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EXTENSION OF SHI'S QUASI-UNIFORMITY TO THE FUZZY SOFT SETS

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Abstract. The purpose of this paper is to introduce Shi's (quasi-)uniformity structure in the context of fuzzy soft sets. We define the notion of a fuzzy soft (quasi-)uniformity in the sense of Shi. We give the relations between a fuzzy soft (quasi-)uniformity and a fuzzy soft cotopology. Also, we investigate the relations between fuzzy soft remote neighborhood structures which are generated by a given fuzzy soft uniformity structure.

1. Introduction

Most of the existing mathematical tools for formal modeling, reasoning and computing are crisp, deterministic and precise in character. In 1999, Molodtsov [12] initiated the theory of soft sets as a new mathematical tool for dealing with uncertainties which traditional mathematical tools cannot handle. Later other authors like Maji et al. [11] further studied the theory of soft sets and also introduced the concept of fuzzy soft set which combines fuzzy sets and soft sets. Soft set and fuzzy soft set theories have rich potential for applications in several directions (see [1, 7, 8, 9]). Furthermore, Aygünoglu et al. [2] studied the topological structure of fuzzy soft sets in the sense of Šostak [14].

The theory of uniform structures is an important area of topology which in a certain sense can be viewed as a bridge linking metrics as well as topological groups with general topological structures. Therefore, it is not surprising that the attention of mathematicians interested in fuzzy topology constantly addressed the problem to give an appropriate definition of a uniformity in fuzzy context and to develop the corresponding theory. Fuzzy versions of (quai-)uniformity theory were established by Hutton [6], Lowen [10], Höhle [5] and Shi [13, 15]. One remarkable advantage of Shi's (quasi-)uniformity is that it can be directly reflect the characteristics of

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pointwise fuzzy topology, i.e., the relations between a point and its quasi-coincident neighborhood or remote neighborhood.

The aim of this study is to handle the extension of Shi's (quasi-)uniformity in the context of fuzzy soft sets. This study is organized as follows. In Section 2, we give some preliminary concepts and properties. In Section 3, we introduce the concept of topological fuzzy soft remote neighborhood system and give the relations between fuzzy soft cotopological spaces. In the last section, we define pointwise fuzzy soft quasi-uniformity, induce two kinds of fuzzy soft remote neighborhood systems from a given pointwise fuzzy soft uniformity and investigate the relations between them.

2. Preliminaries

Throughout this paper, L is a complete lattice with an order reversing involution $'$ and M is a completely distributive lattice. The least element and the greatest element of L is denoted by 0_L and 1_L, respectively. Let $a, b \in L$. An element $a \in L$ is said to be coprime if $a \leq b \vee c$ implies that $a \leq b$ or $a \leq c$. The set of all coprimes of L is denoted by $c(L)$. We say a is way below (wedge below) b, in symbols, $a \ll b$ ($a \lhd b$) or $b \gg a$ ($b \rhd a$), if for every directed (arbitrary) subset $D \subseteq L, \vee D \geq b$ implies $a \leq d$ for some $d \in D$. Clearly if $a \in L$ is coprime, then $a \ll b$ if and only if $a \lhd b$. A complete lattice L is said to be continuous (completely distributive) if every element in L is the supremum of all elements way below (wedge below) it (see [4]).

Let E and K be arbitrary nonempty sets viewed on the sets of parameters. A fuzzy soft set f on X is a mapping from E into L^X, i.e., $f_e := f(e) : X \to L$ is an L-fuzzy set on X, for each $e \in E$. The family of all L-fuzzy soft sets on X is denoted by $(L^X)^E$. By Φ_X and $\widetilde{E}_X$, we denote respectively the null fuzzy soft set and absolute fuzzy soft set. The set of all coprimes of $(L^X)^E$ is denoted by $c((L^X)^E)$.

Definition 2.1. [11] For two L-fuzzy soft sets f and g on X, we say that f is an L-fuzzy soft subset of g and write $f \sqsubseteq g$ if $f_e \leq g_e$, for each $e \in E$. f is said to be equal to g if $f \sqsubseteq g$ and $g \sqsubseteq f$

Definition 2.2. [11] (1) Union of two L-fuzzy soft sets f and g on X is an L-fuzzy soft set $h = f \sqcup g$, where $h_e = f_e \vee g_e$, for each $e \in E$.

(2) Intersection of two L-fuzzy soft sets f and g on X is an L-fuzzy soft set $h = f \sqcap g$, where $h_e = f_e \wedge g_e$, for each $e \in E$.

(3) The complement of an L-fuzzy soft set f is denoted by f', where $f' : E \longrightarrow L^X$ is a mapping given by $f'_e = (f_e)'$, for each $e \in E$. Clearly $(f')' = f$.

