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INTEGRABILITY OF DISTRIBUTIONS IN GCR-LIGHTLIKE SUBMANIFOLDS

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Abstract. We give necessary and sufficient conditions for the integrability of various distributions of GCR-lightlike submanifold of an indefinite Kenmotsu manifold. We also find the conditions for each leaf of holomorphic distribution and radical distribution to be totally geodesic in submanifold.

1. Introduction

Duggal and Şahin [4] introduced the theory of contact CR and contact SCR-lightlike submanifolds of indefinite Sasakian manifolds. To find an umbrella of invariant, screen real, contact CR-lightlike submanifolds and real hypersurfaces, Duggal and Şahin [5] introduced a new class of submanifolds called generalized Cauchy-Riemann (GCR)-lightlike submanifolds of indefinite Sasakian manifolds. Since contact geometry has vital role in the theory of differential equations, optics and phase spaces of a dynamical system, therefore contact geometry with definite and indefinite metric becomes the topic of main discussion.

In [8] Şahin and Güneş studied the necessary and sufficient conditions on integrability of distibutions on CR-lightlike submanifolds in an indefinite Kaehler manifolds. Sangeet et al. [9] and Varun et al. [10] established the conditions for the integrability of various distributions of GCR-lightlike submanifolds of indefinite Kaehler manifolds and indefinite cosymplectic manifolds, respectively. In this paper, we give necessary and sufficient conditions for the integrability of various distributions of GCR-lightlike submanifold of an indefinite Kenmotsu manifold. We also find the conditions for each leaf of holomorphic distribution and radical distribution to be totally geodesic in submanifold.

2. Lightlike Submanifolds

We recall notations and fundamental equations for lightlike submanifolds, which are due to the book [3] by Duggal and Bejancu.

Let $(\bar{M}, \bar{g})$ be a real $(m+n)$-dimensional semi-Riemannian manifold of constant

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index q such that $m, n \geq 1$, $1 \leq q \leq m+n-1$ and (M, g) be an m-dimensional submanifold of $\bar{M}$ and g the induced metric of $\bar{g}$ on M. If $\bar{g}$ is degenerate on the tangent bundle TM of M then M is called a lightlike submanifold of $\bar{M}$. For a degenerate metric g on M, $TM^{\perp}$ is a degenerate n-dimensional subspace of $T_x\bar{M}$. Thus, both T_xM and $T_xM^{\perp}$ are degenerate orthogonal subspaces but no longer complementary. In this case, there exists a subspace $RadT_xM = T_xM \cap T_xM^{\perp}$ which is known as radical (null) subspace. If the mapping $RadTM : x \in M \longrightarrow RadT_xM$ defines a smooth distribution on M of rank $r > 0$ then the submanifold M of $\bar{M}$ is called an r-lightlike submanifold and $RadTM$ is called the radical distribution on M.

Let $S(TM)$ be a screen distribution which is a semi-Riemannian complementary distribution of $Rad(TM)$ in TM, that is,

$$TM = RadTM \perp S(TM), \tag{2.1}$$

and $S(TM^{\perp})$ is a complementary vector subbundle to $RadTM$ in $TM^{\perp}$. Let $tr(TM)$ and $ltr(TM)$ be complementary (but not orthogonal) vector bundles to TM in $T\bar{M} \mid_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$ respectively. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}). \tag{2.2}$$

$$T\bar{M} \mid_M = TM \oplus tr(TM) = (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}). \tag{2.3}$$

Theorem 2.1. *([3]). Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ of $RadTM$ in $S(TM^{\perp})^{\perp}$ and a basis of $\Gamma(ltr(TM) \mid_{\mathrm{u}})$ consisting of smooth section $\{N_i\}$ of $S(TM^{\perp})^{\perp} \mid_{\mathrm{u}}$, where u is a coordinate neighborhood of M, such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad \text{for any} \quad i, j \in \{1, 2, .., r\},$$

where $\{\xi_1, ..., \xi_r\}$ is a lightlike basis of $\Gamma(Rad(TM))$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then according to the decomposition (2.3), the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \bar{\nabla}_X U = -A_U X + \nabla^{\perp}_X U, \tag{2.4}$$

for any $X, Y \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$, where $\{\nabla_X Y, A_U X\}$ and $\{h(X, Y), \nabla^{\perp}_X U\}$ belongs to $\Gamma(TM)$ and $\Gamma(tr(TM))$, respectively. Here ∇ is a torsion-free linear connection on M, h is a symmetric bilinear form on $\Gamma(TM)$ which is called second fundamental form, A_U is linear a operator on M, known as shape operator.

