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BERTRAND PARTNER D-CURVES IN THE MINKOWSKI 3-SPACE E_1^3

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Abstract. In this paper, we consider the idea of Bertrand curves for curves lying on surfaces in the Minkowski 3-space E_1^3. By considering the Darboux frame, we define these curves as Bertrand partner D-curves and give the general characterizations for those curves. Then, we find the relations between the geodesic curvatures, the normal curvatures and the geodesic torsions of these associated curves in some special cases.

1. INTRODUCTION

In the differential geometry of space curves the associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of other curve have an important role for characterizations of space curves. The well-known examples of such curves are Bertrand curves. These special curves are characterized as a kind of corresponding relation between two curves such that the curves have common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. These curves have an important role in the theory of curves. Hereby, from the past to today, a lot of mathematicians have studied on Bertrand curves in different spaces such as Euclidean space, Minkowski space or Galilean space [1-5,9,12,15]. Also these curves have an important role in the theory of ruled surfaces and in the characterizations of some other special curves. In [4], Izumiya and Takeuchi have studied cylindrical helices and Bertrand curves from the view point as curves on ruled surfaces. Also, they have studied generic properties of cylindrical helices and Bertrand curves as applications of singularity theory for plane curves and spherical curves [5]. Furthermore, by considering frames of ruled surfaces, Ravani and Ku extended the notion of Bertrand curve to the ruled surfaces and called Bertrand offsets [11]. The corresponding characterizations of the Bertrand offsets of ruled surfaces in the Minkowski 3-space E_1^3 were given by Kasap and Kuruoğlu [6].

The differential geometry of non-null curves lying fully on a surface in the Minkowski 3-space E_1^3 has been given by Uğurlu and Çalışkan [13]. They have

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given the Darboux frames of curves according to the Lorentzian characters of surfaces and curves.

In this paper, we consider the notion of the Bertrand curve for curves lying on surfaces in the Minkowski 3-space E_1^3. We call these new associated curves as Bertrand D-curves and by using the Darboux frame of curves, we give definition and characterizations of these special curves.

2. PRELIMINARIES

The Minkowski 3-space E_1^3 is the real vector space $\mathbb{R}_1^3$ provided with the standard flat metric given by

$$\langle , \rangle = -dx_1^2 + dx_2^2 + dx_3^2$$

where (x_1, x_2, x_3) is a rectangular coordinate system of E_1^3. An arbitrary vector $\vec{v} = (v_1, v_2, v_3)$ in E_1^3 can have one of three Lorentzian causal characters; it can be spacelike if $\langle \vec{v}, \vec{v} \rangle > 0$ or $\vec{v} = 0$, timelike if $\langle \vec{v}, \vec{v} \rangle < 0$ and null (lightlike) if $\langle \vec{v}, \vec{v} \rangle = 0$ and $\vec{v} \neq 0$. Similarly, an arbitrary curve $\vec{\alpha} = \vec{\alpha}(s)$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\vec{\alpha'}(s)$ are respectively spacelike, timelike or null (lightlike) [8]. We say that a timelike vector is future pointing or past pointing if the first compound of the vector is positive or negative, respectively. For any vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in E_1^3, the Lorentzian vector product of $\vec{x}$ and $\vec{y}$ is defined by

$$\vec{x} \times \vec{y} = \begin{vmatrix} e_1 & -e_2 & -e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2,\ x_1 y_3 - x_3 y_1,\ x_2 y_1 - x_1 y_2)$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \qquad e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})$$

and $e_1 \times e_2 = -e_3, \quad e_2 \times e_3 = e_1, \quad e_3 \times e_1 = -e_2$.

Denote by $\left\{\vec{T},\ \vec{N},\ \vec{B}\right\}$ the moving Frenet frame along the curve $\alpha(s)$ in the Minkowski space E_1^3. For an arbitrary spacelike curve $\alpha(s)$ in E_1^3, the following Frenet formulae are given,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ -\varepsilon k_1 & 0 & k_2 \\ 0 & k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix},$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = 1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \varepsilon = \pm 1$, $\left\langle \vec{B}, \vec{B} \right\rangle = -\varepsilon$, $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$ and k_1 and k_2 are curvature and torsion of spacelike curve $\alpha(s)$, respectively. Here, ε determines the kind of spacelike curve $\alpha(s)$. If $\varepsilon = 1$, then $\alpha(s)$ is a spacelike curve with spacelike principal normal $\vec{N}$ and timelike binormal $\vec{B}$. If $\varepsilon = -1$, then $\alpha(s)$ is a spacelike curve with timelike principal normal $\vec{N}$ and spacelike binormal $\vec{B}$. Furthermore, for a timelike curve $\alpha(s)$ in E_1^3, the following Frenet formulae are given as follows,

$$\begin{bmatrix} \vec{T}' \\ \vec{N}' \\ \vec{B}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 \\ k_1 & 0 & k_2 \\ 0 & -k_2 & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}.$$

where $\left\langle \vec{T}, \vec{T} \right\rangle = -1$, $\left\langle \vec{N}, \vec{N} \right\rangle = \left\langle \vec{B}, \vec{B} \right\rangle = 1$, $\left\langle \vec{T}, \vec{N} \right\rangle = \left\langle \vec{T}, \vec{B} \right\rangle = \left\langle \vec{N}, \vec{B} \right\rangle = 0$ and k_1 and k_2 are curvature and torsion of the timelike curve $\alpha(s)$ respectively (For details see [14]).