Let $p \mid f$ denote the set $\{g \in (L^X)^E \mid p \not\sqsubseteq g \sqsupseteq f\}$ for $p \in c((L^X)^E)$ and $f \in (L^X)^E$.

Proposition 2.1. [11] *Let Δ be an index set and $f, g, h, f_i, g_i \in (L^X)^E$, for all $i \in \Delta$, then we have the following properties:*

(1) $f \sqcap f = f,\ f \sqcup f = f$.
(2) $f \sqcap g = g \sqcap f,\ f \sqcup g = g \sqcup f$.
(3) $f \sqcup (g \sqcup h) = (f \sqcup g) \sqcup h,\ f \sqcap (g \sqcap h) = (f \sqcap g) \sqcap h$.
(4) $f = f \sqcup (f \sqcap g),\ f = f \sqcap (f \sqcup g)$.
(5) $f \sqcap \left(\bigsqcup_{i\in\Delta} g_i\right) = \bigsqcup_{i\in\Delta} (f \sqcap g_i)$.
(6) $f \sqcup (\sqcap_{i\in\Delta} g_i) = \sqcap_{i\in\Delta} (f \sqcup g_i)$.

(7) $\Phi_X \sqsubseteq f \sqsubseteq \widetilde{E}_X$.
(8) $(\sqcap_{i\in\Delta} f_i)' = \bigsqcup_{i\in\Delta} f_i'$.
(9) $\left(\bigsqcup_{i\in\Delta} f_i\right)' = \sqcap_{i\in\Delta} f_i'$.
(10) *If* $f \sqsubseteq g$, *then* $g' \sqsubseteq f'$.

Definition 2.3. [2] A mapping $\tau : K \longrightarrow M^{(L^X)^E}$ is called an (L, M)-fuzzy (E, K)-soft topology on X if it satisfies the following conditions for each $k \in K$ (where, $\tau(k) = \tau_k : (L^X)^E \to M$ is a mapping for each $k \in K$).

(O1) $\tau_k(\Phi_X) = \tau_k(\widetilde{E}_X) = 1_M$.
(O2) $\tau_k(f \sqcap g) \geq \tau_k(f) \wedge \tau_k(g)$, for all $f, g \in (L^X)^E$.
(O3) $\tau_k(\bigsqcup_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \tau_k(f_i)$, for all $f_i \in (L^X)^E, i \in \Delta$.

The pair (X, τ) is called an (L, M)-fuzzy (E, K)-soft topological space. The value $\tau_k(f)$ is interpreted as the degree of openness of a fuzzy soft set f with respect to parameter $k \in K$.

Example 2.1. Let $E = K = \mathbb{N}$ be the set of natural numbers, $I = [0, 1]$ and $\tau : K \to I^{(I^X)^E}$ be defined as follows:

$$\tau_k(f) = \begin{cases} 1, & \text{if } f = \Phi_X, \widetilde{E}_X, \\ \frac{1}{k}, & \text{otherwise.} \end{cases}, \ \forall k \in K.$$

It is easy to testify that τ is a fuzzy soft topology on X.

Definition 2.4. [2] A mapping $\eta : K \longrightarrow M^{(L^X)^E}$ is called an (L, M)-fuzzy (E, K)-soft cotopology on X if it satisfies the following conditions for each $k \in K$:

(C1) $\eta_k(\Phi_X) = \eta_k(\widetilde{E}_X) = 1_M$.
(C2) $\eta_k(f \sqcup g) \geq \eta_k(f) \wedge \eta_k(g)$, for all $f, g \in (L^X)^E$.
(C3) $\eta_k(\sqcap_{i\in\Delta} f_i) \geq \bigwedge_{i\in\Delta} \eta_k(f_i)$, for all $f_i \in (L^X)^E, i \in \Delta$.

The pair (X, η) is called an (L, M)-fuzzy (E, K)-soft cotopological space.

Let τ be a fuzzy soft topology on X, then the mapping $\eta : K \to M^{(L^X)^E}$ defined by $\eta_k(f) = \tau_k(f')$, for all $k \in K$ is a fuzzy soft cotopology on X. Let η be a fuzzy soft cotopology on X, then the mapping $\tau : K \to M^{(L^X)^E}$ defined by $\tau_k(f) = \eta_k(f')$, for all $k \in K$, is a fuzzy soft topology on X.

3. Fuzzy Soft Remote Neighborhood System

In this section, we recall the definition of a fuzzy soft remote neighborhood system and give the relationships between a fuzzy soft remote neighborhood system and a fuzzy soft cotopological space. If the parameter sets E and K are both one pointed, then the results obtained in this section coincide with Yue et al.[16].