According to (2.2), considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and $S(TM^{\perp})$, respectively then (2.4)gives

$$\bar{\nabla}_X Y = \nabla_X Y + h^l(X, Y) + h^s(X, Y), \tag{2.5}$$

$$\bar{\nabla}_X U = -A_U X + D^l_X U + D^s_X U, \tag{2.6}$$

where we put $h^l(X, Y) = L(h(X, Y)), h^s(X, Y) = S(h(X, Y)), D^l_X U = L(\nabla^{\perp}_X U)$, $D^s_X U = S(\nabla^{\perp}_X U)$.

As h^l and h^s are $\Gamma(ltr(TM))$-valued and $\Gamma(S(TM^{\perp}))$-valued respectively, therefore they are called as the lightlike second fundamental form and the screen second fundamental form on M. In particular

$$\bar{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X, N), \tag{2.7}$$

$$\bar{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X, W), \tag{2.8}$$

where $X \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^\perp))$. By using (2.2)-(2.3) and (2.5)-(2.8), we obtain

$$\bar{g}(h^s(X, Y), W) + \bar{g}(Y, D^l(X, W)) = g(A_W X, Y), \tag{2.9}$$

$$\bar{g}(h^l(X, Y), \xi) + \bar{g}(Y, h^l(X, \xi)) + g(Y, \nabla_X \xi) = 0, \tag{2.10}$$

for any $\xi \in \Gamma(RadTM)$ and $W \in \Gamma(S(TM^\perp))$.

Let P be the projection morphism of TM on $S(TM)$. Then using (2.1), we can induce some new geometric objects on the screen distribution $S(TM)$ on M as

$$\nabla_X PY = \nabla^*_X PY + h^*(X, Y), \tag{2.11}$$

$$\nabla_X \xi = -A^*_\xi X + \nabla^{*t}_X \xi, \tag{2.12}$$

for any $X, Y \in \Gamma(TM)$ and $\xi \in \Gamma(RadTM)$, where $\{\nabla^*_X PY, A^*_\xi X\}$ and $\{h^*(X, Y), \nabla^{*t}_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(RadTM)$, respectively. ∇^* and ∇^{*t} are linear connections on complementary distributions $S(TM)$ and $RadTM$, respectively. h^* and A^* are $\Gamma(RadTM)$-valued and $\Gamma(S(TM))$-valued bilinear forms and called as the second fundamental forms of distributions $S(TM)$ and $RadTM$, respectively.
By using above equation we obtain

$$\bar{g}(h^l(X, \bar{P}Y), \xi) = g(A^*_\xi X, \bar{P}Y), \tag{2.13}$$

$$\bar{g}(h^*(X, \bar{P}Y), N) = g(A_N X, \bar{P}Y), \tag{2.14}$$

Next, an odd-dimensional semi-Riemannian manifold $\bar{M}$ is said to be an indefinite almost contact metric manifold if there exist structure tensors $(\phi, V, \eta, \bar{g})$, where ϕ is a $(1, 1)$ tensor field, V is a vector field called structure vector field, η is a 1-form and $\bar{g}$ is the semi-Riemannian metric on $\bar{M}$ satisfying (see [7])

$$\bar{g}(\phi X, \phi Y) = \bar{g}(X, Y) - \eta(X)\eta(Y), \quad \bar{g}(X, V) = \eta(X), \tag{2.15}$$

$$\phi^2 X = -X + \eta(X)V, \quad \eta \circ \phi = 0, \quad \phi V = 0, \quad \eta(V) = 1, \tag{2.16}$$

for any $X, Y \in \Gamma(TM)$.

An indefinite almost contact metric manifold $\bar{M}$ is called an indefinite Kenmotsu manifold if (see [1]),

$$(\bar{\nabla}_X \phi)Y = \bar{g}(\phi X, Y)V + \eta(Y)\phi X. \tag{2.17}$$

$$\bar{\nabla}_X V = X - \eta(X)V. \tag{2.18}$$

3. Generalized Cauchy-Riemann Lightlike Submanifold

Calin[2], proved that if the characteristic vector field V is tangent to $(M, g, S(TM))$ then it belongs to $S(TM)$. We assume the characteristic vector field V is tangent to M throughout this paper.

