Mathematical Sciences And Applications E-Notes
Volume 2 No. 1 pp. 44–54 (2014) ©MSAEN

FRACTIONAL ORDER DERIVATIVE AND RELATIONSHIP BETWEEN DERIVATIVE AND COMPLEX FUNCTIONS

ALI KARCI AND AHMET KARADOGAN

(Communicated by Murat TOSUN)

Abstract. The concept of fractional order derivative can be found in wide range of many different areas. Due to this case, there are many methods about fractional order derivative (FOD). The most of them are Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar-Gangal, and Caputo methods which are fractional order derivatives as mentioned in the literature. However, they are not sound and complete for constant and identity functions. This case means that they are curve fitting or curve approximation methods.

FOD concept was defined in [10]. The deficiencies of Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar-Gangal, and Caputo methods were illustrated in this study. In this study, the concept of FOD and its relationships with complex functions was handled.

1. INTRODUCTION

The fractional order derivatives are different approaches instead of classical derivatives. There are a lot of studies on this subject. The most of these studies being used are Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar- Gangal, and Caputo fractional order derivatives methods. Due to this case, this study focused on Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar- Gangal, and Caputo fractional order derivatives. Some of these studies can be briefly described.

a) Pooseh et al considered finite differences as a subclass of direct methods in the calculus of variations for Riemann-Liouville method [1].
b) Mirevski et al defined new g-Jacobi functions by using Riemann-Liouville operator [2].

Date: Received: November 1, 2013 and Accepted: March 11, 2014.
2010 *Mathematics Subject Classification.* 00A69, 70-08.
Key words and phrases. Derivative, Fractional Order Derivative.
This article is the written version of author's plenary talk delivered on August 26-29, 2013 at 2nd International Eurasian Conference on Mathematical Sciences and Applications IECMSA-2013 at Sarajevo, Bosnia and Herzegovina.

c) Schiavone and Lamb discussed theory of fractional powers of operators on an arbitrary Frechet space, and the authors of this study obtained multivariable fractional integrals and derivatives defined on certain space of test functions and generalized functions [3].
d) Bataineh et al have used the differential equations of fractional order appear in many applications in physics, chemistry and engineering [4].
e) Diethelm et al presented a collection of numerical algorithms for the solution of the various problems arising in derivatives of fractional order [5].
f) Chen and Ye presented some approaches based on piecewise interpolation for fractional calculus, and some improvement based on the Simpson method for the fractional differential equations [6].
g) Li et al (2013) studied on fractional order iterative learning control including many theoretical and experimental results, and these results shown the improvement of transient and steady-state performances [7].
h) Li et al (2011) formulated many models in terms of fractional derivatives and they also studied the important properties of the Riemann-Liouville derivative, one of mostly used fractional derivatives [8].
i) Efe modeled sliding mode control in term of fractional order operators [9].
j) The fractional order derivative was redefined in the papers of Karci [10, 11].

This paper is organized as follow. Section 2 describes the deficiencies and errors for Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar- Gangal, and Caputo methods. Section 3 describes the concept of fractional order derivative and its relationship with complex function. Finally, paper is concluded in Section 4.

2. DEFICIENCIES OF FRACTIONAL ORDER DERIVATIVE METHODS

There are different methods and approximations for fractional order derivatives since 1730. There is a common property related to these methods and approximations. The order of derivative as integer caused gamma function involvement in most of the methods and approximations. We used seven most popular methods in this study.

a) L.Euler (1730) method:

$$\frac{d^n x^m}{dx^n} = m(m-1)\cdots(m-n+1)x^{m-n} \tag{2.1}$$

b) Grunwald-Letnikov method: It is seen that Grunwald-Letnikov method uses discrete values as seen in Eq.2.2, so it cannot be a differential method.

$$ {}_aD_t^{\alpha} f(t) = \lim_{h\to 0} \frac{1}{h^{\alpha}} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh) \tag{2.2}$$

c) Riemann-Liouville method: The method of Riemann-Liouville is given in Eq.2.3.

$$ {}_aD_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}} \tag{2.3}$$

d) Oldham-Spanier method: This method is about fractional derivative scaling property as seen in Eq.2.4.

$$\frac{d^q f(\beta x)}{dx^q} = \beta^q \frac{d^q f(\beta x)}{d(\beta x)^q} \tag{2.4}$$

e) Miller-Ross method: This method is about the fractional orders as seen in Eq.2.5.

