Mathematical Sciences And Applications E-Notes
Volume 1 No. 2 pp. 90–102 (2013) ©MSAEN

SOME PROPERTIES OF POLYNOMIALS G AND N, AND TABLES OF KNOTS AND LINKS

İSMET ALTINTAŞ

(Communicated by Murat TOSUN)

Abstract. In [1], we have constructed a polynomial invariant of regular isotopy, , for oriented knot and link diagrams L. From by multiplying it by normalizing factor, we obtained an ambient isotopy invariant, , for oriented knots and links. In this paper, we give some properties of these polynomials. We also calculate the polynomials and of the knots through nine crossings and the two-component links through eight crossing.

1. Introduction

In [1], we constructed a one-variable Laurent polynomial invariant of oriented knots and links, and denoted it by N_L for an oriented link diagram L. The primary version of this is an invariant of regular isotopy for oriented knot and link diagrams, denoted G_L. We have seen that G_L is a special case of a general polynomial, *[L]*, well defined on equivalence classes of oriented diagrams. Since the polynomial $N_L$obtained from G_L by multiplying it by a normalizing factor is an invariant for oriented knots and links. The polynomial N_L may be compared with the original Jones polynomialcite [2], [3], [4] and with the normalized bracket polynomial [5], [6], [7], [8]. In fact, the polynomial N_L yields the Jones polynomial and the normalized bracket polynomial (See [1], Theorem 2.11). Thus, in principle this gives an oriented state model for the Jones polynomial. In [1], we also used the polynomial G_L to prove that the number of crossings in connected, reduced alternating projection of a link L is a topological invariant of L. This is a remarkable application of the polynomial G_L. It solves some of old conjectures about alternating knots due to Tait, see [9],[10], [11].

This paper is organized as follows. In this section, we give the definitions of the polynomials G_L and N_L for an oriented link diagram L. The polynomial N_L yields the Jones polynomial and the normalized bracket polynomial. In Section

Date: Received: October 9, 2012; Accepted: May 18, 2013.

2000 *Mathematics Subject Classification.* 57M25.

Key words and phrases. G-polynomial, N-polinomial, Jones Polynomial, regular isotopy, ambient isotopy, tables of knots and links.

This article is the written version of author's plenary talk delivered on September 03-07, 2012 at 1st International Euroasian Conference on Mathematical Sciences and Applications IECMSA-2012 at Prishtine, Kosovo.

2, we prove some properties of the polynomials G_L and N_L. In Section 3, we calculate the polynomials G_L and N_L for all the knots through nine crossings and two-component links through eight crossings.

A link L of k components is a subset of $R^3 \subset R^3 \cup \{\infty\} = S^3$, consisting of k disjoint piecewise linear simple closed curves; a knot is a link with one component. Although links live in R^3, we usually represent them by link diagrams: the regular projections of links into with over passing curves specified.

Definition 1.1. Let L denote an oriented knot or link diagram. Then $G_L \in Z[p, p^{-1}]$ is a Laurent polynomial in the variable p assigned to oriented link diagram L. The polynomial satisfies the properties:

(1) $G_O = 1$, $G_{O \sqcup L} = rG_L$,
(2) $p^{-1}G_{L_+} - pG_{L_-} = (p^{-1} - p)G_{L_0}$,
(3) G_L is an invariant of regular isotopy,

where L_+, L_- and L_0 are diagrams in Figure 1, O is the oriented diagram with zero-crossing of the unknot and $\sqcup$ denotes disjoint union and $r = -(p^{-1}+p)$. For detail, see [1].

The three regular diagrams L_+, L_- and L_0 formed in Figure 1 are called skein diagrams and the relation (the axiom (2) of Definition 1.1) between Laurent polynomials of these skein diagrams is called the skein relation. Also, an operation that replaces that one of L_+, L_- and L_0 by the other two is called a skein operation [12].

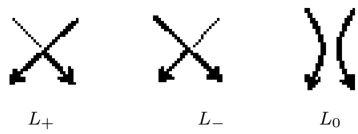


FIGURE 1. Skein diagrams

It is possible to create an invariant of ambient isotopy associated with the polynomial G_L for oriented diagram L. For this we use the writhe, $\omega(L)$, of the oriented link diagram L, ($\omega(L)$ is the sum of all crossing signs). Recall that $\omega(L)$ is also a regular isotopy invariant. Thus we may give the following definition.

Definition 1.2. We define a polynomial $N_L \in Z[p, p^{-1}]$ for an oriented link diagram L by the formula

$$N_L(p) = (-p^{-1})^{-\omega(L)} G_L(p)$$

where $\omega(L)$ is the writhe of the oriented link diagram L. Call a normalized polynomial of the polynomial by the writhe. The normalized polynomial N_L is an invariant of ambient isotopy (see [1], Theorem 2.3).

Proposition 1.1. *The polynomial satisfies the following skein relation:*

$$p^{-2}N_{L_+} - p^2N_{L_-} = (p - p^{-1})N_{L_0} \tag{1.1}$$

where L_+, L_- and L_0 are skein diagrams in Figure 1.

