.(2.7) into Eq.(2.3) yields a polynomial as:

$$\Omega(\Gamma) = \upsilon_s \Gamma^s + \ldots + \upsilon_1 \Gamma + \upsilon_0. \tag{2.8}$$

We can find a relation of θ, ϵ and δ according to the balance principle and compute some values of them.

Step 3. Setting the coefficients of $\Omega(\Gamma)$ to zero yields a system of algebraic equations:

$$\upsilon_i = 0, \quad i = 0, \ldots, s. \tag{2.9}$$

Solving the algebraic eqaution system (2.9), we can determine the values of $\xi_0, \ldots, \xi_\theta$, $\zeta_0, \ldots, \zeta_\epsilon$ and $\tau_0, \ldots, \tau_\delta$.

Step 4. Rewrite the Eq.(2.5) by the integral form:

$$\pm(\eta - \eta_0) = \int \frac{d\Gamma}{\sqrt{\Lambda(\Gamma)}} = \int \sqrt{\frac{\Psi(\Gamma)}{\Phi(\Gamma)}} d\Gamma. \tag{2.10}$$

When we solve Eq.(2.10) by using a complete discrimination system for polynomial to classify the roots of $\Phi(\Gamma)$, we obtain the exact solutions of the Eq.(2.3).

3. Application to the two-dimesional Bratu type Equation

Under the wave transformation (2.2) and the transformation

$$u = (1/s)ln\upsilon, \tag{3.1}$$

Eq.(1.1) reduces to the ordinary differentail equation in the form [21]:

$$(k^2+\omega^2)\upsilon\upsilon'' - (k^2+\omega^2)(\upsilon')^2 + \lambda s\upsilon^3 = 0, \tag{3.2}$$

where prime denotes the derivative with respect to η. Substituting Eq.(2.6) and Eq.(2.7) into Eq.(3.2) and using the balance principle we get

$$\theta = \epsilon + \delta + 2. \tag{3.3}$$

Using the solution procedure of the extended trial equation method, we obtain the results as follows:
Case 1: If we take $\epsilon = 0$,$\delta = 1$ and $\theta = 3$, then we get

$$(\upsilon')^2 = \frac{(\tau_1^2)(\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.4}$$

where $\xi_3 \neq 0$,$\zeta_0 \neq 0$. Solving the algebraic equation system (2.9), we obtain the following results:

$$\xi_0 = \frac{-\xi_3\tau_0^3+\xi_1\tau_0\tau_1^2}{2\tau_1^3},\ \xi_1=\xi_1,\ \xi_2 = \frac{3\xi_3\tau_0}{2\tau_1}+\frac{\xi_1\tau_1}{2\tau_0},\ \xi_3=\xi_3, \tag{3.5}$$

$$\zeta_0=\zeta_0,\ \tau_0=\tau_0,\ \tau_1=\tau_1,\ \omega = -\sqrt{\frac{-k^2\xi_3-2\lambda s\zeta_0\tau_1}{\xi_3}} \tag{3.6}$$

Substituting these results into Eq.(2.5) and Eq.(2.10), we get

$$\pm(\eta-\eta_0) = A\int\frac{d\Gamma}{\sqrt{\Gamma^3+\left(\frac{3\tau_0}{2\tau_1}+\frac{\xi_1\tau_1}{2\xi_3\tau_0}\right)\Gamma^2+\frac{\xi_1}{\xi_3}\Gamma+\frac{-\xi_3\tau_0^3+\xi_1\tau_0\tau_1^2}{2\xi_3\tau_1^3}}}, \tag{3.7}$$

where, $A = \sqrt{(\zeta_0/\xi_3)}$. Integrating Eq.(3.7), we obtain

$$\pm(\eta-\eta_0) = -\frac{2A}{\sqrt{\Gamma-\alpha_1}}, \tag{3.8}$$

$$\pm(\eta-\eta_0) = \frac{2A}{\sqrt{\alpha_2-\alpha_1}}arctan\sqrt{\frac{\Gamma-\alpha_2}{\alpha_2-\alpha_1}},\quad \alpha_2>\alpha_1, \tag{3.9}$$

$$\pm(\eta-\eta_0) = \frac{A}{\sqrt{\alpha_1-\alpha_2}}ln\left|\frac{\sqrt{\Gamma-\alpha_2}-\sqrt{\alpha_1-\alpha_2}}{\sqrt{\Gamma-\alpha_2}+\sqrt{\alpha_1-\alpha_2}}\right|,\quad \alpha_1>\alpha_2, \tag{3.10}$$