Definition 3.1. ([6]) Let $(M, g, S(TM), S(TM^\perp))$ be a real lightlike submanifold of an indefinite Kenmotsu manifold $(\bar{M}, \bar{g})$ then M is called a generalized Cauchy-Riemann (GCR)-lightlike submanifold if the following conditions are satisfied

(A) There exist two subbundles D_1 and D_2 of $Rad(TM)$ such that

$$Rad(TM) = D_1 \oplus D_2, \quad \phi(D_1) = D_1, \quad \phi(D_2) \subset S(TM).$$

(B) There exist two subbundles D_0 and $\bar{D}$ of $S(TM)$ such that

$$S(TM) = \{\phi D_2 \oplus \bar{D}\} \perp D_0 \perp V, \quad \phi(\bar{D}) = L \perp S.$$

where D_0 is invariant non degenerate distribution on M, $\{V\}$ is one dimensional distribution spanned by V and L, S are vector subbundles of $ltr(TM)$ and $S(TM)^{\perp}$, respectively.

Then tangent bundle TM of M is decomposed as

$$TM = \{D \oplus \bar{D} \oplus \{V\}\}, \quad \text{where} \quad D = Rad(TM) \oplus D_0 \oplus \phi(D_2).$$

Let Q, P_1, P_2 be the projection morphism on D, ϕL, ϕS respectively, therefore

$$X = QX + V + P_1X + P_2X, \tag{3.1}$$

for $X \in \Gamma(TM)$. Applying ϕ to (3.1), we obtain

$$\phi X = fX + \omega P_1 X + \omega P_2 X, \tag{3.2}$$

where $fX \in \Gamma(D)$, $\omega P_1 X \in \Gamma(L)$ and $\omega P_2 X \in \Gamma(S)$, or, we can write (3.2), as

$$\phi X = fX + \omega X, \tag{3.3}$$

where fX and ωX are the tangential and transversal components of ϕX, respectively.
Similarly,

$$\phi U = BU + CU, \quad U \in \Gamma(tr(TM)),$$

where BU and CU are the sections of TM and $tr(TM)$, respectively. By using Kenmotsu property of $\bar{\nabla}$ with (2.5) and (2.6), we have the following lemmas.

Lemma 3.1. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then we have*

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X,Y) + g(\phi X, Y)V - \eta(Y)fX,$$

and

$$(\nabla^t_X \omega)Y = Ch(X,Y) - h(X, fY) - \eta(X)\omega X,$$

where $X, Y \in \Gamma(TM)$ and

$$(\nabla_X f)Y = \nabla_X fY - f\nabla_X Y, \quad (\nabla^t_X \omega)Y = \nabla^t_X \omega Y - \omega \nabla_X Y.$$

Lemma 3.2. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$ then we have*

$$(\nabla_X B)U = A_{CU}X - fA_U X + g(\phi X, U)V, \quad \text{and} \quad (\nabla^t_X C)U = -\omega A_U X - h(X, BU),$$

where $X \in \Gamma(TM)$ and $U \in \Gamma(tr(TM))$ and

$$(\nabla_X B)U = \nabla_X BU - B\nabla^t_X U, \quad (\nabla^t_X C)U = \nabla^t_X CU - C\nabla^t_X U.$$

4. Integrability of Distributions

Let $\bar{M}$ be a real $m+n$-dimensional smooth manifold then a distribution of rank t on $\bar{M}$ is a mapping D defined on $\bar{M}$, which assign to each point x of $\bar{M}$ a t-dimensional linear subspace D_x of $T_x(M)$. A vector field X on $\bar{M}$ belongs to D if $X(x) \in D_x$ for each x of $\bar{M}$. The distribution D is said to be involutive if $[X,Y] \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$. Then from page no. 34 of [3], a distribution D is integrable if and only if it is involutive.

Theorem 4.1. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then $\bar{D}$ is integrable if and only if $\nabla_X \phi Y = \nabla_Y \phi X$ for any $X, Y \in \Gamma(\bar{D})$.*

Proof. For any $X, Y \in \Gamma(\bar{D})$ we have

$$h(X, \phi Y) = \bar{\nabla}_X \phi Y - \nabla_X \phi Y.$$

Replacing X by Y and then subtracting the resulting equation from the above equation and using (2.17) and (3.3), we get

$$\begin{aligned} h(X, \phi Y) - h(Y, \phi X) &= \bar{\nabla}_X \phi Y - \bar{\nabla}_Y \phi X - \nabla_X \phi Y + \nabla_Y \phi X \\ &= \phi[X,Y] - \nabla_X \phi Y + \nabla_Y \phi X \\ &= f[X,Y] + \omega[X,Y] - \nabla_X \phi Y + \nabla_Y \phi X. \end{aligned}$$

Taking tangential parts of this equation, we have $f[X,Y] = \nabla_X \phi Y - \nabla_Y \phi X$. Hence the result follows. □

Theorem 4.2. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the distribution D_0 is integrable if and only if*

(i) $\bar{g}(h^*(X,Y), N) = \bar{g}(h^*(Y,X), N)$
(ii) $\bar{g}(h^*(X,\phi Y), N_1) = \bar{g}(h^*(Y,\phi X), N_1)$
(iii) $h^s(X, \phi Y) = h^s(Y, \phi X)$
(iv) $g(\nabla^*_X Y, \phi\xi) = g(\nabla^*_Y X, \phi\xi)$,

for any $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), N_1 \in \Gamma(L), W \in \Gamma(S)$, and $\xi \in \Gamma(D_2)$.

