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GENERAL FIXED POINT THEOREMS OF GENERALIZED GREGUŠ TYPE IN SYMMETRIC SPACES AND APPLICATIONS

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Abstract. In this paper a general fixed point theorem of Greguš type in symmetric space is proved, which generalize Theorems 1 and 3 [19], Theorem 1.2 [23], Theorems 3.1 and 4.1 [28] and we obtain new similar results for strict expansive mappings. In the last part of this paper, as applications, we obtain new results for mappings satisfying contractive (expansive) condition of integral type.

1. Introduction

Let X be a nonempty set and f and g be self mappings of X. We say that $x \in X$ is a coincidence point of f and g if $fx = gx$. The set of all coincidence points of f and g will be denoted by $C(f, g)$. A point $w \in X$ is said to be a point of coincidence of f and g if there exists $x \in X$ such that $w = fx = gx$.

Let (X, d) be a metric space and f, g be two self mappings of X. Jungck [17] defined f and g to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

In 1994, Pant [21] introduced the notion of pointwise R - weakly commuting mappings. It is proved in [22] that pointwise R - weakly commuting is equivalent to commutativity in coincidence points.

Definition 1.1 ([18])**.** f and g are said to be weakly compatible if $fgu = gfu$ for all $u \in C(f, g)$.

Al - Thagafi and Naseer Shahzad [3] introduced the notion of occasionally weakly compatible (owc) mappings.

Definition 1.2 ([3])**.** Two self mappings f and g of a nonempty set X are said to be occasionally weakly compatible (owc) mappings if there exist a coincidence point of f and g at which f and g commute.

Remark 1.1 ([3]). If f and g are weakly compatible, then they are occasionally weakly compatible, but the following example show that the converse is not true.

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Example 1.1 ([2]). Let $X = [1, \infty)$ be with the usual metric. Define $f, g : X \to X$ by: $f(x) = 3x - 2$ and $g(x) = x^2$. We have that $fx = gx$ if $x = 1$ or $x = 2$, and $fg(1) = gf(1) = 1$, $fg(2) \neq gf(2)$. Therefore, f and g are occasionally weakly compatible but are not weakly compatible.

It has been observed in [13] that some of the defining properties of the metric space are not used in the proof of certain metric theorems. Hicks and Rhoades [13] established some common fixed point theorems in symmetric spaces and proved that very general probabilistic structures admit a compatible symmetric.

A symmetric on a non empty set X is a non - negative real function d on $X \times X$ such that

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for every $x, y \in X$.

Some fixed point theorems in symmetric spaces are proved in [28], [19], [23], [20], [15] and in other papers.

The study of fixed points for mappings satisfying an implicit relation is initiated in [26], [27] and in other papers.

Actually the method is used in the study of fixed points in metric spaces, symmetric spaces, quasimetric spaces, Tychonoff spaces, reflexive spaces, convex metric spaces, compact metric spaces, paracompact metric spaces, in two or three metric spaces for single valued functions, hybrid pairs of mappings and set valued functions.

Quite recently, the method is used in the study of fixed points for mappings satisfying a contractive condition of integral type and fuzzy metric spaces. With this method the proofs of some fixed point theorems are more simple. Also, the method allow the study of local and global properties of fixed point structures.

2. Preliminaries

Greguš [12] proved the following theorem:

Theorem 2.1 ([12]). *Let C be a nonempty closed common subset of a Banach space X and let T be a mapping of C into itself satisfying the inequality*

$$\|Tx - Ty\| \leq a \|x - y\| + b \|x - Tx\| + c \|y - Ty\| \tag{2.1}$$

for all $x, y \in X$, where $a > 0$, $b, c \geq 0$ and $a + b + c = 1$. Then T has a unique fixed point.

Some authors have generalized Theorem 2.1 in [6], [7], [8], [9], [11], [24], [25], [28] and in other papers. Quite recently, the following theorem is proved in [23].