Definition 3.1. [3] A topological fuzzy soft remote neighborhood system is a set $\mathcal{R} = \{R^p \mid p \in c((L^X)^E)\}$ of mappings $R^p : K \to M^{(L^X)^E}$ such that for each $k \in K$:

(RN1) $R^p_k(\widetilde{E}_X) = 0_M, R^p_k(\Phi_X) = 1_M$.
(RN2) $R^p_k(f) \neq 0_M$ implies $p \not\sqsubseteq f$.
(RN3) $R^p_k(f \sqcup g) = R^p_k(f) \wedge R^p_k(g)$.
(RN4) $R^p_k(f) = \bigvee_{g \in p|f} \bigwedge_{r \not\sqsubseteq g} R^r_k(g)$.

(Here, $R^p(k) = R^p_k : (L^X)^E \to M$ is a mapping for each $k \in K$.)

Lemma 3.1. [3] *Let $\eta : K \to M^{(L^X)^E}$ be an (L, M)-fuzzy (E, K)-soft cotopology on X. Then we have the following properties.*

(1) $\mathcal{R}_\eta = \{R^p_\eta \mid p \in c((L^X)^E)\}$ is a topological fuzzy soft remote neighborhood system, where R^p_η is defined by

$$(R^p_\eta)_k(f) = \begin{cases} \bigvee\limits_{g \in p|f} \eta_k(g), & \text{if } p \not\sqsubseteq f; \\ 0_M, & \text{otherwise.} \end{cases}$$

for all $k \in K, p \in c((L^X)^E)$ and $f \in (L^X)^E$.

(2) If η and ζ are two (L, M)-fuzzy (E, K)-soft cotopologies on X which determine the same topological fuzzy soft remote neighborhood system, then $\eta = \zeta$.

Lemma 3.2. [3] *Let $\mathcal{R} = \{R^p \mid p \in c((L^X)^E)\}$ be a topological fuzzy soft remote neighborhood system and $\eta : K \to M^{(L^X)^E}$ be defined by follows: for all $k \in K$ and $f \in (L^X)^E$,*

$$\eta_k(f) = \bigwedge_{p \not\sqsubseteq f} R^p_k(f).$$

Then η is an (L, M)-fuzzy (E, K)-soft cotopology on X. Furthermore, if $\mathcal{R}$ and $\mathcal{P}$ are two topological fuzzy soft remote neighborhood systems on X which determine the same (L, M)-fuzzy (E, K)-soft cotopology, then $\mathcal{R} = \mathcal{P}$.

Lemma 3.3. [3] *Let $\mathcal{R} = \{R^p \mid p \in c((L^X)^E)\}$ be a set satisfying the conditions (RN1)-(RN3) of Definition 3.1. Then the following statements are equivalent:*

(RN4) $R^p_k(f) = \bigvee\limits_{g \in p|f} \bigwedge\limits_{r \not\sqsubseteq g} R^r_k(g)$.

(RN4)* $R^p_k(f) = \bigvee\limits_{g \in p|f} (R^p_k(g) \wedge \bigwedge\limits_{r \not\sqsubseteq g} R^r_k(f))$.

4. Fuzzy Soft Quasi-Uniformity in the Sense of Shi

In this section, we consider the extension of Shi's quasi-uniformity structure in the context of fuzzy soft sets.

Let $\mathcal{D}(X, E)$ denote the set of all mappings $\lambda : c((L^X)^E) \to (L^X)^E$ such that $f \not\sqsubseteq \lambda(f)$ for all $f \in c((L^X)^E)$. λ^* is the smallest element of $\mathcal{D}(X, E)$, i.e., $\lambda^*(f) = \Phi_X$ for all $f \in c((L^X)^E)$. For $\lambda, \mu \in \mathcal{D}(X, E)$, we define the operations on $\mathcal{D}(X, E)$ as follows:

(1) $\lambda \leq \mu$ if and only if $\lambda(f) \sqsubseteq \mu(f)$ for all $f \in c((L^X)^E)$.

(2) $(\lambda \vee \mu)(f) = \lambda(f) \sqcup \mu(f)$ for all $f \in c((L^X)^E)$.

(3) $(\lambda \diamond \mu)(f) = \sqcap\{\lambda(g) \mid g \in c((L^X)^E)$ and $g \not\sqsubseteq \mu(f)\}$ for all $f \in c((L^X)^E)$.

Then $\lambda \vee \mu \in \mathcal{D}(X, E), \lambda \diamond \mu \in \mathcal{D}(X, E), \lambda \diamond \mu \leq \lambda, \lambda \diamond \mu \leq \mu$ and the operators $\vee$ and $\diamond$ satisfy the associative law.