Definition 2.1. ([10]) ***i) Hyperbolic angle:*** Let $\vec{x}$ and $\vec{y}$ be future pointing (or past pointing) timelike vectors in E_1^3. Then there is a unique real number $\theta \geq 0$ such that $< \vec{x}, \vec{y} >= -\|\vec{x}\| \|\vec{y}\| \cosh\theta$. This number is called the *hyperbolic angle* between the vectors $\vec{x}$ and $\vec{y}$.

ii) Central angle: Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in E_1^3 that span a timelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $|< \vec{x}, \vec{y} >| = \|\vec{x}\| \|\vec{y}\| \cosh\theta$. This number is called the *central angle* between the vectors $\vec{x}$ and $\vec{y}$.

iii) Spacelike angle: Let $\vec{x}$ and $\vec{y}$ be spacelike vectors in E_1^3 that span a spacelike vector subspace. Then there is a unique real number $\theta \geq 0$ such that $< \vec{x}, \vec{y} >= \|\vec{x}\| \|\vec{y}\| \cos\theta$. This number is called the *spacelike angle* between the vectors $\vec{x}$ and $\vec{y}$.

iv) Lorentzian timelike angle: Let $\vec{x}$ be a spacelike vector and $\vec{y}$ be a timelike vector in E_1^3. Then there is a unique real number $\theta \geq 0$ such that $|< \vec{x}, \vec{y} >| = \|\vec{x}\| \|\vec{y}\| \sinh\theta$. This number is called the *Lorentzian timelike angle* between the vectors $\vec{x}$ and $\vec{y}$.

Definition 2.2. ([13])A surface in the Minkowski 3-space E_1^3 is called a timelike (respectively spacelike) surface if the normal vector of the surface is a spacelike (respectively timelike) vector at each point on the surface.

3. DARBOUX FRAME OF A CURVE LYING ON A SURFACE IN THE MINKOWSKI 3-SPACE E_1^3

Let S be an oriented surface in 3-dimensional Minkowski space E_1^3 and let consider a non-null curve $x(s)$ lying fully on S. Since the curve $x(s)$ is also in space, there exists a Frenet frame $\left\{ \vec{T}, \vec{N}, \vec{B} \right\}$ at each point on the curve where $\vec{T}$ is unit tangent vector field, $\vec{N}$ is principal normal vector field and $\vec{B}$ is binormal vector field, respectively.

Since the curve $x(s)$ lies on the surface S, there exists another frame of the curve $x(s)$ which is called Darboux frame and denoted by $\left\{ \vec{T}, \vec{g}, \vec{n} \right\}$. In this frame $\vec{T}$ is the unit tangent vector field of the curve, $\vec{n}$ is the unit normal vector field of the surface S along the curve and $\vec{g}$ is a unit vector obtained by $\vec{g} = \pm\vec{n} \times \vec{T}$. Since the unit tangent $\vec{T}$ is common in both Frenet frame and Darboux frame, the vectors $\vec{N}$, $\vec{B}$, $\vec{g}$ and $\vec{n}$ lie on same plane. Then, if the surface S is an oriented timelike surface, the relations between these frames can be given as follows:

If the curve $x(s)$ is timelike,

$$\begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}.$$

If the curve $x(s)$ is spacelike,

$$\begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}.$$

If the surface S is an oriented spacelike surface, then the curve $x(s)$ lying on S is a spacelike curve. So, the relations between the frames can be given as follows

$$\begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\varphi & \sinh\varphi \\ 0 & \sinh\varphi & \cosh\varphi \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{N} \\ \vec{B} \end{bmatrix}.$$

In all cases, φ is the angle between the vectors $\vec{g}$ and $\vec{N}$.

According to the Lorentzian causal characters of the surface S and the curve $x(s)$ lying on S, the derivative formulae of the Darboux frame can be changed as follows:

i) If the surface S is a timelike surface, then the curve $x(s)$ lying on S can be a spacelike or a timelike curve. Thus, the derivative formulae of the Darboux frame of $x(s)$ are given by

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varepsilon\tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}, \tag{3.1}$$

$$\left\langle \vec{T}, \vec{T} \right\rangle = \varepsilon = \pm 1, \quad \langle \vec{g}, \vec{g} \rangle = -\varepsilon, \quad \langle \vec{n}, \vec{n} \rangle = 1.$$

ii) If the surface S is a spacelike surface, then the curve $x(s)$ lying on S is a spacelike curve. Thus, the derivative formulae of the Darboux frame of $x(s)$ are given by

$$\begin{bmatrix} \dot{\vec{T}} \\ \dot{\vec{g}} \\ \dot{\vec{n}} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \vec{T} \\ \vec{g} \\ \vec{n} \end{bmatrix}, \tag{3.2}$$

$$\left\langle \vec{T}, \vec{T} \right\rangle = 1, \quad \langle \vec{g}, \vec{g} \rangle = 1, \quad \langle \vec{n}, \vec{n} \rangle = -1.$$

In these formulae, k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. Here and in the following, we use "dot" to denote the derivative with respect to the arc length parameter of a curve.

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows:

i) If both S and $x(s)$ are timelike or spacelike, then

$$k_g = \kappa\cos\varphi, \quad k_n = \kappa\sin\varphi, \quad \tau_g = \tau + \frac{d\varphi}{ds}.$$

ii) If S is timelike and $x(s)$ is spacelike, then

$$k_g = \kappa \cosh \varphi, \quad k_n = \kappa \sinh \varphi, \quad \tau_g = \tau + \frac{d\varphi}{ds}.$$

(For details [13]). Furthermore, the geodesic curvature k_g and geodesic torsion τ_g of any curve $x(s)$ can be calculated as follows

$$k_g = -\left\langle \frac{dx}{ds}, \frac{d^2x}{ds^2} \times \vec{n} \right\rangle, \quad \tau_g = -\varepsilon\mu \left\langle \frac{dx}{ds}, \vec{n} \times \frac{d\vec{n}}{ds} \right\rangle, \tag{3.3}$$

where $\mu = \langle \vec{n}, \vec{n} \rangle = \pm 1$.