Theorem 2.2 ([23]). *Let f and S be occasionally weakly compatible self - mappings of a metric space (X, d) satisfying*

$$\begin{aligned} d(fx, f^2x) \neq \max\{&d(Sx, Sf(x)), d(fx, Sx), \\ &d(f^2x, Sfx), d(fx, Sfx), d(Sx, f^2x)\} \end{aligned} \tag{2.2}$$

where $fx \neq f^2x$.

Then f and S have a common fixed point.

In a recent paper, Branciari [5] established the following result:

Theorem 2.3. *Let (X, d) be a complete metric space, $c \in (0, 1)$ and let $f : X \to X$ be a mapping such that for each $x, y \in X$*

$$\int_0^{d(fx,fy)} h(t)dt \le c \int_0^{d(x,y)} h(t)dt,$$

where $h : [0, \infty) \to [0, \infty)$ is a Lebesque - measurable mapping which is summable (i.e., with finite integral) on each compact subset of $[0, \infty)$, such that, for each $\varepsilon > 0$, $\int_0^{\varepsilon} h(t)dt > 0$. Then f has a unique fixed point $z \in X$ such that, for each $x \in X$, $\lim_{n\to\infty} f^n x = z$.

Theorem 2.3 has been generalized in several papers. Quite recently, [9], [20], [29], [30] and other papers extended Theorem 2.3 for weakly compatible and occasionally weakly compatible mappings.

3. Generalized implicit Greguš type functions

In the following we denote by $\mathfrak{F}_G$ the family of all functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying $F(t, t, 0, 0, t, t) = 0$, $\forall t > 0$ and named this family, generalized implicit Greguš type functions.

Example 3.1. $\phi(t_1, ..., t_6) = t_1 - \max\{t_2, t_3, t_4, t_5, t_6\}$.

Example 3.2. $\phi(t_1, ..., t_6) = t_1^p - at_2^p - (1 - a)\max\{t_3^p, t_4^p, (t_3t_5)^{\frac{p}{2}}, (t_5t_6)^{\frac{p}{2}}\}$, where $0 < a < 1$ and $p \ge 1$.

Example 3.3. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4\} - c\max\{t_2, t_5, t_6\}$, where a, b, $c \ge 0$ and $a + c = 1$.

Example 3.4. $\phi(t_1, ..., t_6) = t_1 - \alpha\max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, where $0 < \alpha < 1$, $a, b \ge 0$ and $a + b = 1$.

Example 3.5. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\dfrac{t_5 + t_6}{1 + t_3 + t_4}$, where a, $b \ge 0$ and $a + 2b = 1$.

Example 3.6. $\phi(t_1, ..., t_6) = t_1 - \max\left\{t_2, \frac{1}{2}(t_3 + t_4), \frac{1}{2}(t_5 + t_6)\right\}$.

Example 3.7. $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c\min\{t_5, t_6\}$, where a, b, $c \ge 0$ and $a + c = 1$.

Example 3.8. $\phi(t_1, ..., t_6) = t_1(1+\alpha t_2) - \alpha(t_3t_4 + t_5t_6) - at_2 - (1-a)\max\{t_3, t_4, (t_5t_6)^{\frac{1}{2}}, (t_3t_6)^{\frac{1}{2}}\}$, where $\alpha > 0$ and $0 < a < 1$.

Example 3.9. $\phi(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, where $0 < c \le 1$, $b \ge 0$ and $a + b = 1$.

Example 3.10. $\phi(t_1, ..., t_6) = t_1 - at_2 - b(t_3 + t_4) - c(t_5 + t_6)$, where $a, b, c \ge 0$ and $a + 2c = 1$.

Example 3.11. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\max\{t_3, t_4, \frac{2t_4+t_6}{2}, \frac{2t_4+t_5}{2}, \frac{t_5+t_6}{2}\}$, where a, $b \ge 0$ and $a + b = 1$.

Example 3.12. $\phi(t_1, ..., t_6) = t_1 - at_2 - b\max\{2t_4 + t_5, 2t_4 + t_6, t_3 + t_5 + t_6\}$, where a, $b \ge 0$ and $a + 2b = 1$.