Proof. We have from the axiom (2) of the definition of G_L the following skein relation:

$$p^{-1}G_{L_+} - pG_{L_-} = (p^{-1}-p)G_{L_0}.$$

Let $\omega(L_0) = \omega$. Since $\omega(L_+) = \omega+1$ and $\omega(L_-) = \omega-1$, with $a = -p^{-1}$, we can write that

$$p^{-1}a^{-\omega}G_{L_+} - pa^{-\omega}G_{L_-} = (p^{-1}-p)a^{-\omega}G_{L_0}.$$

Hence

$$p^{-1}aa^{-(\omega+1)}G_{L_+} - pa^{-1}a^{-(\omega-1)}G_{L_-} =(p^{-1}-p)a^{-\omega}G_{L_0}.$$

or

$$p^{-2}N_{L_+} - p^2N_{L_-} = (p - p^{-1})N_{L_0}. \qquad \square$$

We may show that, in fact the polynomial is an oriented state model for the original Jones polynomial and that it is also a version of the normalized bracket polynomial.

For now it is suffices to say that the Jones polynomial $V_L(t)$ is determined by the axioms:[2],[4].

(1) $V_O = 1,$

(2) $t^{-1}V_{L_+} - tV_{L_-} = (t^{\frac{1}{2}}-t^{-\frac{1}{2}})V_{L_0}$

(3) $V_L(t)$ is an invariant of ambient isotopy

where L_+, L_- and L_0 are diagrams in Figure 1, O is the oriented diagram with zero-crossing of the unknot.

The bracket polynomial $< L > (A)$ is determined by the axioms: [5],[6],[7],[8].

(1) $< L_+ >= A < L_0 > +B < L_\infty >$

$< L_- >= A < L_\infty > +B < L_0 >$

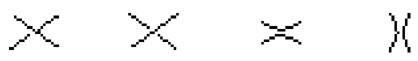
(2) $< O \sqcup L >= d < L >, < O >= 1$

where L_+, L_- and L_0 and L_∞ are diagrams in Figure 2. O denotes the diagram with zero-crossing of the unknot and $\sqcup$ denotes disjoint union. By setting $B = A^{-1}$ and d=-A^2-A^{-2}, $< L >$ is an invariant of regular isotopy [5],[6],[7],[8]. But it is not an invariant of ambient isotopy. The associated invariant of ambient isotopy for is the Laurent polynomial defined by the formula

$$f_L(A) = (-A^3)^{-\omega(L)}$$

where $\omega(L)$ is the writhe of L and is defined for oriented links by forgetting the orientation. Since $f_L(A)$ is an invariant of ambient isotopy [5],[6],[7],[8], it is called the normalized bracket polynomial for oriented links L and it satisfies the following skein relation:

(1.2) $$-A^4f_{L_+} + A^{-4}fL- = (A^2-A^{-2})f_{L_0} .$$



L_+ L_- L_0 L_∞

FIGURE 2. Non-oriented skein diagrams

Proposition 1.2. *$N_L(t^{1/2}) = V_L(t)$ and $N_L(A^{-2}) = f_L(A)$.*

Proof. By taking $p = t^{1/2}$ in the skein relation (1.1) we calculate that

$$t^{-1}N_{L_+}- tN_{L_-} = (t^{\frac{1}{2}}-t^{-\frac{1}{2}})N_{L_0} .$$

This proves the axiom (2) of the Jones polynomial. The axioms (1) and (3) are follows directly from the corresponding facts about N_L.

Similarly, by taking $p = A^{-2}$ in the same relation we have

$$-A^4 N_{L_+} + A^{-4} N_{L_-} = (A^2 - A^{-2}) N_{L_0}.$$

This is identical the relation (1.2). Thus the proof is completed. □

2. THE BASIC CHARACTERISTICS OF POLYNOMIALS G AND N

In this section, we give some properties of the polynomials G_L and N_L. From the skein relation in the axiom (2) of Definition 1.1, we can easily write the following skein operations:

(1.3) $$G_{L_+} = p^2 G_{L_-} + (1-p^2) G_{L_0},$$

(1.4) $$G_{L_-} = p^{-2} G_{L_+} + (1-p^{-2}) G_{L_0}.$$

Similarly, from the skein relation (1.1) we can write the following skein operations:

(1.5) $$N_{L_+} = p^4 N_{L_-} + (p^3-p) N_{L_0},$$

(1.6) $$N_{L_-} = p^{-4} N_{L_+} + (p^{-3}-p^{-1}) N_{L_0}.$$

We use the above equations to prove some properties of the polynomials G_L and N_L. We also use to calculate the polynomials G_L and N_L of knots and links in next section.

Theorem 2.1. *Let $-L$ be the knot with the reverse orientation to on the knot L. Then*

$$G_{-L}(p) = G_L(p) \text{ and } N_{-L}(p) = N_L(p).$$

Proof. In order to calculate $G_{-L}(p)$, we may use the same skein diagram as for the calculation of $G_L(p)$. Indeed, if we change the orientation on the knot L, does not change its sing. Because the polynomial $G_L(p)$ is independent the orientation of the knot. The same thing is also true for the polynomial $N_L(p)$. Hence the results follows. □

As a consequence of the above theorem, the polynomial N_L is not a useful tool in the study of whether or not a knot is invertible. However, the polynomial is a powerful tool in the study of amphicheirality of a knot.

Theorem 2.2. *Let L^* be the mirror image a knot (or link) L. Then*

$$G_{L^*}(p) = G_L(p^{-1}) \text{ and } N_{L^*}(p) = N_L(p^{-1}).$$

Therefore, if a knot is amphicherial, then $N_L(p) = N_L(p^{-1})$, i.a., $N_L(p)$ is symmetric.