Proof. Using the definition of GCR-lightlike submanifold the distribution D_0 is integrable if and only if

$$\bar{g}([X,Y], \phi W) = \bar{g}([X,Y], \phi\xi) = \bar{g}([X,Y], V) = \bar{g}([X,Y], \phi N_1) = \bar{g}([X,Y], N) = 0,$$

for any $X, Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), N_1 \in \Gamma(L), W \in \Gamma(S)$, and $\xi \in \Gamma(D_2)$. Using (2.5), (2.11), (2.17) and (2.18), we have

$$\bar{g}([X,Y], N) = \bar{g}(h^*(X,Y), N) - \bar{g}(h^*(Y,X), N), \tag{4.1}$$

$$\begin{aligned} \bar{g}([X,Y], V) &= -\bar{g}(Y, \bar{\nabla}_X V) + \bar{g}(X, \bar{\nabla}_Y V) = -\bar{g}(Y,X) + \bar{g}(X,Y) \\ &= 0. \end{aligned} \tag{4.2}$$

$$\begin{aligned} \bar{g}([X,Y], \phi N_1) &= \bar{g}(\bar{\nabla}_X Y, \phi N_1) - \bar{g}(\bar{\nabla}_Y X, \phi N_1) \\ &= -\bar{g}(\nabla_X \phi Y, N_1) + \bar{g}(\nabla_Y \phi X, N_1) \\ &= -\bar{g}(h^*(X, \phi Y), N_1) + \bar{g}(h^*(Y, \phi X), N_1). \end{aligned} \tag{4.3}$$

$$\begin{aligned} \bar{g}([X,Y], \phi W) &= \bar{g}(\bar{\nabla}_X Y, \phi W) - \bar{g}(\bar{\nabla}_Y X, \phi W) \\ &= -\bar{g}(\bar{\nabla}_X \phi Y, W) + \bar{g}(\bar{\nabla}_Y \phi X, W) \\ &= -\bar{g}(h^s(X, \phi Y), W) + \bar{g}(h^s(Y, \phi X), W), \end{aligned} \tag{4.4}$$

and

$$\bar{g}([X,Y],\phi\xi) = g(\nabla_X Y,\phi\xi) - g(\nabla_Y X,\phi\xi)$$
$$= g(\nabla^*_X Y,\phi\xi) - g(\nabla^*_Y X,\phi\xi). \quad (4.5)$$

Thus from (4.1)-(4.5) the proof is complete. □

Corollary 4.1. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then the distribution D_0 is integrable if and only if*

(i) $g(Y,A_N X) = g(X,A_N Y)$
(ii) $g(A_{N_1}X,\phi Y) = g(A_{N_1}Y,\phi X)$
(iii) $h^s(X,\phi Y) = h^s(Y,\phi X)$
(iv) $g(h^l(X,\phi Y),\xi) = g(h^l(Y,\phi X),\xi)$,

for any $X,Y \in \Gamma(D_0), N \in \Gamma(ltr(TM)), N_1 \in \Gamma(L), W \in \Gamma(S)$, and $\xi \in \Gamma(D_2)$.

Theorem 4.3. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then $RadTM$ is integrable if and only if*

(i) $\bar{g}(h^l(\xi',Z),\xi'') = \bar{g}(h^l(\xi'',Z),\xi')$
(ii) $\bar{g}(h^l(\xi',\phi\xi),\xi'') = \bar{g}(h^l(\xi'',\phi\xi),\xi')$
(iii) $h^s(\xi',\phi\xi'') = h^s(\xi'',\phi\xi')$
(iv) $g(h^*(\xi',\phi\xi''),N_1) = g(h^*(\xi'',\phi\xi'),N_1)$.

for any $\xi \in \Gamma(D_2)$, $\xi',\xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L)$, $W \in \Gamma(S)$, $Z \in \Gamma(D_0)$.