The following theorem is proved in [28].

Theorem 3.1. *Let (X, d) be a symmetric space and $F \in \mathfrak{F}_G$. Suppose that f, g, S and T are self mappings of X such that each of the pairs $\{f, S\}$ and $\{g, T\}$ is owc and the inequality*

$$F(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) < 0, \tag{3.1}$$

holds for all $x, y \in X$ and least one of $d(Sx, Ty)$, $d(fx, Sx)$, $d(gy, Ty)$, $d(fx, Ty)$, $d(gy, Sx)$ is positive. Then, f, g, S and T have a unique common fixed point.

The purpose of this paper is to prove two general fixed point theorem which generalize the results from Theorems 1 and 3 [19], Theorem 1.2 [23], Theorem 3.1 and 4.1 [28] and we obtain new results for mappings satisfying new implicit relations for strict expansive mappings. As applications, new results for mappings satisfying contractive/expansive conditions of integral type are obtained.

4. Main results

Theorem 4.1. *Let (X, d) be a symmetric space and $F \in \mathfrak{F}_G$. Suppose that f and S are self mappings of X such that each of the pair $\{f, S\}$ is owc and*

$$\begin{gathered} F(d(fx, f^2x), d(Sx, Sfx), d(fx, Sx), \\ d(f^2x, Sfx), d(fx, Sfx), d(f^2x, Sx)) \neq 0, \end{gathered} \tag{4.1}$$

for all $x, y \in X$ with $fx \neq f^2x$. Then, f and S have a common fixed point.

Proof. Since the pair $\{f, S\}$ is owc, there exists $u \in X$ such that $fu = Su$ and $fSu = Sfu$ which implies $f^2u = fSu = Sfu$. If $fu = f^2u$, then $fu = f^2u = fSu = Sfu$ and fu is a common fixed point of f and S. If $fu \neq f^2u$, then by (4.1) we have successively:

$$F(d(fu, f^2u), d(Su, Sfu), d(fu, Su), d(f^2u, Sfu), d(fu, Sfu), d(f^2u, Su)) \neq 0,$$

$$F(d(fu, f^2u), d(fu, f^2u), 0, 0, d(fu, f^2u), d(fu, f^2u)) \neq 0,$$

a contradiction of $F \in \mathfrak{F}_G$. Hence $fu = f^2u$, which implies as in the first point of the proof that fu is a common fixed point of f and S. □

Remark 4.1.
- i) By Theorem 4.1 and Example 3.1 we obtain Theorem 2.2 (Theorem 1.2 [23]) and Corollary 1.1 [23] if instead of " $\neq$ " we have " $<$ ".
- ii) A new result we obtain if instead of " $\neq$ " we have " $>$ " for expansive mappings.
- iii) New results are obtained by Example 3.2 - 3.12 for contractive/extensive mapping.

Example 4.1. Let $X = [0, \infty)$ be and $d(x, y) = |x - y|$, $fx = 2x$ and $Sx = 6x$. Then $f^2x = 4x$, $d(fx, f^2x) = 2x$ and $d(fx, Sx) = 4x$. For $x \neq 0$, $d(fx, f^2x) \neq 0$ and $d(fx, f^2x) = 2x < 4x = d(fx, Sx)$.

Hence,

$$d(fx, f^2x) < \max\{d(Sx, Sfx), d(fx, Sx), d(f^2x, Sfx), d(fx, Sfx), d(f^2x, Sx)\}.$$

Then,

$$\begin{aligned} &F(d(fx, f^2x), d(Sx, Sfx), d(fx, Sx), d(f^2x, Sfx), d(fx, Sfx), d(f^2x, Sx)) \\ &= d(fx, f^2x) - \max\{d(Sx, Sfx), d(fx, Sx), d(f^2x, Sfx), d(fx, Sfx), d(f^2x, Sx)\} \neq 0 \end{aligned}$$

for $fx \neq f^2x$.