Proof. We prove that $N_{L^*}(p) = N_L(p^{-1})$ (Similarly, one prove that $G_{L^*}(p) = G_L(p^{-1})$). Let L be a knot and L^* be the mirror image of L. Then the sings of a crossing C of L and the equivalent crossing of L^* are opposite. If the sing of the crossing C of L is positive, then the sing of the same crossing C of L^* is negative. If we take $L = L_+$ at the crossing C, then $L^* = L_-$ at the same crossing C. Thus, the polynomial N_L can be written as the skein relation (1.5) and the polynomial N_{L^*} can be written as the skein relation (1.6). Since the relation (1.6) is equal the relation (1.5) by replacement of p by p^{-1}, it follows that $N_{L^*}(p) = N_L(p^{-1})$. □

Theorem 2.3. *Suppose that $L_1 \# L_2$ is the connected sum of two links, then*

$$N_{L_1 \# L_2}(p) = N_{L_1}(p) N_{L_2}(p).$$

Proof. The connected sum link $L_1\#L_2$ has a diagram that appears as in Figure 3. Without cutting the strands of L_1 , let's flip that part of the diagram corresponding to in two different ways, to get the two links L_+ and L_- (Figure 4). Note that these diagrams are still diagrams (projections) of $L_1\#L_2$. In addition, L_0 is simply the disjoint union $L_1 \sqcup L_2$.

From the skein relation $p^{-2}N_{L_+} - p^2 N_{L_-} = (p - p^{-1})N_{L_0}$ (see, (1.1)), we can write the following relation:

$$p^{-2}N_{L_1\#L_2} - p^2 N_{L_1\#L_2} = (p - p^{-1})N_{L_1 \sqcup L_2}.$$

We know that as a result of the axiom (1) of the definition 1.1

$$N_{L_1 \sqcup L_2} = (-p^{-1} - p)N_{L_1}(p)N_{L_2}(p).$$

Hence, we have

$$(p^{-2} - p^2)N_{L_1\#L_2} = (p - p^{-1})(-p^{-1} - p)N_{L_1}(p)N_{L_2}(p)$$

or

$$N_{L_1\#L_2} = N_{L_1}(p)N_{L_2}(p). \qquad \square$$

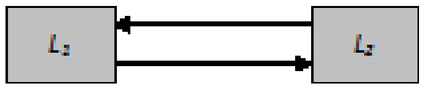


FIGURE 3. A projection for $L_1\#L_2$

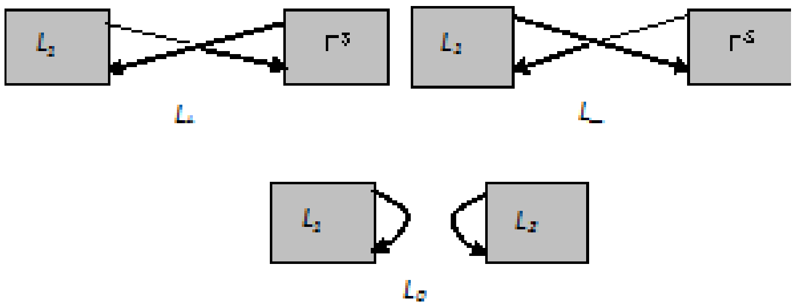


FIGURE 4. Three related links

3. KNOT TABLES, LINK TABLES AND POLYNOMIALS G AND N

In this section we give the polynomials G_L and N_L for all the knots through nine crossings and two-component links through eight crossings. The pictures of knots are the mirror images of knots in [13], (see, Appendix I). The pictures of knots through eight crossings can be found in [12], (also see [14]). The pictures of links taken from[13], (see, Appendix II). For the pictures of some knots and links, see [15].

The polynomials G_L and N_L of the knots of Appendix I are listed in Table 1, and the polynomials G_L and N_L of the links of Appendix II are listed in Table 2. On the first columns of the tables, we give Alexander and Briggs notation of the knot or link. The second columns, the third columns and the last columns contain writhe, the polynomial G_L and the polynomial N_L of the diagram of the knot and link, respectively. For the writhe in Table 1, see [14]. We have written the writhe in Table 2 as depending on chosen orientation of the link.

We use the skein operations (1.3) and (1.4) to calculate the polynomial G_L for knots and links. The polynomial N_L is given by the formula N_L=$(-p^{-1})^{-\omega(L)}G_L$, where ω is the writhe of the oriented diagram of the link L.

We now calculate the polynomials G_L and of the right-hand trefoil knot as an example. In Figure 5 we have drawn the skein diagram for the calculation of the polynomial G_L of the right-hand trefoil knot using the skein operation (1.3). It follows from the skein diagram that

$$\begin{aligned} G_L(p) &= p^2G_O(p) + (1–p^2)[p2G_{OO}(p) + (1–p^2)G_O(p)] \\ &= –p^{-1}–p^3+p^5. \end{aligned}$$

Since the writhe of the right-hand trefoil knot L,ω $=3$, by Definition 1.2, we obtain that

$$N_L(p) = p^2+p^6–p^8.$$

The third columns and the last columns of the Tables 1 and 2 contain a sequence of numbers that denote the polynomial G_L and the polynomial N_L of the knot and link, respectively. The first number, which appears in the curly brackets, is the minimum degree of the polynomial. The next sequence of numbers gives the coefficients of the polynomial, beginning with the coefficient of the minimum degree term. For example, (-5)[-1 2 -3 3 -4 3 -2 1] denotes the polynomial

$$–p^{-5}+2p^{-3}–3p^{-1} \ +3p^1–4p^3+3p^5–2p^7+p^9.$$

Generally, the notation denotes the polynomial

$$p^n(a_o \ +a_1p^2+a_2p^4+...+a_mp^{2m})$$

If the polynomial G_L or of a knot L is given by , then the polynomial G_{L^*} or N_{L^*} of the mirror image L^* of the knot L is given $(-n-2m)[a_m a_{m-1}...a_1 a_0]$.