Definition 4.1. An order-preserving mapping $\lambda \in \mathcal{D}(X, E)$ is said to be symmetric, if for all $f, g \in c((L^X)^E)$, there exists $h \in c((L^X)^E)$ such that $h \not\sqsubseteq f'$ and $g \not\sqsubseteq \lambda(h)$ implies that there exists $p \in c((L^X)^E)$ such that $p \not\sqsubseteq g'$ and $f \not\sqsubseteq \lambda(p)$.

Let $\mathcal{D}_s(X, E)$ denote the set of all symmetric mappings in $\mathcal{D}(X, E)$.

Definition 4.2. A pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity on X is a mapping $\mathcal{U} : K \to M^{\mathcal{D}(X,E)}$ which satisfies the following axioms for each $k \in K$, (where $\mathcal{U}(k) = \mathcal{U}_k : \mathcal{D}(X, E) \to M$ is a mapping for each $k \in K$)

(U1) $\mathcal{U}_k(\lambda^*) = 1_M$.

(U2) $\mathcal{U}_k(\lambda \vee \mu) = \mathcal{U}_k(\lambda) \wedge \mathcal{U}_k(\mu)$ for all $\lambda, \mu \in \mathcal{D}(X, E)$.

(U3) $\mathcal{U}_k(\lambda) = \bigvee\limits_{\mu \diamond \mu \geq \lambda} \mathcal{U}_k(\mu)$ for all $\lambda \in \mathcal{D}(X, E)$.

If $\mathcal{U}$ is a pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity on X, then the pair $(X, \mathcal{U})$ is called an (L, M)-fuzzy (E, K)-soft quasi-uniform space.

Definition 4.3. A mapping $\mathcal{B} : K \to M^{\mathcal{D}(X,E)}$ is called a base of one pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity if it satisfies following conditions for each $k \in K$,

(B1) $\mathcal{B}_k(\lambda^*) = 1_M$.

(B2) $\mathcal{B}_k(\lambda \vee \mu) \geq \mathcal{B}_k(\lambda) \wedge \mathcal{B}_k(\mu)$ for all $\lambda, \mu \in \mathcal{D}(X, E)$.

(B3) $\mathcal{B}_k(\lambda) \leq \bigvee\limits_{\mu \diamond \mu \geq \lambda} \mathcal{B}_k(\mu)$ for all $\lambda \in \mathcal{D}(X, E)$.

Definition 4.4. A mapping $\Theta : K \to M^{\mathcal{D}(X,E)}$ is called a subbase of one pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity if it satisfies following conditions for each $k \in K$,

(SB1) $\Theta_k(\lambda^*) = 1_M$.

(SB2) $\Theta_k(\lambda) \leq \bigvee\limits_{\mu \diamond \mu \geq \lambda} \Theta_k(\mu)$ for all $\lambda \in \mathcal{D}(X, E)$.

Theorem 4.1. *(1) Let $\mathcal{B} : K \to M^{\mathcal{D}(X,E)}$ be an (L, M)-fuzzy (E, K)-soft base and define for each $k \in K$, $\mathcal{U}_k(\lambda) = \bigvee\limits_{\mu \geq \lambda} \mathcal{B}_k(\mu)$. Thus, the mapping $\mathcal{U} : K \to M^{\mathcal{D}(X,E)}$ is a pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity on X, generated by $\mathcal{B}$.*

(2) Let $\Theta : K \to M^{\mathcal{D}(X,E)}$ be a subbase of one pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity on X and define for each $k \in K$, $\mathcal{B}_k(\lambda) = \bigvee\limits_{(\sqcup)_{i \in \Gamma} \mu_i \geq \lambda} \bigwedge\limits_{i \in \Gamma} \Theta_k(\mu_i)$, where $(\sqcup)$ denotes "finite union". Then $\mathcal{B}$ is a base of one pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity on X.

Proof. (1) From the definition of (L, M)-fuzzy (E, K)-soft quasi-uniformity, (U1) is obvious and also (U2) is trivial. Now we prove (U3). From (U2), we know that $\mathcal{U}_k(\lambda) \geq \bigvee_{\mu \diamond \mu \geq \lambda} \mathcal{U}_k(\mu)$. So, we need to prove the converse inequality.