$$D^{\varepsilon} f(t) = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_n} f(t) \tag{2.5}$$

where $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n, \forall i, \alpha_i < 1$

f) Kolwankar-Gangal method: 0<q<1, local fractional order derivative at point $x = y$ for $f : [0,1] \longrightarrow R$ is

$$D^q f(y) = \lim_{x\to y} \frac{d^q\big(f(x) - f(y)\big)}{d(x-y)^q} \tag{2.6}$$

g) Caputo (1967) method: The method of Caputo is seen in Eq.2.7.

$$ {}_a^C D_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(v)dv}{(t-v)^{\alpha+1-n}} \tag{2.7}$$

2.1. Fractional Order Derivatives of $f(x) = cx^0$.

These seven methods for fractional order derivatives are mostly used methods. Due to this case, we will investigate the fractional order derivative of each method in this section for a constant function.

First of all, the results of Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar-Gangal, and Caputo methods for $f(x) = cx^0$ are illustrated in Eq.2.8, Eq.2.9, Eq.2.10, Eq.2.11, Eq.2.12 and Eq.2.13. c is a constant, $n = \frac{1}{2}$ for Euler method.

$$\frac{d^n x^m}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} = \frac{d^{\frac{1}{2}} x^0}{dx^{\frac{1}{2}}} = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)} c \frac{\sqrt{x}}{x} \tag{2.8}$$

$\Gamma(1) = 1$ and $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. If $n = \frac{1}{3}$, then $\Gamma\left(\frac{1}{2}\right)$ is required and it is a transcendental number such as π and base of natural logarithm e= 2,7134884828. Since

$$\Gamma\left(\frac{1}{3}\right) = \int_0^\infty t^{-\frac{2}{3}} e^{-t} dt = 3 \sum_{i=0}^{\infty} \frac{3^i e^{-t} t^{\left(\frac{3i+1}{3}\right)}}{\prod_{j=0}^{i} (3j+1)} \tag{2.9}$$

and Eq.2.9 illustrates that it cannot be computed in a simple way.$f(x) = cx^0$ is a constant function and its change with respect to the change in x is zero. However, Eq.2.8 depicts that the result obtained from Euler method is not zero and it depends on variable x. The two conditions must be taken in care.

i) If $\Gamma\left(\frac{1}{3}\right) \neq 0$, and is finite, then the fractional order derivative of $f(x) = cx^0$ is concluded in dependence on variable x. The change in a constant is zero in all cases.
ii) If $\Gamma\left(\frac{1}{3}\right) = 0$, then $\frac{d^n x^m}{dx^n} = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{3}\right)} c \frac{1}{\sqrt[3]{x^2}}$ for $n = \frac{2}{3}$ is indefinite. The obtained results in both cases are inconsistent.

Grunwald-Letnikov method:In the case of constant function, there are many undetermined terms in the Eq.2.10. For example, it is not clear what the term $[\frac{t-a}{h}]$ is.The fractional order derivative for constant function $f(x) = cx^0$ is

$$ {}_aD_t^\alpha f(x) = \lim_{h \to 0.001} \sum_{j=0}^{\left[\frac{t}{0.001}\right]} (-1)^j \binom{\alpha}{j} c = \frac{1}{(0.001)^\alpha} \left[c - \binom{\alpha}{1} c + \binom{\alpha}{2} c + \cdots + \binom{\alpha}{1000t} c \right] \neq 0 \tag{2.10}$$

and it is not zero. So, the obtained result is not derivative of constant function.

Riemann-Liouville method:Assume that $n = 1$ and $n = \frac{1}{2}$

$$ {}_aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}} = {}_a D_t^{\frac{1}{2}} f(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dt} \int_a^t \frac{cdv}{(t-v)^{\frac{1}{2}}} = \frac{c}{\Gamma\left(\frac{1}{2}\right)} \left(- \frac{1}{(t-a)^{\frac{1}{2}}} \right) \neq 0 \tag{2.11}$$

The obtained result is inconsistent, since the result is a function of x. However, initial function is a constant function.

*Oldham-Spaniermethod:*This method can be performed for function $f(\beta t)$ and it is undefined for constant functions.

*Miller-Ross method:*This method is about the differential operator, so, it can be performed for constant functions.