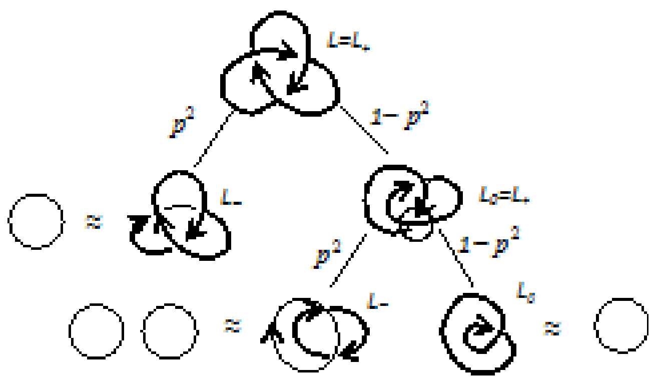


FIGURE 5. The skein operation of trefoil

TABLE 1. The polynomials G_L and N_L of the knots $0_1 - 9_{49}$

L	$\omega(L)$	$G_L(p)$	$N_L(p)$
0_1	0	1	1
3_1	3	(-1)[-1 0 -1 1]	(2)[1 0 1 -1]
4_1	0	(-4)[1 -1 1 -1 1]	(-4)[1 -1 1 -1 1]
5_1	5	(-1)[-1 0 -1 1 -1 1]	(4)[1 0 1 -1 1 -1]
5_2	-5	(7)[-1 1 -2 1 -1 1]	(2)[1 -1 2 -1 1 -1]
6_1	-2	(-2)[1 -1 2 -2 1 -1 1]	(-4)[1 -1 2 -2 1 -1 1]
6_2	2	(-4)[1 -1 2 -2 2 -2 1]	(-2)[1 -1 2 -2 2 -2 1]
6_3	0	(-6)[-1 2 -2 3 -2 2 -1]	(-6)[-1 2 -2 3 -2 2 -1]
7_1	7	(-1)[-1 0 -1 1 -1 1 -1 1]	(6)[1 0 1 -1 1 -1 1 -1]
7_2	-7	(9)[-1 1 -2 2 -2 1 -1 1]	(2)[1 -1 2 -2 2 -1 1 -1]
7_3	7	(-3)[-1 1 -2 2 -3 2 -1 1]	(4)[1 -1 2 -2 3 -2 1 -1]
7_4	-7	(9)[-1 2 -3 2 -3 2 -1 1]	(2)[1 -2 3 -2 3 -2 1 -1]
7_5	7	(-3)[-1 1 -3 3 -4 3 -2 1]	(4)[1 -1 3 -3 4 -3 2 -1]
7_6	3	(-5)[-1 2 -3 3 -4 3 -2 1]	(-2)[1 -2 3 -3 4 -3 2 -1]
7_7	-1	(-5)[-1 3 -3 4 -4 3 -2 1]	(-6)[-1 3 -3 4 -4 3 -2 1]
8_1	-4	(0)[1 -1 2 -2 2 -2 1 -1 1]	(-4)[1 -1 2 -2 2 -2 1 -1 1]
8_2	-4	(4)[1 -1 2 -2 3 -3 2 -2 1]	(0)[1 -1 2 -2 3 -3 2 -2 1]
8_3	0	(-8)[1 -1 2 -3 3 -3 2 -1 1]	(-8)[1 -1 2 -3 3 -3 2 -1 1]
8_4	0	(-6)[1 -1 2 -3 3 -3 3 -2 1]	(-6)[1 -1 2 -3 3 -3 3 -2 1]
8_5	4	(-4)[1 -1 3 -3 3 -4 3 -2 1]	(0)[1 -1 3 -3 3 -4 3 -2 1]
8_6	4	(-6)[1 -1 3 -4 4 -4 3 -2 1]	(-2)[1 -1 3 -4 4 -4 3 -2 1]
8_7	2	(-6)[-1 2 -2 4 -4 4 -3 2 -1]	(-4)[-1 2 -2 4 -4 4 -3 2 -1]
8_8	-2	(-4)[-1 2 -3 5 -4 4 -3 2 -1]	(-6)[-1 2 -3 5 -4 4 -3 2 -1]
8_9	0	(-8)[1 -2 3 -4 5 -4 3 -2 1]	(-8)[1 -2 3 -4 5 -4 3 -2 1]
8_{10}	2	(-6)[-1 2 -3 5 -4 5 -4 2 -1]	(-4)[-1 2 -3 5 -4 5 -4 2 -1]
8_{11}	4	(-6)[1 -2 4 -4 5 -5 3 -2 1]	(-2)[1 -2 4 -4 5 -5 3 -2 1]
8_{12}	0	(-8)[1 -2 4 -5 5 -5 4 -2 1]	(-8)[1 -2 4 -5 5 -5 4 -2 1]
8_{13}	2	(-8)[-1 3 -4 5 -5 5 4 -3 1]	(-6)[-1 3 -4 5 -5 5 4 -3 1]
8_{14}	-4	(-6)[1 -2 4 -5 6 -5 4 -3 1]	(-2)[1 -2 4 -5 6 -5 4 -3 1]
8_{15}	8	(-4)[1 -2 5 -5 6 -6 4 -3 1]	(4)[1 -2 5 -5 6 -6 4 -3 1]
8_{16}	2	(-8)[-1 3 -5 6 -6 6 -4 3 -1]	(-6)[-1 3 -5 6 -6 6 -4 3 -1]
8_{17}	0	(-8)[1 -3 5 -6 7 -6 5 -3 1]	(-8)[1 -3 5 -6 7 -6 5 -3 1]
8_{18}	0	(-8)[1 -4 6 -7 9 -7 6 -4 1]	(-8)[1 -4 6 -7 9 -7 6 -4 1]
8_{19}	-8	(14)[1 0 1 0 0 -1]	(6)[1 0 1 0 0 -1]
8_{20}	2	(-4)[-1 2 -1 2 -1 1 -1]	(-2)[-1 2 -1 2 -1 1 -1]
8_{21}	4	(-2)[2 -2 3 -3 2 -2 1]	(2)[2 -2 3 -3 2 -2 1]
9_1	9	(-1)[-1 0-1 1 -1 1 -1 1 -1 1]	(8)[1 01 -1 1 -1 1 -1 1 -1]
9_2	-9	(11[(-1 1-2 2 -2 2 -2 1 -1 1]	(2)[1 -1 2 -2 2 -2 2 -1 1 -1]
9_3	9	(-3)[-1 1-2 2 -3 3 -3 2 -1 1]	(6)[1 -1 2 -2 3 -3 3 -2 1 -1]
9_4	9	(-5)[-1 1-2 3 -4 3 -3 3 -2 1]	(4)[1 -1 2 -2 3 -3 3 -2 1 -1]
9_5	9	(-7)[-1 2-3 3 -4 3 -3 2 -1 1]	(2)[1 -2 3 -3 4 -3 3 -2 1 -1]
9_6	9	(-3)[-1 1-3 3 -4 5 -4 3 -2 1]	(6)[1 -13 -3 4 -5 4 -3 2 -1]
9_7	9	(-5)[-1 1-3 4 -5 5 -4 3 -2 1]	(4)[1 -13 -4 5 -5 4 -3 2 -1]
9_8	1	(-7)[-1 2-3 4 -5 5 -5 3 -2 1]	(-6)[1 -2 3 -4 5 -5 5 -3 2 -1]
9_9	9	(-3)[-1 1-3 4 -5 5 -5 4 -2 1]	(6)[1 -1 3 -4 5 -5 5 -4 2 -1]
9_{10}	9	(-5)8-1 2 -4 5 -6 5 -5 3 -1 1]	(4)[1 -2 4 -5 6 -5 5 -3 1 -1]