Let for $k \in K$ and $\alpha \in c(M)$, $\alpha \lhd \mathcal{U}_k(\lambda)$. From (B3), we have

$$\alpha \lhd \mathcal{U}_k(\lambda) = \bigvee_{\mu \geq \lambda} \mathcal{B}_k(\mu) \leq \bigvee_{\mu \geq \lambda} \bigvee_{\nu \diamond \nu \geq \mu} \mathcal{B}_k(\nu).$$

Then there exist ν and μ such that $\nu \diamond \nu \geq \mu \geq \lambda$ and $\alpha \leq \mathcal{B}_k(\nu)$. Hence $\alpha \leq \mathcal{U}_k(\nu)$. Thus, $\alpha \leq \bigvee_{\nu \diamond \nu \geq \mu} \mathcal{U}_k(\nu)$. Therefore, $\mathcal{U}_k(\lambda) \leq \bigvee\limits_{\mu \diamond \mu \geq \lambda} \mathcal{U}_k(\mu)$.

(2) Since (B1) is obvious and (B2) is trivial, we only prove (B3). Let $k \in K$ and $\alpha \in c(M)$ such that $\alpha \lhd \mathcal{B}_k(\lambda)$. From (SB2), we have

$$\alpha \lhd \mathcal{B}_k(\lambda) = \bigvee_{(\sqcup)_{i \in \Gamma} \mu_i \geq \lambda} \bigwedge_{i \in \Gamma} \Theta_k(\mu_i) \leq \bigvee_{(\sqcup)_{i \in \Gamma} \mu_i \geq \lambda} \bigwedge_{i \in \Gamma} \bigvee_{\nu_i \diamond \nu_i \geq \mu_i} \Theta_k(\nu_i).$$

Then there exist $\{\mu_i\}_{i \in \Gamma} \subseteq \mathcal{D}(X, E)$ such that

(i) $(\sqcup)_{i \in \Gamma} \mu_i \geq \lambda$,

(ii) For each $i \in \Gamma$, there exists ν_i such that $\nu_i \diamond \nu_i \geq \mu_i$ and $\alpha \leq \Theta_k(\nu_i)$. Let $\nu = (\sqcup)_{i \in \Gamma} \nu_i$. Then we have $\nu \diamond \nu \geq (\sqcup)_{i \in \Gamma} (\nu_i \diamond \nu_i) \geq (\sqcup)_{i \in \Gamma} \mu_i \geq \lambda$. Thus

$$\alpha \leq \bigvee_{\nu\diamond\nu\geq\lambda} \bigvee_{(\sqcup)_{i\in\Gamma}\nu_i\geq\nu} \bigwedge_{i\in\Gamma} \Theta_k(\nu_i) = \bigvee_{\nu\diamond\nu\geq\lambda} \mathcal{B}_k(\nu).$$

Therefore, $\mathcal{B}_k(\lambda) \leq \bigvee\limits_{\mu\diamond\mu\geq\lambda} \mathcal{B}_k(\mu)$ from the arbitrariness of α. □

Definition 4.5. Let $\mathcal{U}$ be a pointwise (L, M)-fuzzy (E, K)-soft quasi-uniformity on X. $\mathcal{U}$ is called a pointwise (L, M)-fuzzy (E, K)-soft uniformity if there exists a mapping $\mathcal{B} : K \to M^{\mathcal{D}(X,E)}$ with $\mathcal{B}_k(\lambda) = 0_M$ for all $\lambda \in \mathcal{D}(X, E) \setminus \mathcal{D}_s(X, E)$ and $k \in K$ such that $\mathcal{B}$ is a base of $\mathcal{U}$, i.e., $\mathcal{B}$ satisfies (B1)-(B3) and $\mathcal{U}_k(\lambda) = \bigvee\limits_{\mu\geq\lambda} \mathcal{B}_k(\mu)$ for all $k \in K$.

Lemma 4.1. *Let $(X,\mathcal{U})$ be an (L, M)-fuzzy (E, K)-soft quasi-uniform space and the mapping $R^p_{(\mathcal{U})} : K \to M^{(L^X)^E}$ be defined as follows for each $k \in K$ and $f \in (L^X)^E$,*

$$(R^p_{(\mathcal{U})})_k(f) = \bigvee_{f\sqsubseteq\lambda(p)} \mathcal{U}_k(\lambda).$$

Then the set $\mathcal{R}_{(\mathcal{U})} = \{R^p_{(\mathcal{U})} \mid p \in c((L^X)^E)\}$ is a topological fuzzy soft remote neighborhood system.

Proof. We need to prove (RN1)-(RN4) of Definition 3.1.

(RN1): $(R^p_{(\mathcal{U})})_k(\Phi_X) = \bigvee\limits_{\Phi_X\sqsubseteq\lambda(p)} \mathcal{U}_k(\lambda) \geq \mathcal{U}_k(\lambda^*) = 1_M$ and $(R^p_{(\mathcal{U})})_k(\widetilde{E}_X) = 0_M$.

