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A STUDY ON SEMI-QUATERNIONS ALGEBRA IN SEMI-EUCLIDEAN 4-SPACE

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Abstract. The aim of this paper is to study the semi-quaternions, and to give some of their basic properties. We express De Moivre's formula for semi-quaternions and find roots of a semi-quaternion using this formula.

1. Introduction

The quaternion was first introduced by William Rowan Hamilton as a successor to complex numbers. The quaternions have been used in various areas of mathematics. Most recently, quaternions have enjoyed prominence in computer science, because they are the simplest algebraic tools for describing rotations in three and four dimensions [8]. The Euler's and De Moivre's formulas for the complex numbers are generalized for quaternions. De Moivre's formula implies that there are uncountably many unit quaternions satisfying $x^n = 1$ for $n > 2$ [1]. Also, using De Moivre's formula to find roots of a quaternion is a more useful way. The Euler's and De Moivre's formulas are also investigated for the case of dual and split quaternions in [3, 4].

A brief introduction of the semi-quaternions is provided in [5]. Dyachkova [2] has showed that the set of all invertible elements of semi-quaternions with the quaternion product is a Lie group. Also, she considered the degenerate scalar product $\langle q, p\rangle = a_\circ b_\circ + a_1 b_1$. Thus, the semi-quaternions algebra with this product has the 4-dimensional semi-Euclidean space structure with rank 2 semimetric.

Here, we investigate some algebraic properties of semi-quaternions. De-Moivre's and Euler's formulas for these quaternions are given. De Moivre's formula implies that there are uncountably many unit semi-quaternions satisfying for $q^n = 1$ for $n \geq 2$. Finally, we give some examples for more clarification.

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2. Preliminaries

In this section, we give a brief summary of the real quaternions. For detailed information about these concepts, we refer the reader to [6].

Definition 2.1. A real quaternion is defined *as*

$$q = a_\circ + a_1 i + a_2 j + a_3 k$$

where $a_\circ, a_1, a_2$ and a_3 are real numbers and $1, i, j, k$ of q may be interpreted as the four basic vectors of Cartesian set of coordinates; and they satisfy the non-commutative multiplication rules

$$\begin{aligned} i^2 &= j^2 = k^2 = ijk = -1 \\ ij &= k = -ji, \quad jk = i = -kj \end{aligned}$$

and

$$ki = j = -ik.$$

A quaternion may be defined as a pair (S_q, V_q), where $S_q = a_\circ \in \mathbb{R}$ is scalar part and $V_q = a_1 i + a_2 j + a_3 k \in \mathbb{R}^3$ is the vector part of q. The quaternion product of two quaternions p and q is defined as

$$pq = S_p S_q - \langle V_p, V_q \rangle + S_p V_q + S_q V_p + V_p \wedge V_q$$

where"$\langle , \rangle$"and "$\wedge$" are the inner and vector products in $\mathbb{R}^3$, respectively. The norm of a quaternion is given by the sum of the squares of its components: $N_q = a_\circ^2 + a_1^2 + a_2^2 + a_3^2$, $N_q \in \mathbb{R}$. It can also be obtained by multiplying the quaternion by its conjugate, in either order since a quaternion and its conjugate commute: $N_q = \overline{q}q = q\overline{q}$. Every non-zero quaternion has a multiplicative inverse given by its conjugate divided by its norm: $q^{-1} = \frac{\overline{q}}{N_q}$. The quaternion algebra H is a normed division algebra, meaning that for any two quaternions p and q, $N_{pq} = N_p N_q$, and the norm of every non-zero quaternion is non-zero (and positive) and therefore the multiplicative inverse exists for any non-zero quaternion. Of course, as is well known, multiplication of quaternions is not commutative, so that in general for any two quaternions p and q, $pq \neq qp$. This can have subtle ramifications, for example: $(pq)^2 = pqpq \neq p^2 q^2$.