In the differential geometry of surfaces, for a curve $x(s)$ lying on a surface S the followings are well-known

i) $x(s)$ is a geodesic curve $\Leftrightarrow k_g = 0$,
ii) $x(s)$ is an asymptotic line $\Leftrightarrow \quad k_n = 0$,
iii) $x(s)$ is a principal line $\Leftrightarrow \tau_g = 0$ [7].

The same results also hold for the surfaces in E_1^3.

4. BERTRAND PARTNER D-CURVES IN THE MINKOWSKI 3-SPACE E_1^3

In this section, by considering the Darboux frame, we define Bertrand D-curves and give the characterizations of these curves in E_1^3.

Definition 4.1. Let S and S_1 be two oriented surfaces in E_1^3 and let consider the arc-length parameter curves $x(s)$ and $x_1(s_1)$ lying fully on S and S_1, respectively. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\left\{\vec{T}, \vec{g}, \vec{n}\right\}$ and $\left\{\vec{T}_1, \vec{g}_1, \vec{n}_1\right\}$, respectively. If there exists a corresponding relationship between curves x and x_1 such that at the corresponding points of curves, the Darboux frame element $\vec{g}$ of x coincides with the Darboux frame element $\vec{g}_1$ of x_1, then x is called a Bertrand D-curve, and x_1 is a Bertrand partner D-curve of x. Then, the pair $\{x, x_1\}$ is said to be a Bertrand D-pair. If there exist such curves lying on the oriented surfaces S and S_1, respectively, we call the surface pair $\{S, S_1\}$ as Bertrand D-pair surfaces.

By considering the non-null Lorentzian casual characters of surfaces and curves, from Definition 4.1, it is easily seen that there are five different types of the Bertrand D-curves in E_1^3. Let the pair $\{x, x_1\}$ be a Bertrand $D-$pair. Then according to the character of surface S we have the followings:

If both the surface S and the curve $x(s)$ lying on S are spacelike, then there are two cases; first one is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are spacelike. In this case we say that the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 1. The second case is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are timelike. Then the pair $\{x, x_1\}$ is called a Bertrand D-pair of the type 2. If both the surface S and the curve $x(s)$ lying on S are timelike, then there are two cases; one is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are timelike. In this case we say that the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 3. The other case is that both the surface S_1 and the curve $x_1(s_1)$ fully lying on S_1 are spacelike then the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 4. If the surface S is timelike and the curve $x(s)$ lying on S is spacelike, then the surface S_1 is timelike and the curve $x_1(s_1)$ fully lying on S_1 is spacelike. In this case we say that the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 5.

Theorem 4.1. *Let S be an oriented surface and $x(s)$ be a curve in E_1^3 with arc length parameter s fully lying on S. If S_1 is another oriented surface and $x_1(s_1)$ is a curve with arc length parameter s_1 fully lying on S_1, then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the normal curvature k_n of $x(s)$ and the geodesic curvature k_{g_1}, the normal curvature k_{n_1} and the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfy the following equations,*

i) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 1, then*

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1-\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2}{(1-\lambda k_{g_1})}\right)\left(-k_{n_1} + k_n\frac{1-\lambda k_{g_1}}{\cosh\theta}\right) - \frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}\right]$$

.

ii) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 2, then*

$$\dot{\tau}_{g_1} = \frac{-1}{\lambda}\left[\left(\frac{(1+\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2}{(1+\lambda k_{g_1})}\right)\left(-k_{n_1} + k_n\frac{1+\lambda k_{g_1}}{\sinh\theta}\right) - \left(\frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1+\lambda k_{g_1}}\right)\right]$$

iii) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 3, then*

$$\dot{\tau}_{g_1} = \frac{-1}{\lambda}\left[\left(\frac{(1+\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2}{(1+\lambda k_{g_1})}\right)\left(-k_{n_1} + k_n\frac{1+\lambda k_{g_1}}{\cosh\theta}\right) - \frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1+\lambda k_{g_1}}\right]$$

iv) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 4, then*

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1-\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2}{(1-\lambda k_{g_1})}\right)\left(-k_{n_1} + k_n\frac{1-\lambda k_{g_1}}{\sinh\theta}\right) - \frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}\right]$$

v) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 5, then*

$$\dot{\tau}_{g_1} = \frac{1}{\lambda}\left[\left(\frac{(1+\lambda k_{g_1})^2 + \lambda^2\tau_{g_1}^2}{(1+\lambda k_{g_1})}\right)\left(k_{n_1} - k_n\frac{1+\lambda k_{g_1}}{\cos\theta}\right) + \frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1+\lambda k_{g_1}}\right]$$

for some nonzero constants λ, where θ is the angle between the tangent vectors $\vec{T}$ and $\vec{T}_1$ at the corresponding points of x and x_1.

Proof. i) Suppose that the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 1. Denote the Darboux frames of $x(s)$ and $x_1(s_1)$ by $\left\{\vec{T}, \vec{g}, \vec{n}\right\}$ and $\left\{\vec{T}_1, \vec{g}_1, \vec{n}_1\right\}$, respectively. Then by the definition we can write

$$x(s_1) = x_1(s_1) + \lambda(s_1)\vec{g}_1(s_1), \tag{4.1}$$

for some functions $\lambda(s_1)$. By taking derivative of (4.1) with respect to s_1 and applying the Darboux formulae (3.1) we have

$$\vec{T}\frac{ds}{ds_1} = (1-\lambda k_{g_1})\vec{T}_1 + \dot{\lambda}\vec{g}_1 + \lambda\tau_{g_1}\vec{n}_1 \tag{4.2}$$

Since the direction of $\vec{g}_1$ coincides with the direction of $\vec{g}$, i.e., the tangent vector $\vec{T}$ of the curve lies on the plane spanned by the vectors $\vec{T}_1$ and $\vec{n}_1$, we get

$$\dot{\lambda}(s_1) = 0.$$

This means that λ is a nonzero constant. Thus, the equality (4.2) can be written as follows

$$\vec{T}\frac{ds}{ds_1} = (1-\lambda k_{g_1})\vec{T}_1 + \lambda\tau_{g_1}\vec{n}_1. \tag{4.3}$$