9_{11}	5	(-14)[1 -2 4 -5 5 -6 4 -3 2 -1]	(-9)[-1 2-4 5 -5 6 -4 3 -2 1]
9_{12}	5	(-7)[-1 2 -4 5 -6 6 -5 3 -2 1]	(-2)[1 -2 4 -5 6 -6 5 -3 2 -1]
9_{13}	9	(-5)[-1 2 -4 5 -7 6 -5 4 -2 1]	(4)[1 -2 4 -5 7 -6 5 -4 2 -1]
9_{14}	-3	(-3)[-1 2 -3 5 -6 6 -6 4 -3 1]	(-6)[1 -2 3 -5 6 -6 6 -4 3 -1]
9_{15}	-5	(-3)[1 -2 4 -6 6 -7 6 -4 2 -1]	(-8)[-1 2 -4 6 -6 7 -6 4 -2 1]
9_{16}	-9	(15)[-1 1 -4 5 -6 7 -6 5 -3 1]	(6)[1 -1 4 -5 6 -7 6 -5 3 -1]
9_{17}	-1	(-7)[-1 2 -4 5 -6 7 -6 4 -3 1]	(-6)[1 -2 4 -5 6 -7 6 -4 3 -1]
9_{18}	9	(-5)[-1 2 -5 6 -7 7 -6 4 -2 1]	(4)[1 -2 5 -6 7 -7 6 -4 2 -1]
9_{19}	-1	(-9)[-1 2 -4 6 -7 7 -6 4 -3 1]	(-8)[1 -2 4 -6 7 -7 6 -4 3 -1]
9_{20}	-5	(5)[-1 2 -4 5 -7 7 -6 5 -3 1]	(0)[1 -2 4 -5 7 -7 6 -5 3 -1]
9_{21}	5	(21)[1 -2 4 -6 7 -8 6 -5 3 -1]	(-16)[-1 2 -4 6 -7 8 -6 5 -3 1]
9_{22}	1	(-7)[-1 2 -4 6 -7 7 -7 5 -3 1]	(-6)[1 -2 4 -6 7 -7 7 -5 3 -1]
9_{23}	-9	(13)[-1 2 -5 6 -8 8 -6 5 -3 1]	(4)[1 -2 5 -6 8 -8 6 -5 3 -1]
9_{24}	1	(-9)[-1 3 -5 7 -8 7 -7 4 -2 1]	(-8)[1 -3 5 -7 8 -7 7 -4 2 -1]
9_{25}	5	(-3)[-1 2 -5 7 -8 8 -7 5 -3 1]	(2)[1 -2 5 -7 8 -8 7 -5 3 -1]
9_{26}	3	(-17)[-1 3 -5 7 -8 8 -7 4 -3 1]	(-14)[1 -3 5 -7 8 -8 7 -4 3 -1]
9_{27}	1	(-9)[-1 3 -5 7 -9 8 -7 5 -3 1]	(-8)[1 -3 5 -7 9 -8 7 -5 3 -1]
9_{28}	-3	(-1)[1 -3 5 -8 8 -9 8 -5 3 -1]	(-4)[-1 3 -5 8 -8 9 -8 5 -3 1]
9_{29}	-1	(-11)[1 -3 6 -8 8 -9 7 -5 3 -1]	(-12)[-1 3 -6 8 -8 9 -7 5 -3 1]
9_{30}	1	(-11)[1 -3 5 -8 9 -9 8 -6 3 -1]	(-10)[-1 3 -5 8 -9 9 -8 6 -3 1]
9_{31}	3	(-7)[1 -3 5 -8 9 -9 8 -6 3 -1]	(-4)[-1 3 -5 8 -9 9 -8 6 -3 1]
9_{32}	3	(-17)[-1 3 -6 9 -10 10 -9 6 -4 1]	(-14)[1 -3 6 -9 10 -10 9 -6 4 -1]
9_{33}	1	(-11)[1 -3 6 -9 10 -11 9 -7 4 -1]	(-10)[-1 3 -6 9 -10 11 -9 7 -4 1]
9_{34}	-1	(-3)[1 -4 7 -10 12 -12 10 -8 4 -1]	(-10)[-1 4 -7 10 -12 12 -10 8 -4 1]
9_{35}	-9	(11)[-1 2 -3 4 -5 3 -4 3 -1 1]	(2)[1 -2 3 -4 5 -3 4 -3 1 -1]
9_{36}	5	(-23)[1 -2 4 -6 6 -6 5 -4 2 -1]	(-18)[-1 2 -4 6 -6 6 -5 4 -2 1]
9_{37}	1	(-9)[-1 2 -5 7 -7 8 -7 4 -3 1]	(-8)[1 -2 5 -7 7 -8 7 -4 3 -1]
9_{38}	9	(-5)[-1 3 7- 8 -10 10 -8 6 -3 1]	(4)[1 -3 7 -8 10 -10 8 -6 3 -1]
9_{39}	-5	(-11)[1 -3 6 -8 9 1-0 8 -6 3 -1]	(-16)[-1 3 -6 8 -9 10 -8 6 -3 1]
9_{40}	3	(1)[1 -5 8 -11 13 -13 11 -8 4 -1]	(4)[-1 5 -8 11 -13 13 -11 8 -4 1]
9_{41}	-3	(-9)[-1 3 -5 7 -8 8 -8 5 -3 1]	(-12)[1 -3 5 -7 8 -8 8 -5 3 -1]
9_{42}	-1	(-5)[-1 1 -1 1 -1 1 -1]	(-6)[1 -1 1 -1 1 -1 1]
9_{43}	5	(-5)[-1 1 -2 2 -2 2 -2 1]	(0)[1 -1 2 -2 2 -2 2 -1]
9_{44}	-1	(-3)[-1 2 -3 3 -3 2 -2 1]	(-4)[1 -2 3 -3 3 -2 2 -1]
9_{45}	-5	(-11)[1 -2 3 -4 4 -4 3 -2]	(-16)[-1 2 -3 4 -4 4 -3 2]
9_{46}	-3	(-9)[-1 1 -1 2 -1 1 -2]	(-12)[1 -1 1 -2 1 -1 2]
9_{47}	-3	(-1)[1 -3 3 -5 5 -4 4 -2]	(-4)[-1 3 -3 5 -5 4 -4 2]
9_{48}	-5	(17)[2 -3 4 -6 4 -4 3 -1]	(12)[-2 3 -4 6 -4 4 -3 1]
9_{49}	-9	(13)[-1 2 -4 4 -5 4 -3 2]	(4)[1 -2 4 -4 5 -4 3 -2]