Kowankar-Gangalmethod: Assume that $q = \frac{1}{2}$

$$D^q f(y) = \lim_{x \to y} \frac{d^q\big(f(x) - f(y)\big)}{d(x-y)^q} = \frac{d^q(c-c)}{d(c-c)^q} = \frac{0}{0} \tag{2.12}$$

is indefinite.
Caputo method: Assume that n=1 and $\alpha = \frac{1}{2}$

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^{(n)}(v)dv}{(t-v)^{\alpha+1-n}} = \frac{1}{\Gamma\big(-\frac{1}{3}\big)} \int_a^t \frac{0dv}{(t-v)^{\frac{1}{2}+1-1}} = 0 \tag{2.13}$$

The result of Caputo method is consistent.

The Euler, Grunwald-Letnikov, Riemann-Liouville, Oldham-Spanier, and Kowankar-Gangal methods do not work for constant functions as seen in Eq.2.8, Eq.2.10, Eq.2.11 and Eq.2.12. There is no change in constant function with respect to any change in the independent variable(s). If there is any change in constant function, it is not a constant function. On contrary, any order derivative of constant function is zero with respect to Miller-Ross and Caputo methods.

2.2. **Fractional Order Derivatives of $f(x) = x$.**

Fractional order derivatives of identity function obtained with respect to Euler, Riemann-Liouville, Grunwald-Letnikov, Oldham-Spanier, Miller-Ross, Kolwankar-Gangal, and Caputo methods in this section. Assume that n=1 and $\alpha = \frac{1}{2}$.

Euler method:

$$\frac{d^{\frac{1}{2}} x^1}{dx^{\frac{1}{2}}} = \frac{\Gamma(1+1)}{\Gamma\big(1 - \frac{1}{2} + 1\big)} x^{1-\frac{1}{2}} = \frac{\Gamma(2)}{\Gamma\big(\frac{3}{2}\big)} x^{\frac{1}{2}} \neq 1 \tag{2.14}$$

Classical definition of derivative is

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

It can be seen from definition, the ratio of the change in dependent variable to the change in independent is always 1 (one). In this case, the derivative must be 1 in any fractional order derivative. However, fractional order derivative of identity function with respect to Euler method is different from 1. This means that Euler method yielded in an inconsistent result.

Grunwald-Letnikov method:Assume that t=x, a=0 and h=0.001.

(2.15)

$$\begin{aligned} {}_a D_t^\alpha f(x) &= \lim_{h \to 0.0001} \sum_{j=0}^{\left[\frac{t}{0.001}\right]} (-1)^j \binom{\alpha}{j} f(t - jh) \\ &= \frac{1}{(0.001)^\alpha} \left[f(t) - \binom{\alpha}{1} f(t - 0.001) + \binom{\alpha}{2} f(t - 0.002) + \cdots + \binom{\alpha}{1000t} f(0) \right] \neq 1 \end{aligned}$$

It is seen that the result is not verified that ${}_a D_t^\alpha f(x)$ is not a derivative.

Riemann-Liouville method: Assume that n=1 and $\alpha = \frac{1}{2}$

(2.16)

$$ {}_aD_t^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t \frac{f(v)dv}{(t-v)^{\alpha-n+1}} =_a D_t^{\frac{1}{2}} f(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dt} \int_a^t \frac{xdx}{(t-x)^{\frac{1}{2}-1+1}} $$

$$ = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \frac{d}{dt} \int_a^t \frac{xdx}{(t-x)^{\frac{1}{2}}} = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{a}{\sqrt{t-a}} - 2\sqrt{t-a}\right) \neq 1 $$

The obtained result with respect to Riemann-Liouville method is an inconsistent result.

Oldham-Spanier method: Assume that $\beta = 1$ for $f(x) = x$.

$$ \frac{d^q f(\beta x)}{dx^q} = \frac{1}{q} x^{1-q} \neq 1 \tag{2.17} $$

It is clear that the result is not consistent.