3. Semi-Quaternions

Definition 3.1. A semi-quaternion q is a expression of form

$$q = a_\circ + a_1 i + a_2 j + a_3 k$$

where $a_\circ, a_1, a_2$ and a_3 are real numbers and i, j, k are quaternionic units which satisfy the equalities

$$\begin{aligned} i^2 &= -1, \quad j^2 = k^2 = 0 \\ ij &= k = -ji\ , \quad jk = 0 = kj \end{aligned}$$

and

$$ki = j = -ik.$$

The set of all semi-quaternions are denoted by H_s. A semi-quaternion q is a sum of a scalar and a vector, called scalar part, $S_q = a_\circ$, and vector part $V_q =$

$a_1 i + +a_2 j + a_3 k$. Also, a semi-quaternion can be represented in the following way; $q = z_1 + iz_2$, where $z_1 = a_\circ + a_2 j$, $z_2 = a_1 + a_3 j$.

The addition rule for semi-quaternions is component-wise addition:

$$\begin{aligned} q + p &= (a_\circ + a_1 i + a_2 j + a_3 k) + (b_\circ + b_1 i + b_2 j + b_3 k) \\ &= (a_\circ + b_\circ) + (a_1 + b_1)i \ + (a_2 + b_2)j + (a_3 + b_3)k. \end{aligned}$$

This rule preserves the associativity and commutativity properties of addition. The product of scalar and a semi-quaternion is defined in a straightforward manner. If c is a scaler and $q \in H_s$,

$$cq = cS_q + cV_q = (ca_\circ)1 + (ca_1)i + (ca_2)j + (ca_3)k.$$

The multiplication rule for semi-quaternions is defined as

$$qp = S_q S_p - g(V_q, V_p) + S_q V_p + S_p V_q + V_p \times V_q,$$

where

$$g(V_q, V_p) = a_1 b_1,\ V_p \times V_q = 0i + (a_3 b_1 - a_1 b_3)j + (a_1 b_2 - a_2 b_1)k.$$

Also, this can be written as

$$pq = \begin{bmatrix} a_\circ & -a_1 & 0 & 0 \\ a_1 & a_\circ & 0 & 0 \\ a_2 & a_3 & a_\circ & -a_1 \\ a_3 & -a_2 & a_1 & a_\circ \end{bmatrix} \begin{bmatrix} b_\circ \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

Obviously, the quaternion multiplication is associative and distributive with respect to addition and subtraction, but the commutativity law does not hold in general.

Corollary 3.1. *H_s with addition and multiplication has all the properties of a number field expect commutativity of the multiplication. It is therefore called the skew field of quaternions.*

4. SOME PROPERTIES OF SEMI-QUATERNIONS

1) The conjugate of $q = a_\circ + a_1 i + a_2 j + a_3 k = S_q + V_q$ is

$$\overline{q} = a_\circ - (a_1 i + a_2 j + a_3 k) = S_q - V_q.$$

It is clear the scalar and vector part of q is denoted by $S_q = \frac{q+\overline{q}}{2}$ and $V_q = \frac{q-\overline{q}}{2}$.

2) The norm of q is defined as $N_q = q\overline{q} = \overline{q}q = a_\circ^2 + a_1^2$.

If $N_q = a_\circ^2 + a_1^2 = 1$, then q is called a unit semi-quaternion. The set S_2^3 containing of all the unit semi-quaternions is the 2-fold cover of the special orthogonal group $SO(3)$. It is an analogue of the Hopf bundle [2].

Proposition 4.1. *Let $p, q \in H_s$ and $\lambda, \delta \in \mathbb{R}$. The conjugate and norm of semi-quaternions satisfies the following properties;*

i) $\overline{\overline{q}} = q$, *ii)* $\overline{pq} = \overline{q}\ \overline{p}$, *iii)* $\overline{\lambda q + \delta p} = \lambda \overline{q} + \delta \overline{p}$,

iv) $N_{qp} = N_q N_p$, *v)* $N_{\lambda q} = \lambda^2 N_q$.

3) The inverse of q is defined as $q^{-1} = \frac{\overline{q}}{N_q}$, $N_q \neq 0$, with the following properties;

i) $(qp)^{-1} = p^{-1}q^{-1}$, *ii)* $(\lambda q)^{-1} = \frac{1}{\lambda} q^{-1}$, *iii)* $N_{q^{-1}} = \frac{1}{N_q}$.