Proof. Using the definition of GCR-lightlike submanifold of an indefinite Kenmotsu manifold, $Rad(TM)$ is integrable if and only if

$$\bar{g}([\xi',\xi''],V) = \bar{g}([\xi',\xi''],Z) = \bar{g}([\xi',\xi''],\phi\xi) = \bar{g}([\xi',\xi''],\phi W) = \bar{g}([\xi',\xi''],\phi N_1) = 0,$$

for any $\xi \in \Gamma(D_2)$, $\xi',\xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L), W \in \Gamma(S)$, and $Z \in \Gamma(D_0)$. Using (2.5), (2.12), (2.13), (2.17) and (2.18), we obtain

$$\bar{g}([\xi',\xi''],V) = \bar{g}(\bar{\nabla}_{\xi'}\xi'' - \bar{\nabla}_{\xi''}\xi',V) = -\bar{g}(\xi'',\bar{\nabla}_{\xi'}V) + \bar{g}(\xi',\bar{\nabla}_{\xi''}V)$$
$$= -g(\xi',\xi'') + g(\xi'',\xi') = 0, \quad (4.6)$$

$$\bar{g}([\xi',\xi''],Z) = -g(A^*_{\xi''}\xi',Z) + g(A^*_{\xi'}\xi'',Z)$$
$$= -\bar{g}(h^l(\xi',Z),\xi'') + \bar{g}(h^l(\xi'',Z),\xi'). \quad (4.7)$$

$$\bar{g}([\xi',\xi''],\phi\xi) = \bar{g}(\bar{\nabla}_{\xi'}\xi'',\phi\xi) - \bar{g}(\bar{\nabla}_{\xi''}\xi',\phi\xi)$$
$$= -\bar{g}(\xi'',\bar{\nabla}_{\xi'}\phi\xi) + \bar{g}(\xi',\bar{\nabla}_{\xi''}\phi\xi)$$
$$= -\bar{g}(h^l(\xi',\phi\xi),\xi'') + \bar{g}(h^l(\xi'',\phi\xi),\xi'). \quad (4.8)$$

$$\bar{g}([\xi',\xi''],\phi W) = \bar{g}(\bar{\nabla}_{\xi'}\xi'',\phi W) - \bar{g}(\bar{\nabla}_{\xi''}\xi',\phi W)$$
$$= -\bar{g}(\phi\bar{\nabla}_{\xi'}\xi'',W) + \bar{g}(\phi\bar{\nabla}_{\xi''}\xi',W)$$
$$= -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'',W) + \bar{g}(\bar{\nabla}_{\xi''}\phi\xi',W)$$
$$= -\bar{g}(h^s(\xi',\phi\xi''),W) + \bar{g}(h^s(\xi'',\phi\xi'),W), \quad (4.9)$$

and

$$\bar{g}([\xi',\xi''],\phi N_1) = -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - (\bar{\nabla}_{\xi'}\phi)\xi'',N_1) + \bar{g}(\bar{\nabla}_{\xi''}\phi\xi' - (\bar{\nabla}_{\xi''}\phi)\xi',N_1)$$
$$= -\bar{g}(\nabla_{\xi'}\phi\xi'',N_1) + \bar{g}(\nabla_{\xi''}\phi\xi',N_1)$$
$$= -\bar{g}(h^*(\xi',\phi\xi''),N_1) + \bar{g}(h^*(\xi'',\phi\xi'),N_1). \quad (4.10)$$

Thus from (4.6)-(4.10), the proof is complete. □

Corollary 4.2. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then $RadTM$ is integrable if and only if*

(i) $g(A^*_{\xi''}\xi', Z) = g(A^*_{\xi'}\xi'', Z)$
(ii) $g(A^*_{\xi''}\xi', \phi\xi) = g(A^*_{\xi'}\xi'', \phi\xi)$
(iii) $h^s(\xi', \phi\xi'') = h^s(\xi'', \phi\xi')$
(iv) $g(h^*(\xi', \phi\xi''), N_1) = g(h^*(\xi'', \phi\xi'), N_1)$,

for any $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L)$, $W \in \Gamma(S)$, and $Z \in \Gamma(D_0)$.

Theorem 4.4. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then D_1 is integrable if and only if*

(i) $\nabla^{*t}_X\phi Y - \nabla^{*t}_Y\phi X \in \Gamma(D_1)$.
(ii) $Bh(X, \phi Y) = Bh(Y, \phi X)$
(iii) $A^*_{\phi Y}X = A^*_{\phi X}Y$ *and belongs to* $\Gamma[(\bar{D} \oplus \phi D_2)\perp D_0]$

for any $X, Y \in \Gamma(D_1)$.