$$\pm(\eta-\eta_0) = 2\sqrt{\frac{\zeta_0/\xi_3}{\alpha_2-\alpha_1}}F(\varphi,l),\quad \alpha_1>\alpha_2>\alpha_3, \tag{3.11}$$

where,
$A = \sqrt{(\zeta_0/\xi_3)}$, $F(\varphi,l) = \int_0^{\varphi}\frac{d\psi}{\sqrt{1-l^2sin^2\psi}}$, $\varphi = arcsin\sqrt{\frac{\alpha_2-\alpha_1}{\Gamma-\alpha_1}}$, $l^2 = \frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}$.
Also, $\alpha_1,\alpha_2,\alpha_3$ are the roots os the polynomial equation:

$$\Gamma^3+\frac{\xi_2}{\xi_3}\Gamma^2+\frac{\xi_1}{\xi_3}\Gamma+\frac{\xi_0}{\xi_3} = 0. \tag{3.12}$$

Substituting the solutions (3.8)-(3.11) into Eq.(2.4) and (3.1), we obtain exact solutions to (1.1) as follows:

$$u(x,t)=ln\left[\tau_0+\tau_1\alpha_1+\frac{4\tau_1(\zeta_0/\xi_3)}{(kx-\sqrt{(-k^2\xi_3-2\lambda s\zeta_0\tau_1)/\xi_3}t-\eta_0)^2}\right]^{\frac{1}{s}}, \tag{3.13}$$

(3.14)

$$u(x,t)=ln\left[\tau_0+\tau_1\alpha_1+\tau_1(\alpha_2-\alpha_1)sec^2\left(\frac{\sqrt{\alpha_2-\alpha_1}}{2\sqrt{\zeta_0/\xi_3}}\left(kx-\frac{\sqrt{-k^2\xi_3-2\lambda s\zeta_0\tau_1}}{\sqrt{\xi_3}}t-\eta_0\right)\right)\right]^{\frac{1}{s}},$$

(3.15)

$$u(x,t)=ln\left[\tau_0+\tau_1\alpha_1+\tau_1(\alpha_1-\alpha_2)cosech^2\left(\frac{\sqrt{\alpha_2-\alpha_1}}{2\sqrt{\zeta_0/\xi_3}}\left(kx-\frac{\sqrt{-k^2\xi_3-2\lambda s\zeta_0\tau_1}}{\sqrt{\xi_3}}t-\eta_0\right)\right)\right]^{\frac{1}{s}},$$

(3.16)

$$u(x,t)=ln\left[\tau_0+\tau_1\alpha_1+\tau_1(\alpha_2-\alpha_1)sn^2\left(\pm\frac{\sqrt{\alpha_2-\alpha_1}}{2\sqrt{\zeta_0/\xi_3}}\left(kx-\frac{\sqrt{-k^2\xi_3-2\lambda s\zeta_0\tau_1}}{\sqrt{\xi_3}}t-\eta_0\right),\frac{\alpha_1-\alpha_3}{\alpha_1-\alpha_2}\right)\right]^{\frac{1}{s}}.$$

Case 2: If we take $\epsilon=0$, $\delta=2$ and $\theta=4$, then we get

$$(v')^2=\frac{(\tau_1+2\tau_2)^2(\xi_4\Gamma^4+\xi_3\Gamma^3+\xi_2\Gamma^2+\xi_1\Gamma+\xi_0)}{\zeta_0}, \tag{3.17}$$

where, $\xi_4\neq0$, $\zeta_0\neq0$. Solving the algebraic Eq.(2.9), we have:

$$\xi_0=\frac{\xi_2\tau_0^2}{\tau_1^2+2\tau_0\tau_2},\ \xi_1=\frac{2\xi_2\tau_0\tau_1}{\tau_1^2+2\tau_0\tau_2},\ \xi_3=\frac{2\xi_2\tau_1\tau_2}{\tau_1^2+2\tau_0\tau_2},\ \xi_4=\frac{\xi_0\tau_2^2}{\tau_1^2+2\tau_0\tau_2}, \tag{3.18}$$

$$\zeta_0=\zeta_0,\ \tau_0=\tau_0,\ \tau_1=\tau_1, \tag{3.19}$$

$$w=-\sqrt{-(\tau_1^2-4\tau_0\tau_2)^2(2k^2\xi_2\tau_2+\lambda s\zeta_0(\tau_1^2+2\tau_0\tau_2))/2\xi_2\tau_2(\tau_1^2-4\tau_0\tau_2)^2}. \tag{3.20}$$