TABLE 2. The polynomials G_L and N_L of the Links 0_1^2–8_{16}^2

L	$w(L)$	$G_L(P)$	$N_L(P)$
0_1^2	0	(-1)[-1 -1]	(-1)[-1 -1]
2_1^2	2	(-1)[-1 0 -1 1]	(1)[-1 0 -1 1]
4_1^2	4	(-1)[-1 0 -1 1 -1]	(3)[-1 0 -1 1 -1]
5_1^2	-1	(-6)[-1 2 -1 2 1 -1]	(-7)[1 -2 1 -2 1 -1]
6_1^2	6	(-1)[-1 0 -1 1 -1 1 -1]	(2)[-1 0 -1 1 -1 1 -1]
6_2^2	-6	(9)[-1 1 -2 2 -2 1 -1]	(3)[-1 1 -2 2 -2 1 -1]
6_3^2	-2	(5)[-1 1 -3 2 -2 2 -1]	(3)[-1 1 -3 2 -2 2 -1]
7_1^2	-5	(-8)[-1 2 -2 3 -2 2 -1 1]	(-13)[1 -2 2 -3 2 -2 1 -1]
7_2^2	1	(-4)[1 -2 3 -3 4 -2 2 -1]	(-3)[-1 2 -3 3 -4 2 -2 1]
7_3^2	-3	(0)[1 -1 3 -3 3 -2 2 -1]	(-3)[-1 1 -3 3 -3 2 -2 1]
7_4^2	3	(-10)[11 2 -3 3 -2 3 -1 1]	(-13)[1 -2 3 -3 2 -3 1 -1]
7_5^2	-7	(10)[1 -2 4 -3 4 -3 2 -1]	(3)[-1 2 -4 3 -4 3 -2 1]
7_6^2	-5	(-4)[1 -3 4 -4 5 -3 3 -1]	(-9)[-1 3 -4 4 -5 3 -3 1]
7_7^2	7	(-10)[-1 1 0 1 0 1]	(-3)[1 -1 0 -1 0 -1]
7_8^2	-3	(-8)[-1 1 -1 2 -1 2]	(-11)[1 -1 1 -2 1 -2]
8_1^2	8	(-1)[-1 0 -1 1 -1 1 -1 1 -1]	(7)[-1 0 -1 1 -1 1 -1 1 -1]
8_2^2	-8	(13)[-1 1 -2 2 -3 3 -2 1 -1]	(5)[-1 1 -2 2 -3 3 -2 1 -1]
8_3^2	4	(1)[-1 1 -3 3 -4 4 -3 2 -1]	(5)[-1 1 -3 3 -4 4 -3 2 -1]
8_4^2	8	(-5)[-1 2 -4 4 -4 4 -3 1 -1]	(3)[-1 2 -4 4 -4 4 -3 1 -1]
8_5^2	-8	(11)[-1 2 -4 4 -5 4 -3 2 -1]	(3)[-1 2 -4 4 -5 4 -3 2 -1]
8_6^2	0	(3)[-1 1 -3 3 -4 3 -2 2 -1]	(3)[-1 1 -3 3 -4 3 -2 2 -1]
