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# APPROXIMATING CLASSES OF FUNCTIONS DEFINED BY A GENERALISED MODULUS OF SMOOTHNESS

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Abstract. In the present paper, we use a generalised shift operator in order to define a generalised modulus of smoothness. By its means, we define generalised Lipschitz classes of functions, and we give their constructive characteristics. Specifically, we prove certain direct and inverse types theorems in approximation theory for best approximation by algebraic polynomials.

## 1. Introduction

In [4], a generalised shift operator was introduced, by its means the generalised modulus of smoothness was defined, and Jackson's and its converse type theorems were proved for this modulus.

In the present paper, we make use of this modulus of smoothness to define generalised Lipschitz classes of functions. We prove the coincidence of such a generalised Lipshcitz class wth the class of functions having a given order of decrease of best approximation by algebraic polynomials.

## 2. Definitions

By $L_p[a,b]$ we denote the set of functions $f$ such that for $1 \le p < \infty$ $f$ is a measurable function on the segment $[a,b]$ and

$$\|f\|_p = \left( \int_a^b |f(x)|^p \, dx \right)^{1/p} < \infty,$$

and for $p = \infty$ the function $f$ is continuos on the segment $[a,b]$ and

$$\|f\|_\infty = \max_{a \le x \le b} |f(x)|.$$

In case that $[a,b] = [-1,1]$ we simply write $L_p$ instead of $L_p[-1,1]$.

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Denote by $L_{p,\alpha}$ the set of functions $f$ such that $f(x)(1-x^2)^\alpha \in L_p$, and put

$$\|f\|_{p,\alpha} = \|f(x)(1-x^2)^\alpha\|_p.$$

Denote by $E_n(f)_{p,\alpha}$ the best approximation of a function $f \in L_{p,\alpha}$ by algebraic polynomials of degree not greater than $n-1$, in $L_{p,\alpha}$ metrics, i.e.,

$$E_n(f)_{p,\alpha} = \inf_{P_n} \|f - P_n\|_{p,\alpha}\,,$$

where $P_n$ is an algebraic polynomial of degree not greater than $n-1$.

By $E(p,\alpha,\lambda)$ we denote the class of functions $f \in L_{p,\alpha}$ satisfying the condition

$$E_n(f)_{p,\alpha} \le Cn^{-\lambda},$$

where $\lambda > 0$ and $C$ is a constant not depending on $n$ ($n \in \mathbb{N}$).

Define generalised shift operator $\hat{\tau}_t\,(f,x)$ by

$$\hat{\tau}_t\,(f,x) = \frac{1}{\pi(1-x^2)\cos^4\frac{t}{2}} \int_0^\pi B_{\cos t}(x,\cos\varphi,R)f(R)\,d\varphi,$$

where

$$R = x\cos t - \sqrt{1-x^2}\sin t\cos\varphi,$$

$$B_y(x,z,R) = 2\Big(\sqrt{1-x^2}y + xz\sqrt{1-y^2} + \sqrt{1-x^2}(1-y)(1-z^2)\Big)^2 - (1-R^2). \tag{2.1}$$

For a function $f \in L_{p,\alpha}$, define the generalised modulus of smoothness by

$$\hat{\omega}(f,\delta)_{p,\alpha} = \sup_{|t|\le\delta} \|\hat{\tau}_t\,(f,x) - f(x)\|_{p,\alpha}\,.$$

Consider the class $H(p,\alpha,\lambda)$ of functions $f \in L_{p,\alpha}$ satisfying the condition

$$\hat{\omega}(f,\delta)_{p,\alpha} \le C\delta^\lambda,$$

where $\lambda > 0$ and $C$ is a constant not depending on $\delta$.

Put $y = \cos t$, $z = \cos\varphi$ in the operator $\hat{\tau}_t\,(f,x)$, denote it by $\tau_y\,(f,x)$ and rewrite it in the form

$$\tau_y\,(f,x) = \frac{4}{\pi(1-x^2)(1+y)^2} \int_{-1}^1 B_y(x,z,R)f(R)\frac{dz}{\sqrt{1-z^2}},$$

where $R$ and $B_y(x,z,R)$ are defined in (2.1).

By $P_\nu^{(\alpha,\beta)}(x)$ ($\nu = 0,1,\dots$) we denote the Jacobi polynomials, i.e., the algebraic polynomials of degree $\nu$, orthogonal with the weight function $(1-x)^\alpha(1+x)^\beta$ on the segment $[-1,1]$, and normed by the condition

$$P_\nu^{(\alpha,\beta)}(1) = 1 \quad (\nu = 0,1,\dots).$$

Denote by $a_n(f)$ the Fourier–Jacobi coefficients of a function $f$, integrable with the weight function $(1-x^2)^2$ on the segment $[-1,1]$, with respect to the system of Jacobi polynomials $\left\{P_n^{(2,2)}(x)\right\}_{n=0}^\infty$, i.e.,

$$a_n(f) = \int_{-1}^1 f(x)P_n^{(2,2)}(x)(1-x^2)^2\,dx \quad (n = 0,1,\dots).$$

## 3. Auxiliary statements

In order to prove our results we need the following theorem.