Proof. For any $X, Y \in \Gamma Rad(TM)$ using (2.17), we have $\bar{\nabla}_X\phi Y = \phi\bar{\nabla}_X Y$, apply ϕ both sides and then using (2.4), (2.12) and (2.16), we obtain

$$\nabla_X Y + h(X, Y) = -\phi(\nabla_X\phi Y + h(X, \phi Y)) + g(A^*_Y X, V)V.$$

Let $X, Y \in \Gamma(D_1)$ and using (2.12) we have

$$\nabla_X Y + h(X, Y) = -\phi(-A^*_{\phi Y}X + \nabla^{*t}_X\phi Y + h(X, \phi Y)) + g(A^*_Y X, V)V,$$

equating the tangential components of above equation both sides, we get

$$\nabla_X Y = fA^*_{\phi Y}X - f\nabla^{*t}_X\phi Y - Bh(X, \phi Y)) + g(A^*_Y X, V)V, \tag{4.11}$$

replacing X by Y and subtracting resulting equation from this equation, we get

$$\begin{aligned}[X, Y] &= f(A^*_{\phi Y}X - A^*_{\phi X}Y) - f(\nabla^{*t}_X\phi Y - \nabla^{*t}_Y\phi X) - Bh(X, \phi Y) + Bh(Y, \phi X) \\ &\quad + g(A^*_Y X, V)V - g(A^*_X Y, V)V,\end{aligned}$$

thus $[X, Y] \in \Gamma(D_1)$ if and only if $\nabla^{*t}_X\phi Y - \nabla^{*t}_Y\phi X \in \Gamma(D_1)$, $Bh(X, \phi Y) = Bh(Y, \phi X)$, $A^*_{\phi Y}X = A^*_{\phi X}Y$ and belongs to $\Gamma[(\bar{D} \oplus \phi D_2)\perp D_0]$, this completes the proof. □

Theorem 4.5. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then D_2 is integrable if and only if*

(i) $\nabla^*_X\phi Y - \nabla^*_Y\phi X \in \Gamma(\phi D_2)$.
(ii) $Bh(X, \phi Y) = Bh(Y, \phi X)$.
(iii) $h^*(X, \phi Y) - h^*(Y, \phi X)$
(iv) $A^*_Y X - A^*_X Y$ *and belongs to* $\Gamma[(\bar{D} \oplus \phi D_2)\perp D_0]$,

for any $X, Y \in \Gamma(D_2)$.

Proof. Let $X, Y \in \Gamma(D_2)$ and using (2.12), we have

$$\nabla_X Y + h(X, Y) = -\phi(\nabla^*_X\phi Y + h^*(X, \phi Y) + h(X, \phi Y)) + g(A^*_Y X, V)V,$$

equating the tangential components of above equation, we get

$$\nabla_X Y = -f\nabla^*_X\phi Y - fh^*(X, \phi Y) - Bh(X, \phi Y)) + g(A^*_Y X, V)V,$$

replacing X by Y and subtracting resulting equation from this equation, we get

$$\begin{aligned}[X, Y] &= -f(\nabla^*_X\phi Y - \nabla^*_Y\phi X) - f(h^*(X, \phi Y) - h^*(Y, \phi X)) - Bh(X, \phi Y) + Bh(Y, \phi X) \\ &\quad + g(A^*_Y X, V)V - g(A^*_X Y, V)V,\end{aligned}$$

thus $[X,Y] \in \Gamma(D_2)$ if and only if $\nabla^*_X \phi Y - \nabla^*_Y \phi X \in \Gamma(\phi D_2)$, $h^*(X,\phi Y) = h^*(Y,\phi X)$, $Bh(X,\phi Y) = Bh(Y,\phi X)$, $A^*_Y X - A^*_X Y$ and belongs to $\Gamma[(\bar{D} \oplus \phi D_2) \perp D_0]$, this proves the theorem. □

Theorem 4.6. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then ϕD_2 is integrable if and only if*

(i) $g(A^*_Y \phi X, \phi Z) = g(A^*_X \phi Y, \phi Z)$
(ii) $h^s(\phi X, Y) = h^s(\phi Y, X)$
(iii) $\bar{g}(h^l(\phi X, Y), \xi) = \bar{g}(h^l(\phi Y, X)$
(iv) $g(\phi Y, A_N \phi X) = g(\phi X, A_N \phi Y)$

for any $X, Y \in \Gamma(D_2), Z \in \Gamma(D_0), W \in \Gamma(S), N \in \Gamma(ltr(TM))$, and $\xi \in \Gamma(D_2)$.