Miller-Ross method: Assume that $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $n = 1$. Then

$$ D^{\alpha} f(x) = D^{\alpha_1} D^{\alpha_2} \cdots D^{\alpha_n} (x) = 1 \tag{2.18} $$

Kolwankar-Gangal method: 0<q<1, local fractional order derivative at point $x = y$ for $f : [0,1] \to R$ is

$$ D^q f(y) = \lim_{x \to y} \frac{d^q\big(f(x) - f(y)\big)}{d(x-y)^q}. $$

f(x) is defined for interval [0,1]. In this case, $x \in [0,1]$ and $f(x) \in [0,1]$. Assume that $q = 1$ and $f(x) = x$, then

$$ D^q f(y) = \lim_{x \to y} \frac{d^q\big(f(x) - f(y)\big)}{d(x-y)^q} = \frac{d(x-y)}{d(x-y)} = 1 \tag{2.19} $$

Caputo method:

(2.20)

$$ {}_a^C D_t^{\alpha} f()t = \frac{1}{\Gamma(\alpha-n)} \int_a^t \frac{f^{(n)}(v)dv}{(t-v)^{\alpha+1-n}} = \frac{1}{\Gamma\left(-\frac{1}{3}\right)} \int_a^t \frac{dv}{(t-v)^{\frac{2}{3}+1-1}} $$

$$ = \frac{1}{\Gamma\left(-\frac{1}{3}\right)} \left(3(t-a)^{\frac{1}{3}}\right) \neq 1 $$

The obtained result with respect to Caputo method is also an inconsistent result.

Euler, Grunwald-Letnikov, Riemann-Liouville, Oldham-Spanier and Caputo have deficiencies for identity function (f(x)=x) and Miller-Ross and Kolwankar-Gangal methods are consistent methods for identity function.

3. A NEW APPROACH FOR FRACTIONAL ORDER DERIVATIVES AND COMPLEX FUNCTIONS

The fractional order derivative methods in the literature can be divided into two groups such as mathematical theory of fractional order derivatives and the applications of fractional order derivatives. There are important shortcomings and deficiencies in both groups of studies. The sources and reasons for these shortcomings and deficiencies can be summarized as follows:

It was assumed that the order of derivative is integer up to a specific step in the literature. While derivation reached to that step, the order of derivative was considered as real number. This process concluded in involvement of gamma functions due to the assumption of order of derivative as integer and extension of continuity. This is deficiencies and shortcoming.

Theorem 3.1. *The fractional order derivative methods do not include gamma functions.*

Proof. The fractional order derivative must have any real order in each derivation step. For example, $f(x) = x^m$ be a function. Assume that the orders of derivations are

$$\frac{\eta_1}{\varphi_1}, \frac{\eta_2}{\varphi_2}, \frac{\eta_3}{\varphi_3}, \cdots, \frac{\eta_k}{\varphi_k}, \eta_i, \varphi_j \in Z, 1 \leq i, j \leq k$$

After the first step derivation, the result

$$f'(x) = mx^{m-\frac{\eta_1}{\varphi_1}}$$

is obtained. Then the second derivation step concludes in

$$f''(x) = m\left(m - \frac{\eta_1}{\varphi_1}\right) x^{m-\frac{\eta_1}{\varphi_1}-\frac{\eta_2}{\varphi_2}}$$

After the third step, the result

$$f^{(3)}(x) = m\left(m - \frac{\eta_1}{\varphi_1}\right)\left(m - \frac{\eta_1}{\varphi_1} - \frac{\eta_2}{\varphi_2}\right) x^{m-\frac{\eta_1}{\varphi_1}-\frac{\eta_2}{\varphi_2}-\frac{\eta_3}{\varphi_3}}$$

is obtained. After the k^{th} step, the derivation result

$$f^{(k)}(x) = m\left(m - \frac{\eta_1}{\varphi_1}\right)\left(m - \frac{\eta_1}{\varphi_1} - \frac{\eta_2}{\varphi_2}\right)\cdots\left(m - \sum_{i=1}^{k-1}\frac{\eta_i}{\varphi_i}\right) x^{m-\sum\limits_{i=1}^{k-1}\frac{\eta_i}{\varphi_i}}$$

is obtained. $f^{(k)}(x)$ does not include gamma function.

The methods about fractional order derivative in the literature have some deficiencies and shortcomings due to some incomplete assumption. Almost all of them involve gamma functions, since there are assumptions up to a specific step that for derivation and extension of continuity. After that point, order of derivation is assumed as real number. This is the source of deficiencies and shortcomings.