By Example 3.1, $F \in \mathfrak{F}_G$ and satisfy (4.1). On the other hand, $fx = Sx$ implies $x = 0$ and $fS0 = Sf0 = 0$. Hence, (f, g) is owc and $x = 0$ is a common fixed point of f and g.

Lemma 4.1 ([19])**.** *Let X be a nonempty set and f and g owc self mappings of X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.*

Theorem 4.2. *Let (X, d) be a symmetric space, $F \in \mathfrak{F}_G$ and let f, g, S and T be self mappings of X such that*

$$F(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)) \neq 0 \tag{4.2}$$

whenever $x, y \in X$ with $fx \neq gy$. If there exist $u, v \in X$ such that $fu = Su$ and $gv = Tv$ then there exists $t \in X$ such that t is the unique point of coincidence of f and S, as well as the unique point of coincidence of g and T.

Proof. First we prove that $fu = gv$. Suppose that $fu \neq gv$. Then by (4.2) we get

$$F(d(fu, gv), d(fu, gv), 0, 0, d(fu, gv), d(fu, gv)) \neq 0.$$

This contradicts $F \in \mathfrak{F}_G$. Hence, $fu = Su = gv = Tv = t$. Assuming that there exists $w \neq u$ such that $fw = Sw$ and $fw \neq fu$, we obtain by (4.1) that

$$F(d(fw, gv), d(fw, gv), 0, 0, d(fw, gv), d(fw, gv)) \neq 0.$$

This contradicts $F \in \mathfrak{F}_G$. It follows that $t = Su = fu$ is the unique point of coincidence of f and S. Similarly, one proves that $t = gv = Tv$ is the unique point of coincidence of g and T. □

Theorem 4.3. *Let (X, d) be a symmetric space and $F \in \mathfrak{F}_G$. Suppose that f, g, S and T are self mappings of X such that each of the pairs $\{f, S\}$ and $\{g, T\}$ is owc and inequality(4.2) holds for all $x, y \in X$ with $fx \neq gy$. Then, f, g, S and T have a unique common fixed point.*

Proof. Since each of the pair $\{f, S\}$ and $\{g, T\}$ is owc, there exist $u, v \in X$ such that $fu = Su$ and $fSu = Sfu$, respectively $gv = Tv$ and $gTv = Tgv$. Since $F \in \mathfrak{F}_G$ by Theorem 4.2 there exists $t \in X$ such that t is the unique point of coincidence of f and S, as well unique point of coincidence of g and T, i.e., $t = fu = Su = gv = Tv$. By Lemma 4.1 t is the unique fixed point of f and S, as well as the unique common fixed point of g and T. Consequently, t is the unique common fixed point of f, g, S and T. □

Corollary 4.1. *Let (X, d) be a symmetric space. Suppose that f, g, S and T are self mappings of X such that each of the pairs $\{f, S\}$ and $\{g, T\}$ is owc. If*

$$d(fx, gy) \neq \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)\} \tag{4.3}$$

whenever $x, y \in X$ with $fx \neq gy$. Then f, g, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 4.3 and Example 3.1. □

Remark 4.2. 1. If in (4.3) we have " $<$ " instead of " $\neq$ " we obtain Theorem 1 [19].

2. If in (4.3) we have " $>$ " instead of " $\neq$ " we obtain a new result for extensive mappings which is different by results from recent paper [16].

Corollary 4.2. *Let (X, d) be a symmetric space, $0 < a < 1$ and $p \geq 1$. Let f, g, S and T be self mappings of X such that each of the pairs $\{f, S\}$ and $\{g, T\}$ is owc. Suppose that*

$$d^p(fx, gy) \neq ad^p(Sx, Ty) + (1-a)M(x, y), \tag{4.4}$$

for all $x, y \in X$ with $fx \neq gy$, where

$M(x, y) = \max\{d^p(fx, Sx), d^p(gy, Ty), d^{\frac{p}{2}}(fx, Sx) \cdot d^{\frac{p}{2}}(fx, Ty), d^{\frac{p}{2}}(fx, Ty) \cdot d^{\frac{p}{2}}(Sx, gy)\}$.