Proof. Using the definition of GCR-lightlike submanifolds, it is clear that ϕD_2 is integrable if and only if

$$\bar{g}([\phi X, \phi Y], Z) = \bar{g}([\phi X, \phi Y], V) = \bar{g}([\phi X, \phi Y], \phi W) = \bar{g}([\phi X, \phi Y], \phi\xi) = \bar{g}([\phi X, \phi Y], N) = 0,$$

for any $X, Y \in \Gamma(D_2), Z \in \Gamma(D_0), W \in \Gamma(S), \xi \in \Gamma(D_2)$ and $N \in \Gamma(ltr(TM))$. Using (2.4) and (2.12)-(2.17), we obtain

$$\begin{aligned} \bar{g}([\phi X, \phi Y], Z) &= \bar{g}(\bar{\nabla}_{\phi X} \phi Y, Z) - \bar{g}(\bar{\nabla}_{\phi Y} \phi X, Z) \\ &= \bar{g}(\phi \bar{\nabla}_{\phi X} Y, Z) - \bar{g}(\phi \bar{\nabla}_{\phi Y} X, Z) \\ &= -\bar{g}(\bar{\nabla}_{\phi X} Y, \phi Z) + \bar{g}(\bar{\nabla}_{\phi Y} X, \phi Z) \\ &= -g(\nabla_{\phi X} Y, \phi Z) + g(\nabla_{\phi Y} X, \phi Z) \\ &= g(A^*_Y \phi X, \phi Z) - g(A^*_X \phi Y, \phi Z), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \bar{g}([\phi X, \phi Y], V) &= \bar{g}(\bar{\nabla}_{\phi X} \phi Y, V) - \bar{g}(\bar{\nabla}_{\phi Y} \phi X, V) \\ &= -\bar{g}(\bar{\nabla}_{\phi X} \phi Y - \bar{\nabla}_{\phi Y} \phi X, V) \\ &= -g(\phi Y, \phi X) + g(\phi X, \phi Y) \\ &= 0. \end{aligned} \tag{4.13}$$

$$\begin{aligned} \bar{g}([\phi X, \phi Y], \phi W) &= \bar{g}(\bar{\nabla}_{\phi X} \phi Y, \phi W) - \bar{g}(\bar{\nabla}_{\phi Y} \phi X, \phi W) \\ &= \bar{g}(\phi \bar{\nabla}_{\phi X} Y, \phi W) - \bar{g}(\phi \bar{\nabla}_{\phi Y} X, \phi W) \\ &= \bar{g}(\bar{\nabla}_{\phi X} Y, W) - \bar{g}(\bar{\nabla}_{\phi Y} X, W) \\ &= \bar{g}(h^s(\phi X, Y), W) - \bar{g}(h^s(\phi Y, X), W), \end{aligned} \tag{4.14}$$

$$\begin{aligned} \bar{g}([\phi X, \phi Y], \phi\xi) &= \bar{g}(\bar{\nabla}_{\phi X} \phi Y, \phi\xi) - \bar{g}(\bar{\nabla}_{\phi Y} \phi X, \phi\xi) \\ &= \bar{g}(\phi \bar{\nabla}_{\phi X} Y, \phi\xi) - \bar{g}(\phi \bar{\nabla}_{\phi Y} X, \phi\xi) \\ &= \bar{g}(\bar{\nabla}_{\phi X} Y, \xi) - \bar{g}(\bar{\nabla}_{\phi Y} X, \xi) \\ &= \bar{g}(h^l(\phi X, Y), \xi) - \bar{g}(h^l(\phi Y, X), \xi), \end{aligned} \tag{4.15}$$

and

$$\begin{aligned} \bar{g}([\phi X, \phi Y], N) &= \bar{g}(\bar{\nabla}_{\phi X} \phi Y, N) - \bar{g}(\bar{\nabla}_{\phi Y} \phi X, N) \\ &= -\bar{g}(\phi Y, \bar{\nabla}_{\phi X} N) + \bar{g}(\phi X, \bar{\nabla}_{\phi Y} N) \\ &= \bar{g}(\phi Y, A_N \phi X) - \bar{g}(\phi X, A_N \phi Y). \end{aligned} \tag{4.16}$$

Thus from (4.12)-(4.16), the proof is complete. □

Corollary 4.3. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then ϕD_2 is integrable if and only if*

(i) $\bar{g}(h^l(\phi X, \phi Z), Y) = \bar{g}(h^l(\phi Y, \phi Z), X)$
(ii) $h^s(\phi X, Y) = h^s(\phi Y, X)$
(iii) $g(A^*_\xi Y, \phi X) = g(A^*_\xi X, \phi Y)$
(iv) $g(h^*(\phi X, \phi Y), N) = g(h^*(\phi Y, \phi X), N)$

for any $X, Y \in \Gamma(D_2), Z \in \Gamma(D_0), W \in \Gamma(S), N \in \Gamma(ltr(TM))$*, and* $\xi \in \Gamma(D_2)$.