4) To divide a semi-quaternion p by the semi-quaternion $q(\neq 0)$, one simply has to resolve the equation

$$xq = p \quad or \quad qy = p,$$

with the respective solutions

$$\begin{aligned} x &= pq^{-1} = p\frac{\overline{q}}{N_q}, \\ y &= q^{-1}p = \frac{\overline{q}}{N_q}p, \end{aligned}$$

and the relation $N_x = N_y = \frac{N_p}{N_q}$.

5) The scalar product of two semi-quaternions $p = S_p + V_p$ and $q = S_q + V_q$ is defined as

$$\begin{aligned} \langle q, p \rangle &= S_q S_p + g(V_q, V_p) \\ &= S\left(q\overline{p}\right) \end{aligned}$$

The algebra H_s has the 4-dimensional semi-Euclidean space structure ${}_2\mathbb{R}^4$ with rank 2 semimetric[2].

Theorem 4.1. *The scalar product has the properties;*
1) $\langle pq_1, pq_2 \rangle = N_p \langle q_1, q_2 \rangle$
2) $\langle q_1 p, q_2 p \rangle = N_p \langle q_1, q_2 \rangle$
3) $\langle pq_1, q_2 \rangle = \langle q_1, \overline{p} q_2 \rangle$
4) $\langle pq_1, q_2 \rangle = \langle p, q_2 \overline{q_1} \rangle$.

Proof. We will proof identities (1) and (3).

$$\begin{aligned} \langle pq_1, pq_2 \rangle &= S(pq_1\overline{pq_2}) = S(pq_1\overline{q}_2\ \overline{p}) \\ &= S(\overline{q}_2\ \overline{p}pq_1) = N_p\ S(\overline{q}_2 q_1) \\ &= N_p\ S(q_1\overline{q}_2) = N_p\ \langle q_1, q_2 \rangle, \end{aligned}$$

and

$$\begin{aligned} \langle pq_1, q_2 \rangle &= S(pq_1\overline{q}_2) = S(q_1\overline{q}_2 p) \\ &= S(q_1\ \overline{\overline{p}q_2}) = \langle q_1, \overline{p}q_2 \rangle. \end{aligned}$$

□

Theorem 4.2. *The algebra H_s is isomorphic to the subalgebra of the algebra $\mathbb{C}_2$ consisting of the (2×2)-matrices*

$$\hat{A} = \begin{bmatrix} A & B \\ 0 & \overline{A} \end{bmatrix},$$

and to the subalgebra of the algebra $\mathbb{C}_2^{\circ}$ consisting of the (2×2)-matrices

$$\tilde{A} = \begin{bmatrix} A & B \\ 0 & A \end{bmatrix},$$

where $A, B \in \mathbb{C}$.

Proof. The proof can be found in [5]. □

5. De Moivre's formula for semi-qauternions

Definition 5.1. Every nonzero semi-quaternion can be written in the polar form

$$\begin{aligned} q &= a_{\circ} + a_1 i + a_2 j + a_3 k \\ q &= r(\cos\varphi + \overrightarrow{w}\sin\varphi), \qquad 0 \leq \varphi \leq 2\pi \end{aligned}$$

where $r = \sqrt{N_q}$ and

$$\cos\varphi = \frac{a_{\circ}}{r}, \ \ \sin\varphi = \frac{\sqrt{a_1^2}}{r} = \frac{|a_1|}{\sqrt{a_{\circ}^2 + a_1^2}}.$$

The unit vector $\overrightarrow{w}$ is given by

$$\overrightarrow{w} = \frac{1}{\sqrt{a_1^2}}(a_1 i + a_2 j + a_3 k), \ \ a_1 \neq 0.$$

Since $\overrightarrow{w}^2 = -1$, We have a natural generalization of Euler's formula for unit semi-quaternions

$$\begin{aligned} e^{\overrightarrow{w}\varphi} &= 1 + \overrightarrow{w}\varphi + \frac{(\overrightarrow{w}\varphi)^2}{2!} + \frac{(\overrightarrow{w}\varphi)^3}{3!} + ... \\ &= 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - ... + \overrightarrow{w}(\varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - ...) \\ &= \cos\varphi + \overrightarrow{w}\sin\varphi. \end{aligned}$$

for any real number φ. For detalied information about Euler's formula, see [7].