A new definition for fractional order derivative is given in this section, and the relationship between derivative and complex function is verified. The order of derivative can be considered as 1 for

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{(x+h) - x}$$

since powers of $f(x+h), f(x), (x+h)$ and x are 1. The powers of all terms in the derivation definition can be considered as $\alpha \in R$. Then derivative of constant function $f(x) = cx^0$ is as follow.

$$\lim_{h\to 0} \frac{f^\alpha(x+h) - f^\alpha(x)}{(x+h)^\alpha - x^\alpha} = \lim h \to 0 \frac{c^\alpha - c^\alpha}{(x+h)^\alpha - x^\alpha} = 0$$

and the derivative of identity function $f(x) = x$ is as follow.

$$\lim_{h\to 0} \frac{f(x+h)^\alpha - f(x)^\alpha}{(x+h)^\alpha - x^\alpha} = \lim_{h\to 0} \frac{(x+h)^\alpha - x^\alpha}{(x+h)^\alpha - x^\alpha} = 1$$

The fractional order can be considered in this perspective. The powers of f(x+h), f(x), (x+h) and x can be considered as a real number for fractional order derivative.

Definition 3.1. $f(x) : R \to R$ is a function, $\alpha \in R$ and the fractional order derivative can be considered as follows.

$$f^{(\alpha)}(x) = \lim_{h\to 0} \frac{f^\alpha(x+h) - f^\alpha(x)}{(x+h)^\alpha - x^\alpha}$$

In the case of very small value of h, the limit in the Definition 3.1 concluded in indefinite limit.

$$f^{(\alpha)}(x) = \lim_{h\to 0} \frac{f^\alpha(x+h) - f^\alpha(x)}{(x+h)^\alpha - x^\alpha} = \frac{0}{0}$$

In this case, the method used for indefinite limit (such as L'Hospital method) can be used, and the fractional order derivative can be redefined as follows.

Definition 3.2. Assume that $f(x) : R \to R$ is a function, $\alpha \in R$ and L(.) be a L'Hospital process. The fractional order derivative of f(x) is

$$f^{(\alpha)}(x) = \lim_{h\to 0} L\left(\frac{f^\alpha(x+h) - f^\alpha(x)}{(x+h)^\alpha - x^\alpha}\right) = \lim_{h\to 0} \frac{\frac{d\left(f^\alpha(x+h) - f^\alpha(x)\right)}{dh}}{\frac{d\left((x+h)^\alpha - x^\alpha\right)}{dh}}$$

The fractional order derivative definition can be demonstrated that it obtained same results as classical derivative definition for $\alpha = 1$. The concepts of first derivative, second derivative, and so on are meaningless in the case of fractional order derivative, since order of derivation is real number. The results of derivation are not scaler, they are vectors, and complex numbers or complex functions are also vectors. So, there must be a relationship between derivative and complex functions (complex numbers). $f^{(\alpha)}(x)$ is a fractional order derivative, so $f^{(\alpha)}(x) = g(x)+ih(x)$ where $g(x)$ is real part of complex number and $h(x)$ is imaginary part of complex number. The important point is that $g(x)$ and $h(x)$ should be determined. The relationship between result of taking derivative and complex number can be specified in the following theorem.

Theorem 3.2. *Assume that $f(x)$ is a function such as $f : R \to R$ and $\alpha \in R$. then $f^{(\alpha)}(x)$ is a function of complex variables.*

Proof. Assume that $\alpha = \frac{\beta}{\delta}$ and $\delta \neq 0$. If $f(x) \geq 0$, any integer root of $f(x)$ is a real number. The fractional order derivative of $f(x)$ is

$$f^{(\alpha)} = \frac{f'(x) f^{\alpha-1}(x)}{x^{\alpha-1}} = f'(x) \sqrt[\delta]{\frac{f^{\beta-\delta}(x)}{x^{\beta-\delta}}} = f'(x) \sqrt[\delta]{\left(\frac{f(x)}{x}\right)^{\beta-\delta}}$$

If the fractional derivative is a function of complex variables, then $f^{(\alpha)}(x) = g(x) + ih(x)$ where i is the $i = \sqrt{-1}$.

If $f(x) < 0$, there will be two cases:

Case 1: Assume that δ is odd.

If $\left(\frac{f(x)}{x}\right)^{\beta-\delta} \geq 0$ or $\left(\frac{f(x)}{x}\right)^{\beta-\delta} < 0$,

then the obtained function $f^{(\alpha)}(x)$ is a real function and $h(x) = 0$ for both cases. Since the multiplication of any negative number in odd steps yields a negative number.