Furthermore, we have

$$\vec{T} = \cosh\theta\vec{T}_1 + \sinh\theta\vec{n}_1, \tag{4.4}$$

where θ is the angle between the tangent vectors $\vec{T}$ and $\vec{T}_1$ at the corresponding points of x and x_1. By differentiating last equation with respect to s_1, we get

$$\begin{aligned}(k_g\vec{g} + k_n\vec{n})\tfrac{ds}{ds_1} &= (\dot{\theta} + k_{n_1})\sinh\theta\vec{T}_1 \\ &+(k_{g_1}\cosh\theta + \tau_{g_1}\sinh\theta)\vec{g}_1 + (\dot{\theta}+k_{n_1})\cosh\theta\vec{n}_1\end{aligned} \tag{4.5}$$

From this equation and the fact that

$$\vec{n} = \sinh\theta\vec{T}_1 + \cosh\theta\vec{n}_1, \tag{4.6}$$

we get

$$\begin{aligned}(k_n\sinh\theta\vec{T}_1 + k_g\vec{g} + k_n\cosh\theta\vec{n}_1)\tfrac{ds}{ds_1} &= (\dot{\theta} + k_{n_1})\sinh\theta\vec{T}_1 \\ &+(k_{g_1}\cosh\theta + \tau_{g_1}\sinh\theta)\vec{g}_1 \\ &+(\dot{\theta}+k_{n_1})\cosh\theta\vec{n}_1\end{aligned} \tag{4.7}$$

Since the direction of $\vec{g}_1$ coincides with $\vec{g}$ we have

$$\dot{\theta} = -k_{n_1} + k_n\frac{ds}{ds_1}. \tag{4.8}$$

Using the fact that $\vec{T}_1$ is orthogonal to $\vec{g}_1$, from (4.3) and (4.4) we obtain

$$\frac{ds}{ds_1} = \frac{1-\lambda k_{g_1}}{\cosh\theta} = \frac{\lambda\tau_{g_1}}{\sinh\theta}. \tag{4.9}$$

Equality (4.9) gives us

$$\tanh\theta = \frac{\lambda\tau_{g_1}}{1-\lambda k_{g_1}}. \tag{4.10}$$

By taking the derivative of this equation and applying (4.8) we get

$$\dot{\tau}_{g_1} = -\frac{1}{\lambda}\left[\left(\frac{(1-\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2}{(1-\lambda k_{g_1})}\right)\left(-k_{n_1} + k_n\frac{1-\lambda k_{g_1}}{\cosh\theta}\right) - \frac{\lambda^2\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}\right], \tag{4.11}$$

that is desired.

Conversely, assume that equation (4.11) holds for some non-zero constants λ. Then by using (4.9) and (4.10), (4.11) gives us

$$k_n\left(\frac{ds}{ds_1}\right)^3 = \lambda\dot{\tau}_{g_1}(1-\lambda k_{g_1}) + \lambda^2\tau_{g_1}\dot{k}_{g_1} + \left((1-\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2\right)k_{n_1} \tag{4.12}$$

Let define a curve

$$x(s_1) = x_1(s_1) + \lambda(s_1)\vec{g}_1(s_1). \tag{4.13}$$

We will prove that x is a Bertrand D-curve and x_1 is the Bertrand partner D-curve of x. By taking the derivative of (4.13) with respect to s_1 twice, we get

$$\vec{T}\frac{ds}{ds_1} = (1-\lambda k_{g_1})\vec{T}_1 + \lambda\tau_{g_1}\vec{n}_1, \tag{4.14}$$

and

$$(k_g\vec{g} + k_n\vec{n})\left(\frac{ds}{ds_1}\right)^2 + \vec{T}\frac{d^2s}{ds_1^2} = (-\lambda\dot{k}_{g_1} + \lambda\tau_{g_1}k_{n_1})\vec{T}_1 + \left((1-\lambda k_{g_1})k_{g_1} + \lambda\tau_{g_1}^2\right)\vec{g}_1 + \left((1-\lambda k_{g_1})k_{n_1} + \lambda\dot{\tau}_{g_1}\right)\vec{n}_1 \tag{4.15}$$

respectively. Taking the cross product of (4.14) and (4.15) we have

$$\begin{aligned}[k_g\vec{n} + k_n\vec{g}]\left(\frac{ds}{ds_1}\right)^3 &= \left[\lambda\tau_{g_1}k_{g_1}(1-\lambda k_{g_1}) + \lambda^2\tau_{g_1}^3\right]\vec{T}_1 \\ &+ \left[(1-\lambda k_{g_1})^2k_{n_1} + \lambda\dot{\tau}_{g_1}(1-\lambda k_{g_1}) + \lambda^2\tau_{g_1}\dot{k}_{g_1} - \lambda^2\tau_{g_1}^2k_{n_1}\right]\vec{g}_1 \\ &+ \left[k_{g_1}(1-\lambda k_{g_1})^2 + \lambda\tau_{g_1}^2(1-\lambda k_{g_1})\right]\vec{n}_1\end{aligned} \tag{4.16}$$

By substituting (4.12) in (4.16) we get

$$\begin{aligned}[k_g\vec{n} + k_n\vec{g}]\left(\frac{ds}{ds_1}\right)^3 &= \left(\lambda\tau_{g_1}k_{g_1}(1-\lambda k_{g_1}) + \lambda^2\tau_{g_1}^3\right)\vec{T}_1 + k_n\left(\frac{ds}{ds_1}\right)^3\vec{g}_1 \\ &+ \left(k_{g_1}(1-\lambda k_{g_1})^2 + \lambda\tau_{g_1}^2(1-\lambda k_{g_1})\right)\vec{n}_1\end{aligned} \tag{4.17}$$