(RN2): If $(R^p_{(\mathcal{U})})_k(f) > 0_M$, then there exists $\lambda \in \mathcal{D}(X, E)$ such that $\lambda(p) \sqsupseteq f$ and $\mathcal{U}_k(\lambda) > 0_M$. Hence, $p \not\sqsubseteq f$ by $\lambda \in \mathcal{D}(X, E)$.

(RN3): From the definition of $R^p_{(\mathcal{U})}$, we know that $(R^p_{(\mathcal{U})})_k(f \sqcup g) \leq (R^p_{(\mathcal{U})})_k(f) \wedge (R^p_{(\mathcal{U})})_k(g)$. Conversely, let $\alpha \lhd ((R^p_{(\mathcal{U})})_k(f) \wedge (R^p_{(\mathcal{U})})_k(g))$. Hence clearly, $\alpha \lhd (R^p_{(\mathcal{U})})_k(f)$ and $\alpha \lhd (R^p_{(\mathcal{U})})_k(g)$. Since $(R^p_{(\mathcal{U})})_k(f) = \bigvee_{f\sqsubseteq\lambda(p)} \mathcal{U}_k(\lambda)$, there exists $\lambda \in \mathcal{D}(X, E)$ such that $f \sqsubseteq \lambda(p)$ and $\alpha \leq \mathcal{U}_k(\lambda)$.

Similarly, there exists $\mu \in \mathcal{D}(X, E)$ such that $\mu(p) \sqsupseteq f$ and $\alpha \leq \mathcal{U}_k(\mu)$. Therefore, we have $(\lambda \vee \mu)(p) \sqsupseteq f \sqcup g$ and $\alpha \leq \mathcal{U}_k(\lambda) \wedge \mathcal{U}_k(\mu) = \mathcal{U}_k(\lambda \vee \mu)$. Thus $\alpha \leq (R^p_{(\mathcal{U})})_k(f \sqcup g)$. So, $(R^p_{(\mathcal{U})})_k(f \sqcup g) \geq (R^p_{(\mathcal{U})})_k(f) \wedge (R^p_{(\mathcal{U})})_k(g)$.

(RN4): From Lemma 3.3, it is enough to check (RN4*). Since for each $k \in K$,

$(R^p_{(\mathcal{U})})_k(f) \geq \bigvee_{g\in p|f}((R^p_{(\mathcal{U})})_k(g) \wedge \bigwedge_{r\not\sqsubseteq g}(R^r_{(\mathcal{U})})_k(f))$ is obvious. It is sufficient to show that the converse inequality.

Let $\alpha \in c(M)$ such that $\alpha \lhd (R^p_{(\mathcal{U})})_k(f) = \bigvee\limits_{f\sqsubseteq\lambda(p)} \mathcal{U}_k(\lambda) = \bigvee\limits_{f\sqsubseteq\lambda(p)} \bigvee\limits_{\mu\diamond\mu\geq\lambda} \mathcal{U}_k(\mu)$.

There exist $\lambda, \mu \in \mathcal{D}(X, E)$ such that $f \sqsubseteq \lambda(p), \mu \diamond \mu \geq \lambda$ and $\alpha \leq \mathcal{U}_k(\mu)$. Let $g = \mu(p)$. Then $g \in p \mid f$. Furthermore, we get

$$(R^p_{(\mathcal{U})})_k(g) = \bigvee_{g\sqsubseteq\nu(p)} \mathcal{U}_k(\nu) \geq \mathcal{U}_k(\mu) \geq \alpha.$$

Since $\sqcap\{\mu(r) \mid r \in c((L^X)^E), r \not\sqsubseteq \mu(p)\} = \mu \diamond \mu(p) \sqsupseteq \lambda(p) \sqsupseteq f$, we have $\mu(r) \sqsupseteq f$ for all $r \not\sqsubseteq g = \mu(p)$. Hence,

$\bigwedge_{r\not\sqsubseteq g}(R^r_{(\mathcal{U})})_k(f) = \bigwedge_{r\not\sqsubseteq g} \bigvee_{\nu(r)\sqsupseteq f} \mathcal{U}_k(\nu) \geq \bigwedge_{r\not\sqsubseteq g} \mathcal{U}_k(\mu) \geq \alpha$. Then,

$\alpha \leq (R^p_{(\mathcal{U})})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g}(R^r_{(\mathcal{U})})_k(f)$. Therefore, $\alpha \leq \bigvee_{g \in p|f}((R^p_{(\mathcal{U})})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g}(R^r_{(\mathcal{U})})_k(f))$. From the arbitrariness of α, we have for all $k \in K$,
$(R^p_{(\mathcal{U})})_k(f) \leq \bigvee_{g \in p|f}((R^p_{(\mathcal{U})})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g}(R^r_{(\mathcal{U})})_k(f))$. □