Case 2: Assume that δ is even.

If $\left(\frac{f(x)}{x}\right)^{\beta-\delta} \geq 0$, then $h(x) = 0$ and $f^{(\alpha)}(x)$ is a real function.

If $\left(\frac{f(x)}{x}\right)^{\beta-\delta} < 0$,

then the multiplication of any number in even steps yields a positive number for real numbers. However, it yields a negative result for complex numbers, so, $h(x) \neq 0$. This means that $f^{(\alpha)}(x)$ is a complex function. In fact, $f^{(\alpha)}(x)$ is a complex function for both cases. The $h(x) = 0$ for some situations.

$f^{(\alpha)}(x) = g(x) + ih(x)$ is a function and assume that $\alpha = \frac{\beta}{\delta}$ where $\delta \neq 0$. So,

$$f^{(\alpha)} = \frac{f'(x)f^{\alpha-1}(x)}{x^{\alpha-1}} = f'(x)\sqrt[\delta]{\left(\frac{f(x)}{x}\right)^{\beta-\delta}}$$

$f'(x)$ is a real number and the important point is that the parts of $\sqrt[\delta]{\left(\frac{f(x)}{x}\right)^{\beta-\delta}}$ must be determined. This case is subject to future research, since we cannot determine the parts (real and imaginary) of $f^{(\alpha)}(x)$ in this study.
The $f^{(\alpha)}(x)$ can be investigated for polynomial, exponential, trigonometric and logarithmic functions.

Theorem 3.3. *Assume that $f(x) = x^n, n \in Z^+$, and $\alpha = \frac{\beta}{\delta}, \beta, \delta \in Z$ and $\delta \neq 0$. Then $f^{(\alpha)}(x) = nx^{\frac{\beta(n-1)}{\delta}}$.*

Proof. Assume that $f(x)$ is any function and the fractional order derivative of $f(x)$ is

$$f^{(\alpha)}(x) = \left(\frac{f(x)}{x}\right)^{\alpha-1} f'(x) = \left(\frac{x^n}{x}\right)^{\frac{\beta}{\delta}-1} nx^{n-1} = nx^{\frac{\beta(n-1)}{\delta}}$$

Theorem 3.4. *Assume that $f(x) = \sin^n(x)$ and $g(x) = \cos^n(x), n \in Z^+$, and $\alpha = \frac{\beta}{\delta}, \beta, \delta \in Z$ and $\delta \neq 0$. Then*
$f^{(\alpha)}(x) = \frac{n \sin^{\frac{\beta n-\delta}{\delta}}(x)\cos(x)}{x^{\frac{\beta-\delta}{\delta}}}$ *and* $g^{(\alpha)}(x) = -\frac{n \cos^{\frac{\beta n-\delta}{\delta}}(x)\sin(x)}{x^{\frac{\beta-\delta}{\delta}}}$

Proof. The fractional order derivatives for both functions are as follows.
$f^{(\alpha)}(x) = \left(\frac{f(x)}{x}\right)^{\alpha-1} f'(x) = \left(\frac{\sin^n}{x}\right)^{\frac{\beta}{\delta}-1} n\sin^{n-1}(x)\cos(x) = n\frac{\sin^{\frac{\beta n-\delta}{\delta}}(x)\cos(x)}{x^{\frac{\beta-\delta}{\delta}}}$ and
$g^{(\alpha)}(x) = \left(\frac{g(x)}{x}\right)^{\alpha-1} g'(x) = \left(\frac{\cos^n}{x}\right)^{\frac{\beta}{\delta}-1} n\cos^{n-1}(x)(-\sin(x)) = -n\frac{\cos^{\frac{\beta n-\delta}{\delta}}(x)\sin(x)}{x^{\frac{\beta-\delta}{\delta}}}$

Theorem 3.5. *Assume that* $f(x) = (\ln(x))^n$ *and* $g(x) = e^{mx}, m, n \in Z^+$*, and* $\alpha = \frac{\beta}{\delta}, \beta, \delta \in Z$ *and* $\delta \neq 0$*. Then*
$f^{(\alpha)}(x) = \frac{n(\ln(x))^{\frac{n\beta-\delta}{\delta}}}{x^{\frac{\beta}{\delta}}}$ *and* $g^{(\alpha)}(x) = \frac{ne^{\frac{\beta}{\delta}mx}}{x^{\frac{\beta-\delta}{\delta}}}$.