Then f, g, S and T have a unique common fixed point.

Proof. The proof it follows by Theorem 4.3 and Example 3.2. □

Remark 4.3. 1. If in (4.4) we have " $<$ " instead of " $\neq$ " we obtain the correct form of Theorem 2 [19].

2. If in (4.4) we have " $>$ " instead of " $\neq$ " we obtain a new result which is different by results from [16].

Example 4.2. Let $X = [0, \infty)$ be and $d(x, y) = |x - y|$, $fx = x$, $Sx = 2x$, $gx = 2x$ and $Tx = 4x$. Then $d(fx, gy) = |x - 2y|$. If $d(fx, gy) = |x - 2y| > 0$, then $d(fx, gy) = |x - 2y| < 2\,|x - 2y| = d(Sx, Ty)$.

Hence,

$$d(fx, gy) < \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\}$$

and

$$\begin{aligned} &F(d(fx, gy), d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(gy, Sx)) \\ &= d(fx, gy) - \max\{d(Sx, Ty), d(fx, Sx), d(gy, Ty), d(fx, Ty), d(Sx, gy)\} \neq 0. \end{aligned}$$

By Example 3.1, $F \in \mathfrak{F}_G$ and satisfy (4.2). The fact that (f, S) and (g, T) are owc is proved similar as in Example 4.1 and $x = 0$ is a common fixed point of f, g, S and T.

If $f = g$ and $S = T$ by Theorem 4.3 we obtain

Theorem 4.4. *Let (X, d) be a symmetric space and $F \in \mathfrak{F}_G$. Suppose that f and S are self mappings of X such that the pair $\{f, S\}$ is owc and*

$$F(d(fx, fy), d(Sx, Sy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(fy, Sx)) \neq 0 \tag{4.5}$$

for all $x, y \in X$ with $fx \neq fy$. Then f and S have a unique common fixed point.

Corollary 4.3. *Let (X, d) be a symmetric space and let f and S be self mappings of X such that the pair $\{f, S\}$ is owc and*

$$\begin{aligned} d(fx, fy) \neq ad(Sx, Sy) + b\max\{d(fx, Sx), d(fy, Sy)\} + \\ + c\max\{d(Sx, Sy), d(fx, Sy), d(fy, Sx)\} \end{aligned} \tag{4.6}$$

for all $x, y \in X$ with $fx \neq fy$ and $a, b, c \geq 0$ and $a + c = 1$. Then f and S have an unique common fixed point.

The proof it follows by Theorem 4.4 and Example 3.3.

Remark 4.4. 1. If in (4.6) we have " $<$ " instead of " $\neq$ " then we obtain a variant of Theorem 2 [19].

2. If in (4.6) we have " $>$ " instead of " $\neq$ " we obtain a new result which is different by results from [16].

Example 4.3. Let $X = [0, \infty)$ be and $d(x, y) = |x - y|$, $fx = x$ and $Sx = 2x$; $d(fx, fy) = |x - y|$ and $d(Sx, Sy) = 2|x - y|$. If $fx \neq fy$, then $d(fx, fy) < d(Sx, Sy)$ and $d(fx, fy) < \max\{d(Sx, Sy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(fy, Sx)\}$ and

$$\begin{aligned} &F(d(fx, fy), d(Sx, Sy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(fy, Sx)) \\ &= d(fx, Sy) - \max\{d(Sx, Sy), d(fx, Sx), d(fy, Sy), d(fx, Sy), d(fy, Sx)\} \neq 0. \end{aligned}$$

By Example 3.1, $F \in \mathfrak{F}_G$ and satisfy (4.6). As in Example 4.1, (f, S) is owc and $x = 0$ is a common fixed point of f and S.

5. Applications

Let (X, d) be a metric space and $D : X \times X \to \mathbb{R}_+$ defined by $D(x, y) = \int_0^{d(x,y)} h(t)dt$, where $h : [0, \infty) \to [0, \infty)$ as in Theorem 2.3.