Theorem 4.2. *If (X, η) is an (L, M)-fuzzy (E, K)-soft cotopological space, then there is one (L, M)-fuzzy (E, K)-soft quasi-uniformity $\mathcal{U}^\eta$ on X such that the generated fuzzy soft cotopology by $\mathcal{U}^\eta$ is just η, i.e., $\eta = \eta^{\mathcal{U}^\eta}$.*

Proof. Let $f \in (L^X)^E$ and $\lambda_f : c((L^X)^E) \to (L^X)^E$ be defined as follows:

$$\lambda_f(p) = \begin{cases} f, & \text{if } p \not\sqsubseteq f, \\ \Phi_X, & \text{otherwise.} \end{cases}.$$

Then $\lambda_f \in \mathcal{D}(X, E)$ and $\lambda_f \diamond \lambda_f = \lambda_f$. Define $\mathcal{U}^\eta : K \to M^{\mathcal{D}(X,E)}$ by

$$\mathcal{U}^\eta_k(\lambda) = \bigvee\{\bigwedge_{i=1}^{n} \eta_k(f_i) \mid \lambda \leq \vee_{i=1}^n \lambda_{f_i}, n \in \mathbb{N}\}.$$

It is easy to verify that $\mathcal{U}^\eta$ is an (L, M)-fuzzy (E, K)-soft quasi-uniformity on X.

Now we prove that $\eta = \eta^{\mathcal{U}^\eta}$. Noting that $\lambda_g(p) = g$ whenever $p \not\sqsubseteq g$, from the definition of $\eta^{\mathcal{U}^\eta}$, we have

$$\begin{aligned} \eta_k^{\mathcal{U}^\eta}(g) &= \bigwedge_{p \not\sqsubseteq g} \bigvee_{g \sqsubseteq \lambda(p)} \bigvee\{\bigwedge_{i=1}^{n} \eta_k(f_i) \mid \lambda \leq \vee_{i=1}^n \lambda_{f_i}, n \in \mathbb{N}\} \\ &\geq \bigwedge_{p \not\sqsubseteq g} \eta_k(g) = \eta_k(g), \text{ for each } k \in K. \end{aligned}$$

This is to say $\eta^{\mathcal{U}^\eta} \geq \eta$. On the other hand, we have

$$\begin{aligned} \eta_k^{\mathcal{U}^\eta}(g) &= \bigwedge_{p \not\sqsubseteq g} \bigvee_{g \sqsubseteq \lambda(p)} \bigvee\{\bigwedge_{i=1}^{n} \eta_k(f_i) \mid \lambda \leq \vee_{i=1}^n \lambda_{f_i}, n \in \mathbb{N}\} \\ &\leq \bigwedge_{p \not\sqsubseteq g} \bigvee\{\bigwedge_{i=1}^{n} \eta_k(f_i) \mid g \sqsubseteq \sqcup_{i=1}^n \lambda_{f_i}(p), n \in \mathbb{N}\} \\ &= \bigwedge_{p \not\sqsubseteq g} \bigvee\{\bigwedge_{i=1}^{m} \eta_k(f_i) \mid g \sqsubseteq \sqcup_{i=1}^m f_i \not\sqsupseteq p, m \in \mathbb{N}\} \\ &\leq \bigwedge_{p \not\sqsubseteq g} \bigvee\{\eta_k(\sqcup_{i=1}^m f_i) \mid g \sqsubseteq \sqcup_{i=1}^m f_i \not\sqsupseteq p, m \in \mathbb{N}\} \\ &\leq \bigwedge_{p \not\sqsubseteq g} \bigvee\{\eta_k(f) \mid g \sqsubseteq f \not\sqsupseteq p\} = \eta_k(g), \text{ for all } k \in K. \end{aligned}$$

□

Theorem 4.3. *Let $(X, \mathcal{U})$ be an (L, M)-fuzzy (E, K)-soft quasi-uniform space and $R^p_{\mathcal{U}} : K \to M^{(L^X)^E}$ be defined by follows: for each $k \in K$ and $f \in (L^X)^E$,*

$$(R^p_{\mathcal{U}})_k(f) = \bigvee_{p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)} \mathcal{U}_k(\lambda).$$

Then the set $\mathcal{R}_{\mathcal{U}} = \{R^p_{\mathcal{U}} \mid p \in c((L^X)^E)\}$ is a topological (L, M)-fuzzy (E, K)-soft remote neighborhood system.