Proof. The fractional order derivative of logarithmic function $f(x) = (\ln(x))$ is
$f^{(\alpha)}(x) = \left(\frac{f(x)}{x}\right)^{\alpha-1} f'(x) = \left(\frac{\left(\ln(x)\right)^n}{x}\right)^{\frac{\beta}{\delta}-1} \frac{n\left(\ln(x)\right)^{n-1}}{x} = n\frac{\left(\ln(x)\right)^{\frac{n\beta-\delta}{\delta}}}{x^{\frac{\beta}{\delta}}}$ and
the fractional order derivative of exponential function $g(x) = e^{mx}$ is
$g^{(\alpha)}(x) = \left(\frac{g(x)}{x}\right)^{\alpha-1} g'(x) = \left(\frac{e^{mx}}{x}\right)^{\frac{\beta}{\delta}-1} me^{mx} = m\frac{e^{\frac{\beta}{\delta}mx}}{x^{\frac{\beta-\delta}{\delta}}}$

4. CONCLUSIONS

The methods in the literature for fractional order derivatives have important deficiencies and shortcomings. These are illustrated in this paper, so, it is a requirement to redefine the concept of fractional order derivation.

Thus the fractional order derivative was redefined in this paper. Then the obtained results have magnitudes and directions. This means that the result of derivation is a vector. So, there must be a relationship between derivative and complex function. This important point was verified in this paper.

The fractional order derivatives of polynomial, trigonometric, logarithmic and exponential functions were obtained and illustrated in this paper.

References

[1] S. Pooseh, R. Almeida, D.F.M. Torres, *Discrete direct methods in the fractional calculus of variations*, Computers and Mathematics with Applications, doi:10.1016/j.camwa.2013.01.045.

[2] S.P.Mirevski, L.Boyadjiev, R.Scherer, *On the Riemann-Liouville Fractional Calculus, g-Jacobi Functions and F.Gauss Functions*, Applied Mathematics and Computation, 187; 315-325 (2007).

[3] S.E.Schiavone, W.Lamb, *A Fractional Power Approach to Fractional Calculus*, Journal of Mathematical Analysis and Applications, 149; 377-401 (1990).

[4] A.S. Bataineh · A.K. Alomari · M.S.M. Noorani · I. Hashim · R. Nazar, *Series Solutions of Systems of Nonlinear Fractional Differential Equations*, Acta Applied Mathematics, 105; 189-198 (2009).

[5] K.Diethelm, N.J. Ford, A.D. Freed, Yu Luchko, *Algorithms for the Fractional Calculus: A Selection of Numerical Methods*, Computer Methods in Applied Mechanics and Engineering, 194; 743-773 (2005).

[6] C.Li, A.Chen, J.Ye, *Numerical Approaches to Fractional Calculus and Fractional Ordinary Differential Equation*, Journal of Computational Physics, 230; 3352-3368 (2011).

[7] Y.Li, Y.Q. Chen, H.-S. Ahn, G.Tian, *A Survey on Fractional-Order Iterative Learning Control*, Journal of Optimal Theory and Applications, 156; 127-140 (2013).

[8] C.Li, D.Qian, Y.-Q. Chen, *On Riemann-Liouville and Caputo Derivatives*, Discrete Dynamics in Nature and Society, 2011, DOI: 10.1155/2011/562494.

[9] M.. Efe, *Fractional Order Sliding Mode Control with Reaching Law Approach*, Turkish Journal of Electrical Engineering and Computer Science, 18; 731-747 (2010).

[10] A. Karci, *Kesirli Turev icin Yapilan Tanimlamalarin Eksiklikleri ve Yeni Yaklasim*, TOK-2013 Turkish Automatic Control National Meeting and Exhibition.

[11] A. Karci, *A New Approach for Fractional Order Derivative and Its Applications*, Universal Journal of Engineering Sciences, vol.1, pp:110-117, 2013.

Inonu University, Department of Computer Engineering, 44280, Malatya / Turkey
E-mail address: ali.karci@inonu.edu.tr, ahmet.karadogan@inonu.edu.tr