Lemma 5.1 ([20], [29]). *D is a symmetric on X.*

Let (X, D) be the symmetric space determined by D.

Remark 5.1. The contractive condition of integral type in a metric space (X, d), used in fixed point theory can be written as usual contractive conditions in symmetric space (X, D) [20], [29].

The following theorem is proved in [28].

Theorem 5.1 (Theorem 4.1 [28]). *Let (X, d) be a metric space and $h : [0, \infty) \to [0, \infty)$ be a function as in Theorem 2.3. Suppose that A, B, S and T are self mappings of X such that each of the pairs $\{A, S\}$ and $\{B, T\}$ is owc. If $F \in \mathfrak{F}_G$ and*

$$\begin{aligned} &F\Big(\int_0^{d(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt, \\ &\int_0^{d(Ty,By)} h(t)dt, \int_0^{d(Ax,Ty)} h(t)dt, \int_0^{d(By,Sx)} h(t)dt\Big) < 0 \end{aligned} \tag{5.1}$$

whenever $x, y \in X$ and at least one of the distances $d(Sx, Ty)$, $d(Sx, Ax)$, $d(Ty, By)$, $d(Ax, Ty)$, $d(By, Sx)$ is positive. Then A, B, S and T have a unique common fixed point.

We prove a new general fixed point theorem.

Theorem 5.2. *Let (X, d) be a metric space and $h : [0, \infty) \to [0, \infty)$ be a function as in Theorem 2.3. Suppose that each of the pairs $\{A, S\}$ and $\{B, T\}$ is owc. If $F \in \mathfrak{F}_G$ and*

$$\begin{aligned} &F\Big(\int_0^{d(Ax,By)} h(t)dt, \int_0^{d(Sx,Ty)} h(t)dt, \int_0^{d(Sx,Ax)} h(t)dt, \\ &\int_0^{d(Ty,By)} h(t)dt, \int_0^{d(Ax,Ty)} h(t)dt, \int_0^{d(By,Sx)} h(t)dt\Big) \neq 0 \end{aligned} \tag{5.2}$$

for all $x, y \in X$ with $Ax \neq By$. Then A, B, S and T have a unique common fixed point.

Proof. Let D be as in Lemma 5.1. Then we have

$$\begin{aligned}
D(Ax,By) &= \int\limits_0^{d(Ax,By)} h(t)dt, D(Sx,Ty) = \int\limits_0^{d(Sx,Ty)} h(t)dt,\\
D(Sx,Ax) &= \int\limits_0^{d(Sx,Ax)} h(t)dt, D(Ty,By) = \int\limits_0^{d(Ty,By)} h(t)dt, \qquad (5.3)\\
D(Ax,Ty) &= \int\limits_0^{d(Ax,Ty)} h(t)dt, D(By,Sx) = \int\limits_0^{d(By,Sx)} h(t)dt.
\end{aligned}$$

By (5.2) and (5.3) we obtain

$$F(D(Ax,By),D(Sx,Ty),D(Sx,Ax),D(Ty,By),D(Ax,Ty),D(Sx,By)) \neq 0$$

for all $Ax \neq By$ and $F \in \mathfrak{F}_G$.

Since $\{A,S\}$ and $\{B,T\}$ are owc then Theorem 5.2 it follows by Theorem 4.3. □

Remark 5.2. 1. If in (5.2) we have " $<$ " instead of " $\neq$ " then we obtain similar results to the Theorem 5.1.

2. If in (5.2) we have " $>$ " instead of " $\neq$ " we obtain new general results for extensive mappings.

3. By Examples 3.1 - 3.12 we obtain new particular results for contractive/extensive conditions.

Remark 5.3. Similar results we obtain by Theorem 4.1.