8_7^2	2	(-9)[-1 2 -4 4 -6 5 -4 3 -1]	(-7)[-1 2 -4 4 -6 5 -4 3 -1]
8_8^2	-2	(-3)[1 -3 4 -6 6 -6 4 -3 1]	(-5)[1 -3 4 -6 6 -6 4 -3 1]
8_9^2	-4	(1)[-1 2 -4 5 -5 4 -4 2 -1]	(-3)[-1 2 -4 5 -5 4 -4 2 -1]
8_{10}^2	0	(-9)[-1 3 -5 5 -6 5 -4 2 -1]	(-9)[-1 3 -5 5 -6 5 -4 2 -1]
8_{11}^2	8	(3)[-1 1 -4 4 -5 5 -4 3 -1]	(5)[-1 1 -4 4 -5 5 -4 3 -1]
8_{12}^2	-4	(-7)[1 -2 4 -6 5 -6 4 -3 1]	(-11)[1 -2 4 -6 5 -6 4 -3 1]
8_{13}^2	-2	(-3)[1 -4 5 -7 7 -7 5 -3 1]	(-5)[1 -4 5 -7 7 -7 5 -3 1]
8_{14}^2	8	(-5)[-1 3 -6 5 -7 6 -4 3 -1]	(3)[-1 3 -6 5 -7 6 -4 3 -1]
8_{15}^2	0	(-5)[-1 1 -2 1 -1 1 -1]	(-5)[-1 1 -2 1 -1 1 -1]
8_{16}^2	4	(-3)[-2 2 -2 2 -2 1 -1]	(1)[-2 2 -2 2 -2 1 -1]

REFERENCES

[1] Altintas, I., An oriented state model for the Jones polynomial and its applications alternating links, *Applied Mathematics and Computation*, 194, (2007), 168-178.

[2] Jones, V.F.R., A new knot polynomial and Von Neuman algebras, *Notices. Amer. Math. Soc.*, (1985).

[3] Jones, V.F.R., A new knot polynomial and Von Neuman algebras, *Bul. Amer. Math. Soc.* 12, (1985), 103-111.

[4] Jones, V.F.R., Hecke algebra representations of braid groups and link polynomial, *Ann. Math.*, 126, (1987), 335-388.

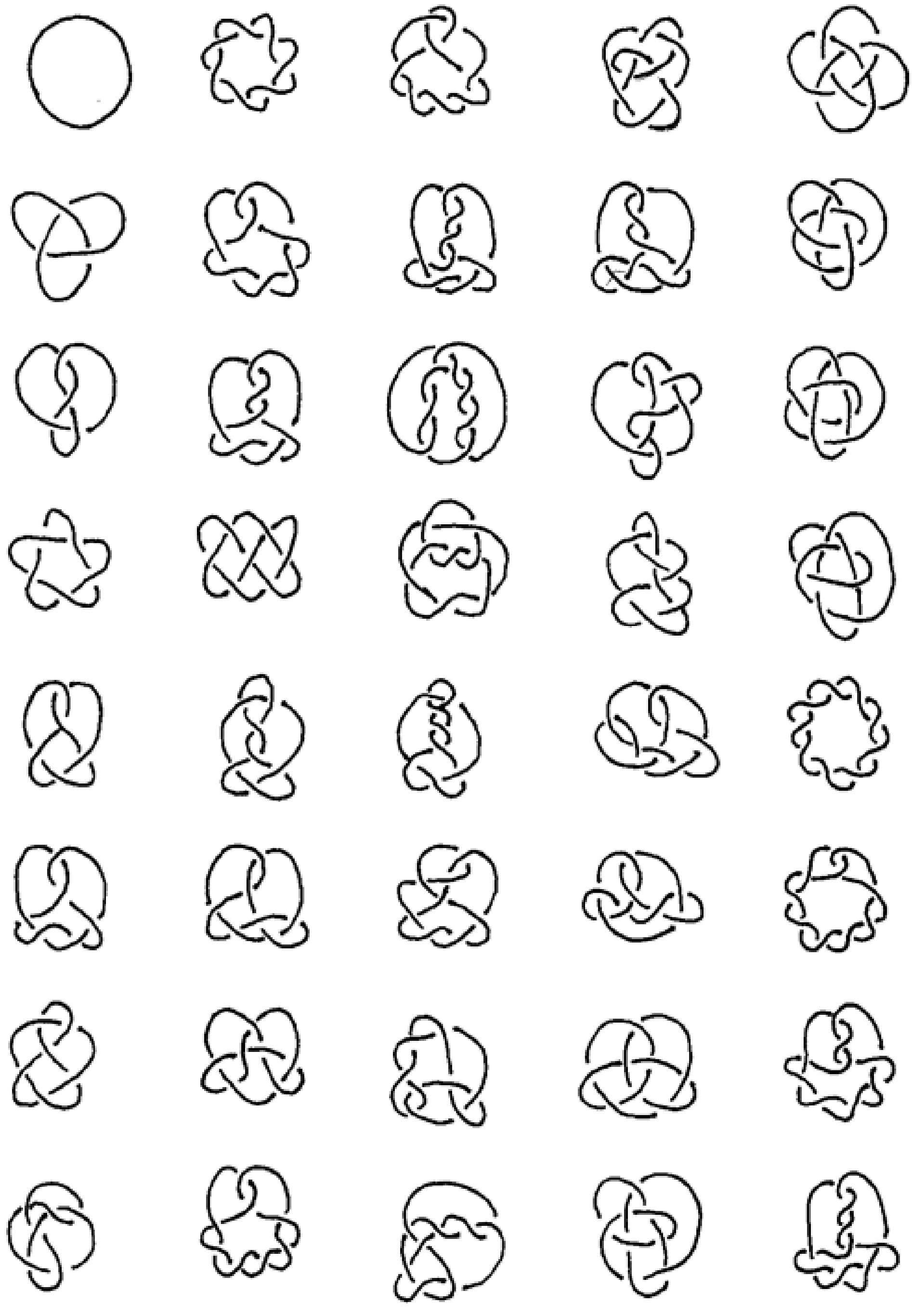
[5] Kauffman, L.H., State models and the Jones polynomial, *Topology*, 26, (1987), 395-407.