Taking the cross product of (4.14) with (4.17) we have

$$\begin{aligned}[k_g\vec{g} + k_n\vec{n}]\left(\frac{ds}{ds_1}\right)^4 &= k_n\left(\frac{ds}{ds_1}\right)^3\lambda\tau_{g_1}\vec{T}_1 \\ &+ \left((1-\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2\right)\left(\lambda\tau_{g_1}^2 + k_{g_1}(1-\lambda k_{g_1})\right)\vec{g}_1 \\ &+ k_n\left(\frac{ds}{ds_1}\right)^3(1-\lambda k_{g_1})\vec{n}_1\end{aligned} \tag{4.18}$$

From (4.17) and (4.18) we obtain

(4.19)

$$\begin{aligned}(k_g^2 - k_n^2)\left(\frac{ds}{ds_1}\right)^4\vec{n} &= \left[\lambda k_g k_{g_1}\tau_{g_1}(1-\lambda k_{g_1})\frac{ds}{ds_1} + \lambda^2k_g\tau_{g_1}^3\frac{ds}{ds_1} - \lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{T}_1 \\ &+ k_n\left(\frac{ds}{ds_1}\right)^2\left[k_g\left(\frac{ds}{ds_1}\right)^2 - \lambda\tau_{g_1}^2 - k_{g_1}(1-\lambda k_{g_1})\right]\vec{g}_1 \\ &+ \left[k_gk_{g_1}(1-\lambda k_{g_1})^2\frac{ds}{ds_1} + \lambda\tau_{g_1}^2k_g(1-\lambda k_{g_1})\frac{ds}{ds_1} - k_n^2(1-\lambda k_{g_1})\left(\frac{ds}{ds_1}\right)^3\right]\vec{n}_1\end{aligned}$$

Furthermore, from (4.14) and (4.17) we get

$$(4.20)\qquad \begin{cases} \left(\frac{ds}{ds_1}\right)^2 = (1-\lambda k_{g_1})^2 - \lambda^2\tau_{g_1}^2, \\ k_g\left(\frac{ds}{ds_1}\right)^2 = k_{g_1}(1-\lambda k_{g_1}) + \lambda\tau_{g_1}^2, \end{cases}$$

respectively. Substituting (4.20) in (4.19) we obtain

(4.21)

$$(k_g^2 - k_n^2)\left(\frac{ds}{ds_1}\right)^4 \vec{n} = \left[\lambda k_g k_{g_1}\tau_{g_1}(1-\lambda k_{g_1})\frac{ds}{ds_1} + \lambda^2 k_g \tau_{g_1}^3 \frac{ds}{ds_1} - \lambda\tau_{g_1}k_n^2\left(\frac{ds}{ds_1}\right)^3\right]\vec{T}_1 + \left[k_g k_{g_1}(1-\lambda k_{g_1})^2\frac{ds}{ds_1} + \lambda\tau_{g_1}^2 k_g(1-\lambda k_{g_1})\frac{ds}{ds_1} - k_n^2(1-\lambda k_{g_1})\left(\frac{ds}{ds_1}\right)^3\right]\vec{n}_1$$

Equality (4.14) and (4.21) shows that the vectors $\vec{T}$ and $\vec{n}$ lie on the plane $sp\left\{\vec{T}_1, \vec{n}_1\right\}$. So, at the corresponding points of the curves, the Darboux frame element $\vec{g}$ of x coincides with the Darboux frame element $\vec{g}_1$ of x_1, i.e, the curves x and x_1 are Bertrand D-curves. □

Let now give the characterizations of Bertrand partner D-curves of the type 1 in some special cases. Assume that $x(s)$ be an asymptotic line. Then, from (4.11) we have the following special cases:

i) Consider that $x_1(s_1)$ is a geodesic curve. Then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the following equation holds,

$$\lambda\dot{\tau}_{g_1} = k_{n_1}\left(1 - \lambda^2\tau_{g_1}^2\right)$$

ii) Assume that $x_1(s_1)$ is also an asymptotic line. Then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the geodesic torsion τ_{g_1} of $x_1(s_1)$ satisfies the following equation,

$$\dot{\tau}_{g_1} = \frac{\lambda\tau_{g_1}\dot{k}_{g_1}}{1-\lambda k_{g_1}}.$$

iii) If $x_1(s_1)$ is a principal line then $x_1(s_1)$ is Bertrand partner D-curve of $x(s)$ if and only if the geodesic curvature k_{g_1} and the normal curvature k_{n_1} of $x_1(s_1)$ satisfy the following equality,

$$k_{n_1}(1-\lambda k_{g_1}) = 0$$

The proofs of the statement (ii), (iii), (iv) and (v) of Theorem 4.1 and the particular cases given above can be given by the same way of the proof of statement (i).

Theorem 4.2. *Let the pair $\{x, x_1\}$ be a Bertrand D-pair. Then the relations between geodesic curvature k_g, geodesic torsion τ_g of $x(s)$and the geodesic curvature k_{g_1}, the geodesic torsion τ_{g_1} of $x_1(s_1)$ are given as follows,*

i) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 1, then*

$$k_g - k_{g_1} = \lambda(k_g k_{g_1} + \tau_g\tau_{g_1})$$

ii) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 2, then*

$$k_g + k_{g_1} = -\lambda(k_g k_{g_1} + \tau_g\tau_{g_1})$$

iii) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 3, then*

$$k_g - k_{g_1} = -\lambda(k_g k_{g_1} - \tau_g \tau_{g_1})$$

iv) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 4, then*

$$k_g + k_{g_1} = \lambda(k_g k_{g_1} + \tau_g \tau_{g_1})$$

v) *if the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 4, then*

$$k_g - k_{g_1} = \lambda(\tau_g \tau_{g_1} - k_g k_{g_1}).$$

Proof. i) Suppose that the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 1. Then by definition from (4.13) we can write

$$x_1(s_1) = x(s_1) - \lambda(s_1)\vec{g}(s_1) \tag{4.22}$$

for some constants λ. By differentiating (4.22) with respect to s_1 we have

$$\vec{T}_1 = (1 + \lambda k_g)\frac{ds}{ds_1}\vec{T} - \lambda\tau_g\frac{ds}{ds_1}\vec{n} \tag{4.23}$$