Lemma 5.1. *Let $\overrightarrow{w}$ be a unit vector, then we have*

$$(\cos\varphi + \overrightarrow{w}\sin\varphi)(\cos\psi + \overrightarrow{w}\sin\psi) \quad = \quad \cos(\varphi + \psi) + \overrightarrow{w}\sin(\varphi + \psi).$$

Theorem 5.1. *(De-Moivre's formula) Let $q = e^{\overrightarrow{w}\varphi} = \cos\varphi + \overrightarrow{w}\sin\varphi$ be a unit semi-quaternion. Then for every integer n;*

$$q^n = \cos n\varphi + \overrightarrow{w}\sin n\varphi.$$

Proof. We use induction on positive integers n. Assume that $q^n = \cos n\varphi + \overrightarrow{w}\sin n\varphi$ holds. Then

$$\begin{aligned} q^{n+1} &= (\cos\varphi + \overrightarrow{w}\sin\varphi)^n(\cos\varphi + \overrightarrow{w}\sin\varphi) \\ &= (\cos n\varphi + \overrightarrow{w}\sin n\varphi)(\cos\varphi + \overrightarrow{w}\sin\varphi) \\ &= \cos(n\varphi + \varphi) + \overrightarrow{w}\ \sin(n\varphi + \varphi) \\ &= \cos(n+1)\varphi + \overrightarrow{w}\ \sin(n+1)\varphi. \end{aligned}$$

The formula holds for all integer n, since

$$\begin{aligned} q^{-1} &= \cos\varphi - \overrightarrow{w}\sin\varphi, \\ q^{-n} &= \cos(-n\varphi) + \overrightarrow{w}\sin(-n\varphi) \\ &= \cos n\varphi - \overrightarrow{w}\sin n\varphi. \end{aligned}$$

□

Example 5.1. Let $q = -1 + i - j + 2k = \sqrt{2}(\cos\frac{3\pi}{4} + \overrightarrow{w}\sin\frac{3\pi}{4})$ be a semi-quaternion. Every powers of this quaternion are found to be with the aid of Theorem 5.2 , for example, 10-th power is

$$\begin{aligned} q^{10} &= 2^5[\cos 10(\frac{3\pi}{4}) + \overrightarrow{w}\sin 10(\frac{3\pi}{4})] \\ &= 2^5(0 - \overrightarrow{w}). \end{aligned}$$

Corollary 5.1. *There are uncountably many unit semi-quaternions satisfying $q^n = 1$ for every integer $n \geq 3$.*

Proof. For every unit vector $\overrightarrow{w}$, the quaternion $q = \cos\frac{2\pi}{n} + \overrightarrow{w}\sin\frac{2\pi}{n}$ is of order n. For $n = 1$ or $n = 2$, the quaternion q is independent of $\overrightarrow{w}$.

□

Example 5.2. $q = \frac{1}{\sqrt{2}} + (\frac{1}{\sqrt{2}}, 1, -1) = \cos\frac{\pi}{4} + \overrightarrow{w}\sin\frac{\pi}{4}$ is of order 8 and $q = \frac{1}{2} + (\frac{\sqrt{3}}{2}, 1, 1) = \cos\frac{\pi}{3} + \overrightarrow{w}\sin\frac{\pi}{3}$ is of order 6.