**Theorem 3.1.** *Let the numbers $p$ and $\alpha$ be such that $1 \le p \le \infty$;*

$$\begin{aligned} 1/2 < \alpha \le 1 & \qquad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \qquad \text{for } 1 < p < \infty, \\ 1 \le \alpha < 3/2 & \qquad \text{for } p = \infty. \end{aligned}$$

*If $f \in L_{p,\alpha}$, then for every natural number $n$*

$$C_1 E_n(f)_{p,\alpha} \le \hat{\omega}(f, 1/n)_{p,\alpha},$$

*where the positive constant $C_1$ does not depend on $f$ and $n$.*

Theorem 3.1 was proved in [4] and, in more general form, in [5]. It is known as a Jackson's type theorem.

We also need the following lemmas.

**Lemma 3.1.** *The operator $\tau_y(f, x)$ has the following properties:*

1) *it is linear,*
2) $\tau_1(f, x) = f(x)$,
3) $\tau_y\left(P_\nu^{(2,2)}, x\right) = P_\nu^{(2,2)}(x) P_\nu^{(0,4)}(y) \quad (\nu = 0, 1, \dots)$,
4) $\tau_y(1, x) = 1$,
5) $a_n(\tau_y(f, x)) = a_n(f) P_n^{(0,4)}(y) \quad (n = 0, 1, \dots)$.

Lemma 3.1 was proved in [4]

**Lemma 3.2.** *Let the numbers $p$ and $\alpha$ be such that $1 \le p \le \infty$;*

$$\begin{aligned} 1/2 < \alpha \le 1 & \qquad \text{for } p = 1, \\ 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \qquad \text{for } 1 < p < \infty, \\ 1 \le \alpha < 3/2 & \qquad \text{for } p = \infty. \end{aligned}$$

*If $f \in L_{p,\alpha}$, then*

$$\|\hat{\tau}_t(f, x)\|_{p,\alpha} \le \frac{C}{\cos^4 \frac{t}{2}} \|f\|_{p,\alpha},$$

*where constant $C$ does not depend on $f$ and $t$.*

Lemma 3.2 was proved in [4].

## 4. Statement of results

**Theorem 4.1.** *Let $p$, $\alpha$ and $\lambda$ be given numbers such that $1 \le p \le \infty$;*

$$\begin{aligned} 1 - \frac{1}{2p} < \alpha < \frac{3}{2} - \frac{1}{2p} & \qquad \text{for } 1 \le p < \infty, \\ 1 \le \alpha < \frac{3}{2} & \qquad \text{for } p = \infty. \end{aligned}$$

*and $0 < \lambda < 2$. Let $f \in L_{p,\alpha}$. If*

$$E_n(f)_{p,\alpha} \le M n^{-\lambda},$$

*then*

$$\hat{\omega}(f,\delta)_{p,\alpha} \leq CM\delta^{\lambda},$$

*where constant $C$ does not depend on $f$, $M$ and $\delta$.*

*Proof.* Let $P_n(x)$ be an algebraical polynomial of degree not greater than $n-1$ such that

$$\|f - P_n\|_{p,\alpha} = E_n(f)_{p,\alpha} \quad (n = 1, 2, \ldots).$$

We define algebraical polynomials $Q_k(x)$ by

$$Q_k(x) = P_{2^k}(x) - P_{2^{k-1}}(x) \quad (k = 1, 2, \ldots)$$

and $Q_0(x) = P_1(x)$. Since for $k \geq 1$

$$\|Q_k\|_{p,\alpha} = \left\|P_2^k - P_{2^{k-1}}\right\|_{p,\alpha} \leq \|P_{2^k} - f\|_{p,\alpha} + \|f - P_{2^{k-1}}\|_{p,\alpha} = E_{2^k}(f)_{p,\alpha} + E_{2^{k-1}}(f)_{p,\alpha},$$

then by the conditions of the theorem we have

$$\|Q_k\|_{p,\alpha} \leq C_1 M 2^{-k\lambda}. \tag{4.1}$$

Taking into consideration property 4) in Lemma 3.1 of the operator $\tau_y$, without loss of generality we may suppose that $t \neq 0$. For $0 < |t| \leq \delta$ we estimate

$$I = \|\hat{\tau}_t(f,x) - f(x)\|_{p,\alpha}.$$

For every positive integer $N$, taking into account property 1) in Lemma 3.1 and the linearity of the operator $\tau_t(f,x)$, we get

$$I \leq \|\hat{\tau}_t(f - P_{2^N}, x) - (f(x) - P_{2^N}(x))\|_{p,\alpha} + \|\hat{\tau}_t(P_{2^N}, x) - P_{2^N}(x)\|_{p,\alpha}.$$