Theorem 4.7. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. Then each leaf of radical distribution is totally geodesic in M if and only if*

(i) $A^*_\xi \xi' \notin \Gamma(D_0 \perp M_1)$
(ii) $\bar{g}(h^s(\xi', \phi\xi''), W) = 0$
(iii) $\bar{g}(h^*(\xi', \phi\xi''), N_1) = 0$,

where $M_1 = \phi(L)$ *for any* $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L)$, $W \in \Gamma(S)$ *and* $Z \in \Gamma(D_0)$.

Proof. Using the definition of GCR-lightlike submanifold of an indefinite Kenmotsu manifold, each leaf of $Rad(TM)$ defines totally geodesic foliation in M if and only if

$$\bar{g}(\nabla_{\xi'}\xi'', V) = \bar{g}(\nabla_{\xi'}\xi'', Z) = \bar{g}(\nabla_{\xi'}\xi'', \phi\xi) = \bar{g}(\nabla_{\xi'}\xi'', \phi W) = \bar{g}(\nabla_{\xi'}\xi'', \phi N_1) = 0,$$

for any $\xi \in \Gamma(D_2)$, $\xi', \xi'' \in \Gamma(Rad(TM))$, $N_1 \in \Gamma(L), W \in \Gamma(S)$, and $Z \in \Gamma(D_0)$. Using (2.5), (2.11), (2.12), (2.15), (2.17) and (2.18), we obtain

$$g(\nabla_{\xi'}\xi'', V) = \bar{g}(\bar{\nabla}_{\xi'}\xi'', V) = -\bar{g}(\xi'', \bar{\nabla}_{\xi'}V) = -g(\xi'', \xi') = 0. \tag{4.17}$$

$$g(\nabla_{\xi'}\xi'', Z) = -g(A^*_{\xi''}\xi', Z). \tag{4.18}$$

$$g(\nabla_{\xi'}\xi'', \phi\xi) = -g(A^*_{\xi''}\xi', \phi\xi). \tag{4.19}$$

$$\begin{aligned} g(\nabla_{\xi'}\xi'', \phi W) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi W) = -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'', W) \\ &= -\bar{g}(h^s(\xi', \phi\xi''), W), \end{aligned} \tag{4.20}$$

and

$$\begin{aligned} g(\nabla_{\xi'}\xi'', \phi N_1) &= \bar{g}(\bar{\nabla}_{\xi'}\xi'', \phi N_1) = -\bar{g}(\bar{\nabla}_{\xi'}\phi\xi'' - (\bar{\nabla}_{\xi'}\phi)\xi'', N_1) \\ &= -\bar{g}(\nabla_{\xi'}\phi\xi'', N_1) = -\bar{g}(h^*(\xi', \phi\xi''), N_1). \end{aligned} \tag{4.21}$$

Hence form (4.17)-(4.21), the assertion follows. □

Theorem 4.8. *Let M be a GCR-lightlike submanifold of an indefinite Kenmotsu manifold $\bar{M}$. If M is D-geodesic then each leaf of holomorphic distribution is totally geodesic foliation in M.*

Proof. For $X, Y \in \Gamma(D)$ using (2.4) and (2.16), we have

$$\begin{aligned} h(X, Y) &= \bar{\nabla}_X Y - \nabla_X Y \\ &= \bar{\nabla}_X(-\phi^2 Y + \eta(Y)V) - \nabla_X Y \\ &= -(\bar{\nabla}_X \phi)(\phi Y) - \phi\bar{\nabla}_X \phi Y - \nabla_X Y \\ &= \bar{g}(X, \phi Y)V - \phi\bar{\nabla}_X \phi Y - \nabla_X Y \\ &= \bar{g}(X, \phi Y)V - \phi\nabla_X \phi Y - \phi h(X, \phi Y) - \nabla_X Y \\ &= \bar{g}(X, \phi Y)V - f\nabla_X \phi Y - \omega\nabla_X \phi Y - Bh(X, \phi Y) - Ch(X, \phi Y) - \nabla_X Y. \end{aligned}$$

Equating the transversal parts both sides we get

$$h(X,Y) = -\omega\nabla_X\phi Y - Ch(X,\phi Y),$$

Since by the hypothesis M is D-geodesic therefore we obtain $-\omega\nabla_X\phi Y = 0$ this implies that $\nabla_X\phi Y \in D$. Hence the proof is complete. □

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