Example 5.1. Let $X = [0,\infty)$ be and $d(x,y) = |x-y|$, $Ax = x$, $Sx = 2x$, $Bx = 2x$, $Tx = 4x$. Then $d(Ax,By) = |x-2y|$, $d(Sx,Ty) = 2|x-2y|$. If $Ax \neq By$, then $|x-2y| > 0$ and $d(Ax,By) < d(Sx,Ty)$. Suppose $h(t) = t$. Because t is strict increasing,

$$D(Ax,By) = \int_0^{d(Ax,By)} tdt = \int_0^{|x-2y|} tdt < \int_0^{2|x-2y|} tdt = \int_0^{d(Sx,Ty)} tdt = D(Sx,Ty).$$

Hence,

$$D(Ax,By) < \max\{D(Sx,Ty),D(Sx,Ax),D(Ty,By),D(Ax,Ty),D(By,Sx)\}.$$

Then

$$\begin{aligned}
&F(\int\limits_0^{d(Ax,By)} tdt, \int\limits_0^{d(Sx,Ty)} tdt, \int\limits_0^{d(Sx,Ax)} tdt, \int\limits_0^{d(Ty,By)} tdt, \int\limits_0^{d(Ax,Ty)} tdt, \int\limits_0^{d(By,Sx)} tdt) =\\
&\int\limits_0^{d(Ax,By)} tdt - \max\{\int\limits_0^{d(Sx,Ty)} tdt, \int\limits_0^{d(Sx,Ax)} tdt, \int\limits_0^{d(Ty,By)} tdt, \int\limits_0^{d(Ax,Ty)} tdt, \int\limits_0^{d(By,Sx)} tdt) \neq 0.
\end{aligned}$$

By Example 3.1, $F \in \mathfrak{F}_G$ and satisfy (5.2). The fact that (A,S) and (B,T) are owc is proved as in Example 4.2 and $x = 0$ is a common fixed point of A, B, S and T.

Remark 5.4. Quite recently in [10], [1], [4] it is proved that if a pair of mappings have a unique point of coincidence that it is weakly compatible if and only if is occasionally weakly compatible (as in [14]). Because in Theorem 4.2 is proved that (f,S) and (g,H) have a unique point of coincidence, it follows that in Theorem 4.3, Corollary 4.1, Corollary 4.2, Theorem 4.4 and Theorem 5.2 we can put weakly compatible instead of owc.

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References

[1] Alghamdi, M. A., Radenović, S., Shahzad, N., On some generalizations of commuting mappings. *Abstr. Appl. Anal.*, Volume 2012, Article ID 952052, 6 pages.

[2] Aliouche, A., Popa, V., General common fixed point for occasionally weakly compatible hybrid mappings and applications. *Novi Sad J. Math.* 39 (2009), No.1, 89-109.

[3] Al-Thagafi, M. A., Shahzad, N., Generalized I-nonexpansive maps and invariant approximations. *Acta Math. Sinica* 24 (2008), No.5, 867-876.

[4] Bisht, R. K., Pant, R. P., A critical remark on fixed point theorems for occasionally weakly compatible mappings. *J. Egypt. Math. Soc.* 21 (2013), 273-275.

[5] Branciari, A., Fixed point theorems for mappings satisfying a general contractive condition of integral type. *Int. J. Math. Math. Sci.* 9 (2002), No. 2, 531-536.

[6] Ćirić, L., A generalization of Greguš fixed point theorem. *Czechoslovak Math. J.* 50 (2000), No.3, 449-458.

[7] Divicaro, M. L., Fisher, B., Sessa, S., A common fixed point theorem of Greguš type. *Publ. Math. Debrecen* 84 (1987), 83-89.

[8] Djoudi, A., Nisse, L., Greguš type fixed points for weakly compatible maps. *Bull. Belg. Math. Soc. Simon Stevin* 10 (2003), 369-378.

[9] Djoudi, A., Aliouche, A., A common fixed point theorem of Greguš type for weakly compatible mappings satisfying contractive condition of integral type. *J. Math. Anal. Appl.* 329 (2007), 31-45.

[10] Dorić, D., Kadelburg, Z., Radenović, S., A note on occasionally weakly compatible mappings. *Fixed Point Theory*, 13 (2012), No.2, 475-480.