Substituting these results into Eq. (2.5) and (2.10), we get

$$\pm(\eta-\eta_0)=B\int\frac{d\Gamma}{\sqrt{\Gamma^4+\frac{2\xi_2\tau_1}{\xi_0\tau_2}\Gamma^3+\frac{\tau_1^2+2\tau_0\tau_2}{\xi_0\tau_2^2}\xi_2\Gamma^2+\frac{2\xi_2\tau_0\tau_1}{\xi_0\tau_2^2}\Gamma+\frac{\xi_2\tau_0^2}{\xi_0\tau_2^2}}}, \tag{3.21}$$

where, $B=\sqrt{\zeta_0(\tau_1^2+2\tau_0\tau_2)/\xi_0\tau_2^2}$.

While $\alpha_1,\alpha_2,\alpha_3$ are the roots of the polynomial equation:

$$\Gamma^4+\frac{\xi_3}{\xi_4}\Gamma^3+\frac{\xi_2}{\xi_4}\Gamma^2+\frac{\xi_1}{\xi_4}\Gamma+\frac{\xi_0}{\xi_4}=0, \tag{3.22}$$

integrating Eq. (3.21) and substituting these results into Eq. (2.4) and (3.1), we obtain exact solutions to Eq. (1.1) as follows:

$$u(x,t)=ln\left[\tau_0+\tau_1\alpha_1+\frac{\tau_1B}{kx+wt-\eta_0}+\tau_2\left(\alpha_1\pm\frac{B}{kx+wt-\eta_0}\right)^2\right]^{\frac{1}{s}}, \tag{3.23}$$

$$u(x,t) = ln\left[\tau_0+\tau_1\alpha_1+\frac{4B^2(\alpha_2-\alpha_1)\tau_1}{4B^2-[(\alpha_1-\alpha_2)(kx+wt-\eta_0)]^2} + \tau_2\left(\alpha_1+\frac{4B^2(\alpha_2-\alpha_1)}{4B^2-[(\alpha_1-\alpha_2)(kx+wt-\eta_0)]^2}\right)^2\right]^{\frac{1}{s}} \tag{3.24}$$

(3.25)

$$u(x,t) = ln\left[\tau_0+\tau_1\alpha_1+\frac{(\alpha_1-\alpha_2)\tau_1}{exp\left[\frac{(\alpha_1-\alpha_2)(kx+\omega t-\eta_0)}{B}\right]-1}+\tau_2\left(\alpha_1+\frac{(\alpha_1-\alpha_2)}{exp\left[\frac{(\alpha_1-\alpha_2)(kx+\omega t-\eta_0)}{B}\right]-1}\right)^2\right]^{\frac{1}{s}},$$

$$u(x,t) = ln\left[\tau_0+\tau_1\alpha_1-\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)\tau_1}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)cosh\left[\left(\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B\right)C\right]} +\tau_2\left(\frac{2(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}{2\alpha_1-\alpha_2-\alpha_3+(\alpha_3-\alpha_2)cosh\left[\left(\sqrt{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}/B\right)C\right]}\right)^2\right]^{\frac{1}{s}}, \tag{3.26}$$

where,

$$\omega = -\sqrt{-(\tau_1^2-4\tau_0\tau_2)^2(2k^2\xi_2\tau_2+\lambda s\zeta_0(\tau_1^2+2\tau_0\tau_2))/2\xi_2\tau_2(\tau_1^2-4\tau_0\tau_2)^2},$$

$$B = \sqrt{\zeta_0(\tau_1^2+2\tau_0\tau_2)/\xi_0\tau_2^2},$$

$$C = kx - \sqrt{-(\tau_1^2+4\tau_0\tau_2)^2(2k^2\xi_2\tau_2+\lambda s\zeta_0(\tau_1^2+2\tau_0\tau_2))/2\xi_2\tau_2(\tau_1^2-4\tau_0\tau_2)^2}t.$$

4. Conclusion

In this study, the extended trial equation method is applied to the two-dimensional Bratu type equation. By using this method we have obtained exact solutions of the two-dimensional Bratu type equation. The results show that the extended trial equation method is a powerful and effective mathematical tool for solving nonlinear partial differential equations in science and engineering.

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