Since

$$P_{2^N}(x) = \sum_{k=0}^{N} Q_k(x),$$

we have

$$I \leq \|\hat{\tau}_t(f - P_{2^N}, x) - (f(x) - P_{2^N}(x))\|_{p,\alpha} + \sum_{k=0}^{N} \|\hat{\tau}_t(Q_k, x) - Q_k(x))\|_{p,\alpha} = J + \sum_{k=1}^{N} I_k.$$

Let $N$ be chosen in such a way that

$$\frac{\pi}{2^N} < \delta \leq \frac{\pi}{2^{N-1}}. \tag{4.2}$$

We prove the following inequalities

$$J \leq C_2 M \delta^{\lambda} \tag{4.3}$$

and

$$I_k \leq C_3 M 2^{-k\lambda}, \tag{4.4}$$

where constants $C_2$ and $C_3$ do not depend on $f$, $M$, $\delta$ and $k$.

First we consider $J$. By Lemma 3.2, taking into account that $|t| \leq \delta$, we have

$$\left\|\hat{\tau}_t\left(f-P_{2^N}, x\right)-\left(f(x)-P_{2^N}(x)\right)\right\|_{p, \alpha} \leq \frac{C_4}{\left(\cos \frac{t}{2}\right)^4}\left\|f-P_{2^N}\right\|_{p, \alpha} = C_5 E_{2^N}(f)_{p, \alpha}$$

Therefore, the condition of the theorem and inequality (4.2) yield

$$\left\|\hat{\tau}_t\left(f-P_{2^N}, x\right)-\left(f(x)-P_{2^N}(x)\right)\right\|_{p, \alpha} \leq C_6 M 2^{-N \lambda} \leq C_7 M \delta^\lambda,$$

which proves inequality (4.3).

Now we prove inequality (4.4). Note that, taking into consideration Lemma 3.2, we have

$$\left\|\hat{\tau}_t\left(Q_k\right)\right\|_{p, \alpha} \leq \frac{C_8}{\left(\cos \frac{t}{2}\right)^4}\left\|Q_k\right\|_{p, \alpha}.$$

Hence,

$$I_k \leq \frac{C_9}{\left(\cos \frac{t}{2}\right)^4} M 2^{-k \lambda},$$

which proves inequality (4.4).

Inequalities (4.3), (4.4) and (4.2) yield

$$I \leq C_{10} M\left(\delta^\lambda+\sum_{k=1}^N 2^{-k \lambda}\right) \leq C_{11} M\left(\delta^\lambda+2^{-N \lambda}\right) \leq C_{12} M \delta^\lambda.$$

Theorem 4.1 is proved. □

**Theorem 4.2.** *Let $p$, $\alpha$ and $\lambda$ be given numbers such that $1 \leq p \leq \infty$, $\lambda > 0$;*

$$1-\frac{1}{2p}<\alpha<\frac{3}{2}-\frac{1}{2p} \quad \text{for } 1 \leq p<\infty,$$
$$1 \leq \alpha<\frac{3}{2} \quad \text{for } p=\infty.$$

*Let $f \in L_{p,\alpha}$. If*

$$\hat{\omega}(f, \delta)_{p, \alpha} \leq M \delta^\lambda,$$

*then*

$$E_n(f)_{p, \alpha} \leq C M n^{-\lambda},$$

*where constant $C$ does not depend on $f$, $M$ and $n$.*

*Proof.* Let $\delta = \frac{1}{n}$. Then, taking into account Theorem 3.1, we obtain

$$E_n(f)_{p, \alpha} \leq \frac{1}{C_1} \hat{\omega}\left(f, \frac{1}{n}\right)_{p, \alpha} \leq C M n^{-\lambda}.$$

Theorem 4.2 is proved. □

**Theorem 4.3.** *Let $p$, $\alpha$ and $\lambda$ be given numbers such that $1 \leq p \leq \infty$;*

$$1-\frac{1}{2p}<\alpha<\frac{3}{2}-\frac{1}{2p} \quad \text{for } 1 \leq p<\infty,$$
$$1 \leq \alpha<\frac{3}{2} \quad \text{for } p=\infty.$$

*Then for $0 < \lambda < 2$ the classes of functions $H(p, \alpha, \lambda)$ coincide with the class $E(p, \alpha, \lambda)$.*

*Proof.* Note that, under the condition of the theorem, Theorem 4.2 implies the inclusion

$$H(p, \alpha, \lambda) \subseteq E(p, \alpha, \lambda),$$

while Theorem 4.1 implies the converse inclusion

$$E(p, \alpha, \lambda) \subseteq H(p, \alpha, \lambda).$$

Hence we conclude that the assertion of Theorem 4.3 is implied by Theorems 4.2 and 4.1. □

Note that analogues of Theorems 4.2, 4.1 and 4.3 for another generalised shift operator were proved in [1] and, in more general forms, in [3, 2].

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