By the definition we have

$$\vec{T}_1 = \cosh\theta\vec{T} - \sinh\theta\vec{n} \tag{4.24}$$

From (4.23) and (4.24) we obtain

$$\cosh\theta = (1 + \lambda k_g)\frac{ds}{ds_1}, \ \ \sinh\theta = \lambda\tau_g\frac{ds}{ds_1} \tag{4.25}$$

Using (4.9) and (4.25) it is easily seen that

$$k_g - k_{g_1} = \lambda(k_g k_{g_1} + \tau_g \tau_{g_1}).$$

The proofs of the statements (ii), (iii), (iv) and (v) of Theorem 4.2 can be given by the same way of the proof of statement (i). □

From Theorem 4.2, we obtain the following special cases:

Let the pair $\{x, x_1\}$ be a Bertrand D-pair of the type 1. Then,

i) if one of the curves x and x_1 is a principal line, then the relation between the geodesic curvatures k_g and k_{g_1} is

$$k_g - k_{g_1} = \lambda k_g k_{g_1}$$

ii) if x_1 is a geodesic curve, then the geodesic curvature of the curve x is given by

$$k_g = \lambda\tau_g\tau_{g_1}$$

iii) if x is a geodesic curve, then the geodesic curvature of the curve x_1 is given by

$$k_{g_1} = -\lambda\tau_g\tau_{g_1}$$

Theorem 4.3. *Let $\{x, x_1\}$ be Bertrand D-pair of the type 1. Then the following relations hold:*

i) $k_{n_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}$

ii) $\tau_g \frac{ds}{ds_1} = k_{g_1} \sinh\theta - \tau_{g_1} \cosh\theta$

iii) $k_g \frac{ds}{ds_1} = k_{g_1} \cosh\theta + \tau_{g_1} \sinh\theta$

iv) $\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta) \frac{ds}{ds_1}$

Proof. i) Since the pair $\{x, x_1\}$ is a Bertrand D-pair of the type 1, we have $\left\langle \vec{T}, \vec{T}_1 \right\rangle = \cosh\theta$. By differentiating this equation with respect to s_1 we have

$$\left\langle (k_g \vec{g} + k_n \vec{n}) \frac{ds}{ds_1}, \vec{T}_1 \right\rangle + \left\langle \vec{T}, k_{g_1} \vec{g}_1 + k_{n_1} \vec{n}_1 \right\rangle = \sinh\theta \frac{d\theta}{ds_1}.$$

Using the fact that the direction of $\vec{g_1}$ coincides with the direction of $\vec{g}$ and

$$\vec{T}_1 = \cosh\theta \vec{T} - \sinh\theta \vec{n}, \quad \vec{n}_1 = -\sinh\theta \vec{T} + \cosh\theta \vec{n} \tag{4.26}$$

we easily get that

$$k_{n_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}.$$

ii) By definition we have $\langle \vec{n}, \vec{g}_1 \rangle = 0$. Differentiating this equation with respect to s_1 we have

$$\left\langle (k_n \vec{T} + \tau_g \vec{g}) \frac{ds}{ds_1}, \vec{g}_1 \right\rangle + \left\langle \vec{n}, -k_{g_1} \vec{T}_1 + \tau_{g_1} \vec{n}_1 \right\rangle = 0.$$

By (4.26) we obtain

$$\tau_g \frac{ds}{ds_1} = k_{g_1} \sinh\theta - \tau_{g_1} \cosh\theta$$

iii) By differentiating the equation $\left\langle \vec{T}, \vec{g}_1 \right\rangle = 0$ with respect to s_1 we get

$$\left\langle (k_g \vec{g} + k_n \vec{n}) \frac{ds}{ds_1}, \vec{g}_1 \right\rangle + \left\langle \vec{T}, (-k_{g_1} \vec{T}_1 + \tau_{g_1} \vec{n}_1 \right\rangle = 0.$$

From (4.26) it follows that

$$k_g \frac{ds}{ds_1} = k_{g_1} \cosh\theta + \tau_{g_1} \sinh\theta.$$

iv) By differentiating the equation $\langle \vec{n}_1, \vec{g} \rangle = 0$ with respect to s_1 we obtain

$$\left\langle (k_{n_1} \vec{T}_1 + \tau_{g_1} \vec{g}_1, \vec{g} \right\rangle + \left\langle \vec{n}_1, (-k_g \vec{T} + \tau_g \vec{n}) \frac{ds}{ds_1} \right\rangle = 0,$$

and using the fact that direction of $\vec{g_1}$ coincides with the direction of $\vec{g}$ and

$$\vec{T} = \cosh\theta \vec{T}_1 + \sinh\theta \vec{n}_1, \quad \vec{n} = \sinh\theta \vec{T}_1 + \cosh\theta \vec{n}_1$$

we get

$$\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta) \frac{ds}{ds_1}.$$

□

The statements of Theorem 4.3 for the pairs $\{x, x_1\}$ of the type 2, 3, 4, and 5 can be given as follows and the proofs can be easily done by the same way of the case the pair $\{x, x_1\}$ is of the type 1.