Proof. By Lemma 3.3, we only need to check (RN4*). Since for each $k \in K$, $(R^p_{\mathcal{U}})_k(f) \geq \bigvee_{g \in p|f} ((R^p_{\mathcal{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathcal{U}})_k(f))$ is obvious, it is sufficient to prove the converse inequality. Let $k \in K$ and $\alpha \in c(M)$ such that

$\alpha \lhd (R^p_{\mathcal{U}})_k(f) = \bigvee_{p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)} \mathcal{U}_k(\lambda) = \bigvee_{p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)} \bigvee_{\mu \diamond \mu \geq \lambda} \mathcal{U}_k(\mu).$

Then there exist $\lambda, \mu \in \mathcal{D}(X,E)$ with $p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)$ such that $\mu \diamond \mu \geq \lambda$ and $\alpha \leq \mathcal{U}_k(\mu)$. Let $g = \sqcup_{s \not\sqsubseteq f'} \mu'(s)$. Then $g \in p \mid f$. Since

$\sqcup_{t \not\sqsubseteq g'} \mu'(t) = \sqcup_{t \not\sqsubseteq \sqcap_{s \not\sqsubseteq f'} \mu(s)} \mu'(t) = \sqcup_{s \not\sqsubseteq f'} \sqcup_{t \not\sqsubseteq \mu(s)} \mu'(t) = \sqcup_{s \not\sqsubseteq f'} (\mu \diamond \mu(s))' \leq \sqcup_{s \not\sqsubseteq f'} (\lambda(s))'$,

we have $p \not\sqsubseteq \sqcup_{t \not\sqsubseteq g'} \mu'(t)$. Hence,

$(R^p_{\mathcal{U}})_k(f) = \bigvee_{p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)} \mathcal{U}_k(\lambda) \geq \mathcal{U}_k(\mu) \geq \alpha$, for each $k \in K$. Furthermore, we have

$\bigwedge_{r \not\sqsubseteq g} (R^r_{\mathcal{U}})_k(f) = \bigwedge_{r \not\sqsubseteq g} \bigvee_{r \not\sqsubseteq \sqcup_{s \not\sqsubseteq f'} \lambda'(s)} \mathcal{U}_k(\lambda) \geq \bigwedge_{r \not\sqsubseteq g} \mathcal{U}_k(\mu) \geq \alpha$, for each $k \in K$. Then

$\alpha \leq (R^p_{\mathcal{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathcal{U}})_k(f)$. Therefore, $\alpha \leq \bigvee_{g \in p|f} ((R^p_{\mathcal{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathcal{U}})_k(f))$,

for each $k \in K$. From the arbitrariness of α, we have

$(R^p_{\mathcal{U}})_k(f) \leq \bigvee_{g \in p|f} ((R^p_{\mathcal{U}})_k(g) \wedge \bigwedge_{r \not\sqsubseteq g} (R^r_{\mathcal{U}})_k(f)).$ □

Theorem 4.4. *If $(X,\mathcal{U})$ is a pointwise (L,M)-fuzzy (E,K)-soft uniform space, then $(R^p_{\mathcal{U}})_k \leq (R^p_{(\mathcal{U})})_k$ for each $k \in K$ and $p \in c((L^X)^E)$.*

Proof. Since $\mathcal{U}$ is a pointwise (L,M)-fuzzy (E,K)-soft uniformity, there exists a base $\mathcal{B}$ of $\mathcal{U}$ such that $\mathcal{B}_k(\lambda) = \Phi_X$ for all $\lambda \in \mathcal{D}(X,E) \setminus \mathcal{D}_s(X,E)$ and $k \in K$. Let $k \in K$ and $\alpha \in c(M)$ with

$\alpha \lhd (R^p_{\mathcal{U}})_k(f) = \bigvee_{p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)} \mathcal{U}_k(\lambda) = \bigvee_{p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \lambda'(r)} \bigvee_{\mu \geq \lambda} \mathcal{B}_k(\mu).$

Then there exist $\lambda \in \mathcal{D}(X,E)$ and $\mu \in \mathcal{D}_s(X,E)$ such that $p \not\sqsubseteq \sqcup_{r \not\sqsubseteq f'} \mu'(r)$. Since μ is symmetric, we have $\mu(r) \sqsupseteq f$. Therefore,

$$\alpha \leq \mathcal{B}_k(\mu) \leq \bigvee_{f \sqsubseteq \lambda(r)} \mathcal{U}_k(\lambda) = (R^p_{(\mathcal{U})})_k(f).$$

From the arbitrariness of α, we have for each $k \in K$ and $f \in ((L^X)^E)$,

$$(R^p_{\mathcal{U}})_k(f) \leq (R^p_{(\mathcal{U})})_k(f).$$

Hence, we obtain the desired inequality. □

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