[11] Fisher, B., Sessa, S., On fixed point theorem of Greguš type. *Int. J. Math. Math. Sci.* 9 (1986), 23-28.

[12] Greguš, M., A fixed point theorem in Banach spaces. *Boll. Un. Mat. Ital.* 17 (1980), No.5, 193-198.

[13] Hicks, T. L., Rhoades, B. E., Fixed point theorems in symmetric spaces with applications to probabilistic spaces. *Nonlinear Anal.* 36 (1999), 331-344.

[14] Imdad, M., Chauhan, S., Employing common limit range property to prove unified metrical common fixed point theorems. *International of Analysis* Volume 2013, Article ID 763261, 10 pages.

[15] Imdad, M., Hasan, M., Nashine, H. K., Murthy, P. P., Employing an implicit function to prove unified common fixed point theorems for expansive type mappings in symmetric spaces. *J. Nonlinear Anal. Appl.*, Volume 2013 (2013), Article ID jnaa-00132, 13 Pages.

[16] Jain, M. K., Rhoades, B. E., Sabya, A. S., Fixed point theorems for occasionally weakly compatible expansive mappings. *J. Adv. Math. Stud.* 5 (2012), No. 2, 54-58.

[17] Jungck, G., Compatible mappings and common fixed points. *Int. J. Math. Math. Sci.* 9 (1986), 771-779.

[18] Jungck, G., Common fixed points for noncontinuous nonself maps on a nonnumeric space. *Far. East J. Math. Sci.* 4 (1996), 192-215.

[19] Jungck, G., Rhoades, B. E., Fixed points for occasionally weakly compatible mappings. *Fixed Point Theory* 7 (2006), 287-296.

[20] Mocanu, M., opa, V., Some fixed point theorems for mappings satisfying implicit relations in symmetric spaces, *Libertas Math.* 28 (2008), 1-13.

[21] Pant, R. P., Common fixed points for noncommuting mappings. *J. Math. Anal. Appl.* 188 (1999), 436-440.

[22] Pant, R. P., Common fixed points for four mappings. *Bull. Calcutta Math. Soc.* 9 (1999), 281-287.

[23] Pant, R. P., Bisht, R. K., Occasionally weakly compatible mappings and fixed points. *Bull. Belg. Math. Soc. Simon Stevin* 19 (2012), 655-661.

[24] Pathak, H. K., Khan, M. S., Compatible mappings of type (B) and common fixed point theorems of Greguš type. *Czechoslovak Math. J.* 45 (1995), no. 4, 685-698.

[25] Pathak, H. K., Cho, Y. J., Kang, S. M., Madharia, B., Compatible mappings of type (C) and common fixed point theorems of Greguš type. *Demonstratio Math. J.* 31 (1998), 499-518.

[26] Popa, V., Fixed point theorems for implicit contractive mappings. *Stud. Cerc. St. Ser. Math. Univ. Bacău* 7 (1997), 129-133.

[27] Popa, V., Some fixed point theorem for compatible mappings satisfying an implicit relation. *Demonstratio Math.* 32 (1999), 157-163.

[28] Popa, V., Symmetric spaces and fixed point theorems for occasionally weakly compatible mappings satisfying contractive conditions of integral type. *Bull. Inst. Politehn. Iaşi, Math., Theoret. Mech. Phys.* 55 (59) (2009), No.3, 11-21.

[29] Popa, V., Mocanu, M., A new viewpoint in the study of fixed points for mappings satisfying a contractive condition of integral type. *Bull. Inst. Politeh. Iaşi, Math., Theoret. Mech. Phys.* 53 (57) (2007), No.5, 269-286.

[30] Popa, V., Mocanu, M., Altering distance and common fixed points under implicit relations, *Hacettepe J. Math. Statistics* 38 (2009), No.3, 329-337.

[31] Rhoades, B. E., Two fixed point theorems for mappings satisfying a general contractive condition of integral type, *Int. J. Math. Math. Sci.* 63 (2003), 4007-4019.

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