For the pair $\{x, x_1\}$ of the type 2

i) $k_{n_1} = \frac{d\theta}{ds_1} + k_n \frac{ds}{ds_1}$ **ii)** $\tau_g \frac{ds}{ds_1} = k_{g_1} \cosh\theta - \tau_{g_1} \sinh\theta$

iii) $k_g \frac{ds}{ds_1} = k_{g_1} \sinh\theta + \tau_{g_1} \cosh\theta$ **iv)** $\tau_{g_1} = (k_g \cosh\theta - \tau_g \sinh\theta)\frac{ds}{ds_1}$

For the pair $\{x, x_1\}$ of the type 3

i) $k_{n_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ **ii)** $\tau_g \frac{ds}{ds_1} = k_{g_1} \sinh\theta - \tau_{g_1} \cosh\theta$

iii) $k_g \frac{ds}{ds_1} = k_{g_1} \cosh\theta + \tau_{g_1} \sinh\theta$ **iv)** $\tau_{g_1} = (-k_g \sinh\theta + \tau_g \cosh\theta)\frac{ds}{ds_1}$

For the pair $\{x, x_1\}$ of the type 4

i) $k_{n_1} = k_n \frac{ds}{ds_1} - \frac{d\theta}{ds_1}$ **ii)** $\tau_g \frac{ds}{ds_1} = k_{g_1} \cosh\theta + \tau_{g_1} \sinh\theta$

iii) $k_g \frac{ds}{ds_1} = k_{g_1} \sinh\theta + \tau_{g_1} \cosh\theta$ **iv)** $\tau_{g_1} = (k_g \cosh\theta - \tau_g \sinh\theta)\frac{ds}{ds_1}$

For the pair $\{x, x_1\}$ of the type 5

i) $k_{n_1} = k_n \frac{ds}{ds_1} + \frac{d\theta}{ds_1}$ **ii)** $\tau_g \frac{ds}{ds_1} = -k_{g_1} \sin\theta + \tau_{g_1} \cos\theta$

iii) $k_g \frac{ds}{ds_1} = k_{g_1} \cos\theta + \tau_{g_1} \sin\theta$ **iv)** $\tau_{g_1} = (k_g \sin\theta + \tau_g \cos\theta)\frac{ds}{ds_1}$

Let now $\{x, x_1\}$ be a Bertrand D-pair of the type 1. From the first equation of (3.3) and by using the fact that $\vec{n}_1 = -\sinh\theta\vec{T} + \cosh\theta\vec{n}$ we have

$$k_{g_1} = [(1+\lambda k_g)\cosh\theta - \lambda\tau_g \sinh\theta]\left[k_g + \lambda k_g^2 - \lambda\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3. \tag{4.27}$$

Then we can give the following corollary.

Corollary 4.1. *Let $\{x, x_1\}$ be a Bertrand D-pair of the type 1. Then the relations between the geodesic curvature k_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g and the geodesic torsion τ_g of $x(s)$ are given as follows.*

i) *If x is a geodesic curve, then the geodesic curvature k_{g_1} of $x_1(s_1)$is*

$$k_{g_1} = -\lambda\tau_g^2\left(\frac{ds}{ds_1}\right)^3(\cosh\theta - \lambda\tau_g \sinh\theta). \tag{4.28}$$

ii) *If x is a principal line, then the relation between the geodesic curvatures k_{g_1} and k_g is given by*

$$k_{g_1} = (k_g + 2\lambda k_g + \lambda^2 k_g^3)\left(\frac{ds}{ds_1}\right)^3 \cosh\theta. \tag{4.29}$$

If the pair $\{x, x_1\}$ is of the type 2, 3, 4 or 5 then the geodesic curvature of the curve $x_1(s_1)$ is given as follows

If the pair $\{x, x_1\}$is of the type 2

$$k_{g_1} = [(1+\lambda k_g)\sinh\theta - \lambda\tau_g \cosh\theta]\left[-k_g - \lambda k_g^2 + \lambda\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3$$

If the pair $\{x, x_1\}$ is of the type 3

$$k_{g_1} = [(1-\lambda k_g)\cosh\theta + \lambda\tau_g \sinh\theta]\left[k_g - \lambda k_g^2 + \lambda\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3$$

If the pair $\{x, x_1\}$is of the type 4

$$k_{g_1} = [(1-\lambda k_g)\sinh\theta + \lambda\tau_g\cosh\theta]\left[-k_g + \lambda k_g^2 - \lambda\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3$$

If the pair $\{x, x_1\}$ is of the type 5

$$k_{g_1} = [(1-\lambda k_g)\cos\theta + \lambda\tau_g\sin\theta]\left[k_g - \lambda k_g^2 - \lambda\tau_g^2\right]\left(\frac{ds}{ds_1}\right)^3$$

and the statements in Corollary 4.1 are obtained by the same way.

Similarly, From the second equation of (3.3) and by using the fact that $\vec{g}$ coincides with $\vec{g_1}$, i.e., $\vec{n}_1 = -\sinh\theta\vec{T} + \cosh\theta\vec{n}$, the geodesic torsion $\tau_{g_1}$of x_1 is given by

$$\begin{aligned}\tau_{g_1} = &\left[(\tau_g + \lambda k_g\tau_g)\cosh^2\theta + \left(-k_g - \lambda k_g^2 + \lambda\tau_g^2\right)\sinh\theta\cosh\theta\right.\\ &\left.+\lambda\tau_g k_g\sinh^2\theta\right]\left(\tfrac{ds}{ds_1}\right)^2\end{aligned} \tag{4.30}$$

From (4.30) we can give the following corollary.

Corollary 4.2. *Let $\{x, x_1\}$ be a Bertrand D-pair of the type 1. Then the relations between the geodesic torsion τ_{g_1} of $x_1(s_1)$ and the geodesic curvature k_g and the geodesic torsion τ_g of $x(s)$ are given as follows.*

i) *If x is a geodesic curve then the geodesic torsion of x_1 is*

$$\tau_{g_1} = \left(\tau_g\cosh^2\theta + \lambda\tau_g^2\sinh\theta\cosh\theta\right)\left(\frac{ds}{ds_1}\right)^2. \tag{4.31}$$

ii) *If x is a principal line then the relation between τ_{g_1} and k_g is*

$$\tau_{g_1} = -(k_g + \lambda k_g^2)\sinh\theta\cosh\theta\left(\frac{ds}{ds_1}\right)^2. \tag{4.32}$$

Furthermore, by using (4.9) and (4.10), from (4.31) and (4.32) we have the following corollary.