Theorem 5.2. *Let $q = \cos\varphi + \overrightarrow{w}\sin\varphi$ be a unit semi-quaternion. The equation $x^n = q$ has n roots*

$$x_k = \cos\frac{\varphi + 2k\pi}{n} + \overrightarrow{w}\sin\frac{\varphi + 2k\pi}{n}, \quad k = 0, 1, ..., n-1.$$

Proof. If $x^n = q$, q will have the same unit vector as x. So, we assume that $x = \cos\chi + \overrightarrow{w}\sin\chi$ is a root of the equation $x^n = q$. From Theorem 5.1, we have

$$x^n = \cos n\chi + \overrightarrow{w}\sin n\chi,$$

Thus, we find

$$\cos n\chi = \cos\varphi \ \ \& \quad \sin n\chi = \sin\varphi.$$

So, the $n-$th roots of q are $x = \cos\frac{\varphi+2k\pi}{n} + \overrightarrow{w}\sin\frac{\varphi+2k\pi}{n}$ for $k = 0, 1, ..., n-1$. □

Example 5.3. Let $q = 1+i-2j+2k = \sqrt{2}(\cos\varphi + \overrightarrow{w}\sin\varphi)$ be a semi-quaternion. The equation $x^3 = q$ has 3 roots and these are

$$x_k = \sqrt[6]{2}(\cos\frac{\varphi + 2k\pi}{3} + \overrightarrow{w}\sin\frac{\varphi + 2k\pi}{3}), \quad k = 0, 1, 2.$$

So, $x_0 = \sqrt[6]{2}(\cos\frac{\pi}{12} + \overrightarrow{w}\sin\frac{\pi}{12})$, $x_1 = \sqrt[6]{2}(\cos\frac{3\pi}{4} + \overrightarrow{w}\sin\frac{3\pi}{4})$, $x_2 = \sqrt[6]{2}(\cos\frac{17\pi}{12} + \overrightarrow{w}\sin\frac{17\pi}{12})$ are the cube roots of q.

Theorem 5.3. *Let q be a unit semi-quaternion with the polar form $q = \cos\varphi + \overrightarrow{w}\sin\varphi$. If $m = \frac{2\pi}{\varphi} \in \mathbb{Z}^+ - \{1\}$, then $n \equiv p(\mod m)$ is possible if and only if $q^n = q^p$.*

Proof. Let $n \equiv p(\mod m)$. Then we have $n = am + p$, where $a \in \mathbb{Z}$.

$$\begin{aligned} q^n &= \cos n\varphi + \overrightarrow{w}\sin n\varphi \\ &= \cos(am+p)\varphi + \overrightarrow{w}\sin(am+p)\varphi \\ &= \cos(a\frac{2\pi}{\varphi} + p)\varphi + \overrightarrow{w}\sin(a\frac{2\pi}{\varphi} + p)\varphi \\ &= \cos(p\varphi + 2\pi a) + \overrightarrow{w}\sin(p\varphi + 2\pi a) \\ &= \cos p\varphi + \overrightarrow{w}\sin p\varphi \\ &= q^p. \end{aligned}$$

Now suppose $q^n = \cos n\varphi + \overrightarrow{w}\sin n\varphi$ and $q^p = \cos p\varphi + \overrightarrow{w}\sin p\varphi$. Since $q^n = q^p$, we have $\cos n\varphi = \cos p\varphi$ and $\sin n\varphi = \sin p\varphi$, which means $n\varphi = p\varphi + 2\pi a$, $a \in \mathbb{Z}$. Thus $n = a\frac{2\pi}{\varphi} + p$, $n \equiv p(\mod m)$. □

Example 5.4. Let $q = \frac{1}{2} + (\frac{\sqrt{3}}{2}, -1, 2)$ be a unit semi-quaternion. From the theorem 5.5, $m = \frac{2\pi}{\pi/3} = 6$, so we have

$$\begin{aligned} q &= q^7 = q^{13} = ... \\ q^2 &= q^8 = q^{14} = ... \\ q^3 &= q^9 = q^{15} = ... = -1 \\ &\;\;\, ... \\ q^6 &= q^{12} = q^{18} = ... = 1. \end{aligned}$$

6. Conclusion

In this paper, we give some of algebraic properties of the semi-quaternions and investigate the Euler's and De Moivre's formulas for these quaternions. De Moivre's formula implies that there are uncountably many unit semi-quaternions satisfying for $q^n = 1$ for $n > 2$. The relation between the powers of semi-quaternions is given in Theorem 5.3.

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