Corollary 4.3. ***i)*** *Let $\{x, x_1\}$ be a Bertrand D-pair of the type 1 and let x be a geodesic line. Then the geodesic torsion τ_{g_1} of $x_1(s_1)$ is given by*

$$\tau_{g_1} = \tau_g(1-\lambda k_{g_1})\left[(1-\lambda k_{g_1}) + \lambda^2\tau_g\tau_{g_1}\right]. \tag{4.33}$$

ii) *Let $\{x, x_1\}$ be a Bertrand D-pair of the type 1 and let x be a principal line. Then the relation between the geodesic curvatures k_g and $k_{g_1}$is given as follows*

$$k_g(1+\lambda k_g)(1-\lambda k_{g_1}) = -\frac{1}{\lambda} = constant. \tag{4.34}$$

When the pair $\{x, x_1\}$ is of the type 2, 3, 4 or 5, then the relations which give the geodesic torsion $\tau_{g_1}$of $x_1(s_1)$ are given as follows.

For the pair $\{x, x_1\}$ of the type 2

$$\tau_{g_1} = \left[\tau_g\sinh^2\theta - \lambda\tau_g k_g + \left(\lambda\tau_g^2 - k_g - \lambda k_g^2)\right)\sinh\theta\cosh\theta\right]\left(\frac{ds}{ds_1}\right)^2$$

For the pair $\{x, x_1\}$ of the type 3

$$\tau_{g_1}=\left[(\tau_g-\lambda k_g\tau_g)\cosh^2\theta+\left(-k_g+\lambda k_g^2+\lambda\tau_g^2\right)\sinh\theta\cosh\theta-\lambda\tau_g k_g\sin^2\theta\right]\left(\frac{ds}{ds_1}\right)^2$$

For the pair $\{x,x_1\}$ of the type 4

$$\tau_{g_1}=\left[(\tau_g+\lambda k_g\tau_g)\sinh^2\theta-\left(\lambda\tau_g^2+k_g+\lambda k_g^2\right)\sinh\theta\cosh\theta+\lambda\tau_g k_g\cosh^2\theta\right]\left(\frac{ds}{ds_1}\right)^2$$

For the pair $\{x,x_1\}$ of the type 5

$$\tau_{g_1}=\left[(\lambda k_g\tau_g-\tau_g)\cos^2\theta+\left(-k_g+\lambda k_g^2-\lambda\tau_g^2\right)\sin\theta\cos\theta-\lambda\tau_g k_g\sin^2\theta\right]\left(\frac{ds}{ds_1}\right)^2$$

5. CONCLUSIONS

In this paper, the definition and characterizations of Bertrand partner D-curves in E_1^3 are given which is a new study of associated curves lying on surfaces. The relations between the geodesic curvatures, the normal curvatures and the geodesic torsions of these curves are given. Moreover, for a special case such as one of the curves is a geodesic line, principal line or asymptotic line, some special relationships are obtained in E_1^3.

References

[1] Burke, J.F., Bertrand curves associated with a pair of curves. *Mathematics Magazine.* 34 (1960), no.1, 60-62.

[2] Görgülü, E., Özdamar, E., A generalizations of the Bertrand curves as general inclined curves in E^n. *Communications de la Fac. Sci. Uni. Ankara, Series A1.* 35 (1986), 53-60.

[3] Ersoy, S., İnalcık, A., On the generalized timelike Bertrand curves in 5-dimensional Lorentzian space. *Differential Geometry-Dynamical Systems.* 13 (2011), 78-88.

[4] Izumiya, S., Takeuchi, N., Special Curves and Ruled surfaces. *Beitrage zur Algebra und Geometrie Contributions to Algebra and Geometry.* 44 (2003), no. 1, 203-212.

[5] Izumiya, S., Takeuchi, N., Generic properties of helices and Bertrand curves. *Journal of Geometry.* 74 (2002), 97-109.

[6] Kasap, E., Kuruoğlu, N., The Bertrand offsets of ruled surfaces in R_1^3. *Acta Mathematica Vietnamica.* 31 (2006), no. 1, 39-48.

[7] O'Neill, B., Elemantery Differential Geometry. Academic Press Inc. New York, 1966.

[8] O'Neill, B., Semi-Riemannian Geometry with Applications to Relativity. Academic Press, London, 1983.

[9] Öğrenmiş, A.O., Öztekin, H., Ergüt, M., Bertrand curves in Galilean space and their characterizations. *Kragujevac J. Math.* 32 (2009), 139-147.

[10] Ratcliffe, J.G., Foundations of Hyperbolic Manifolds. Springer, 2006.

[11] Ravani, B., Ku, T.S., Bertrand Offsets of ruled and developable surfaces. *Comp. Aided Geom. Design.* 23 (1991), no. 2, 145-152.

[12] Struik, D.J., Lectures on Classical Differential Geometry. 2^{nd} ed. Addison Wesley, Dover, 1988.

[13] Uğurlu, H.H., Çalışkan, A., Darboux Ani Dönme Vektörleri ile Spacelike ve Timelike Yüzeyler Geometrisi. Celal Bayar Üniversitesi Yayınları, Yayın No: 0006, 2012.

[14] Walrave, J., Curves and Surfaces in Minkowski space. PhD. thesis, K.U. Leuven, Fac. of Science, Leuven, 1995.

[15] Whittemore, J.K., Bertrand curves and helices. *Duke Math. J.* 6 (1940), no. 1, 235-245.

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