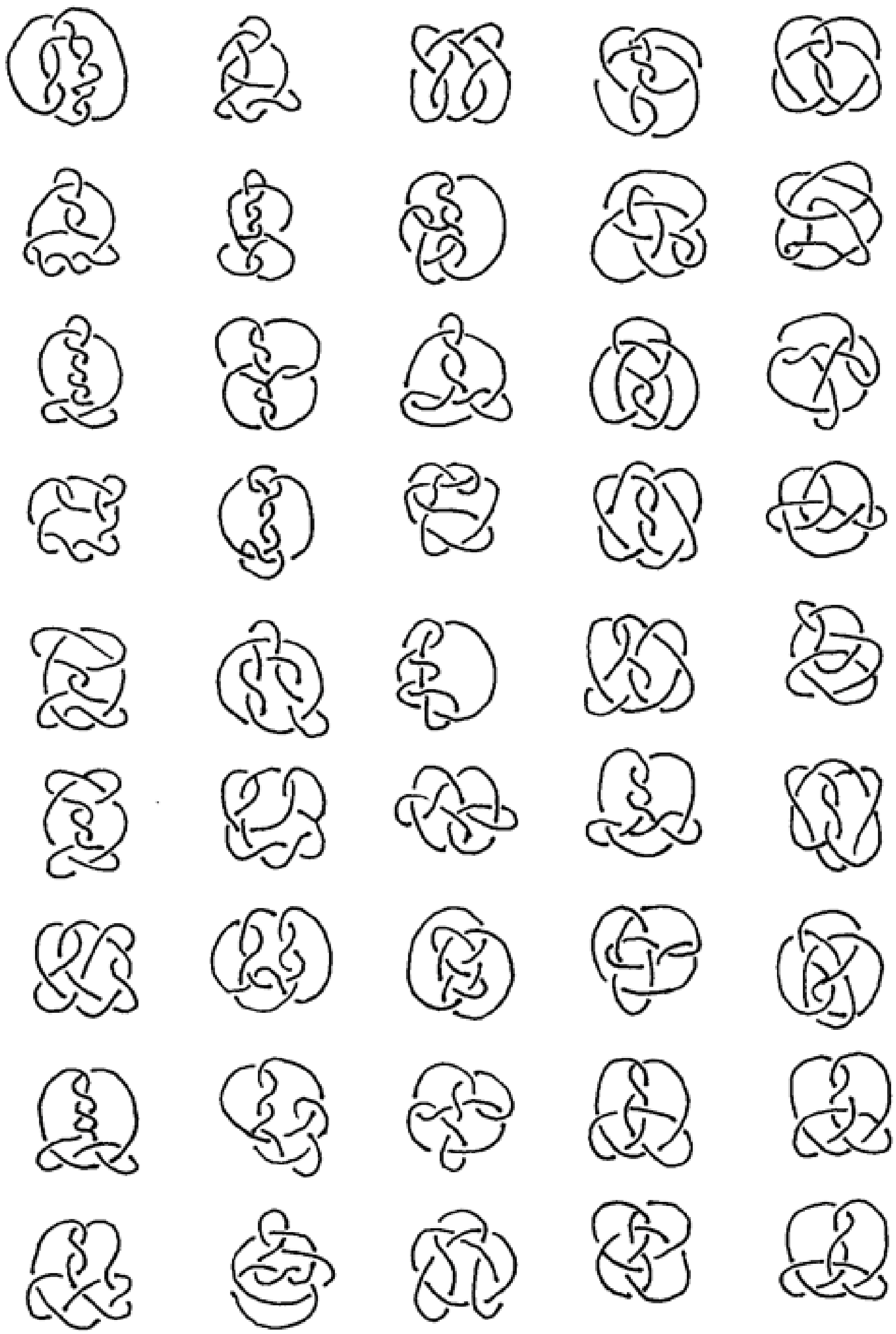
[6] Kauffman, L.H., New invariants in the theory of knots, *Amer. Math. Monthly* vol. 95, (1988), 195-242.

[7] Kauffman, L.H., An invariant of regular isotopy, Trans. Amer. Math. Soc. 318, (1990), 417-471.

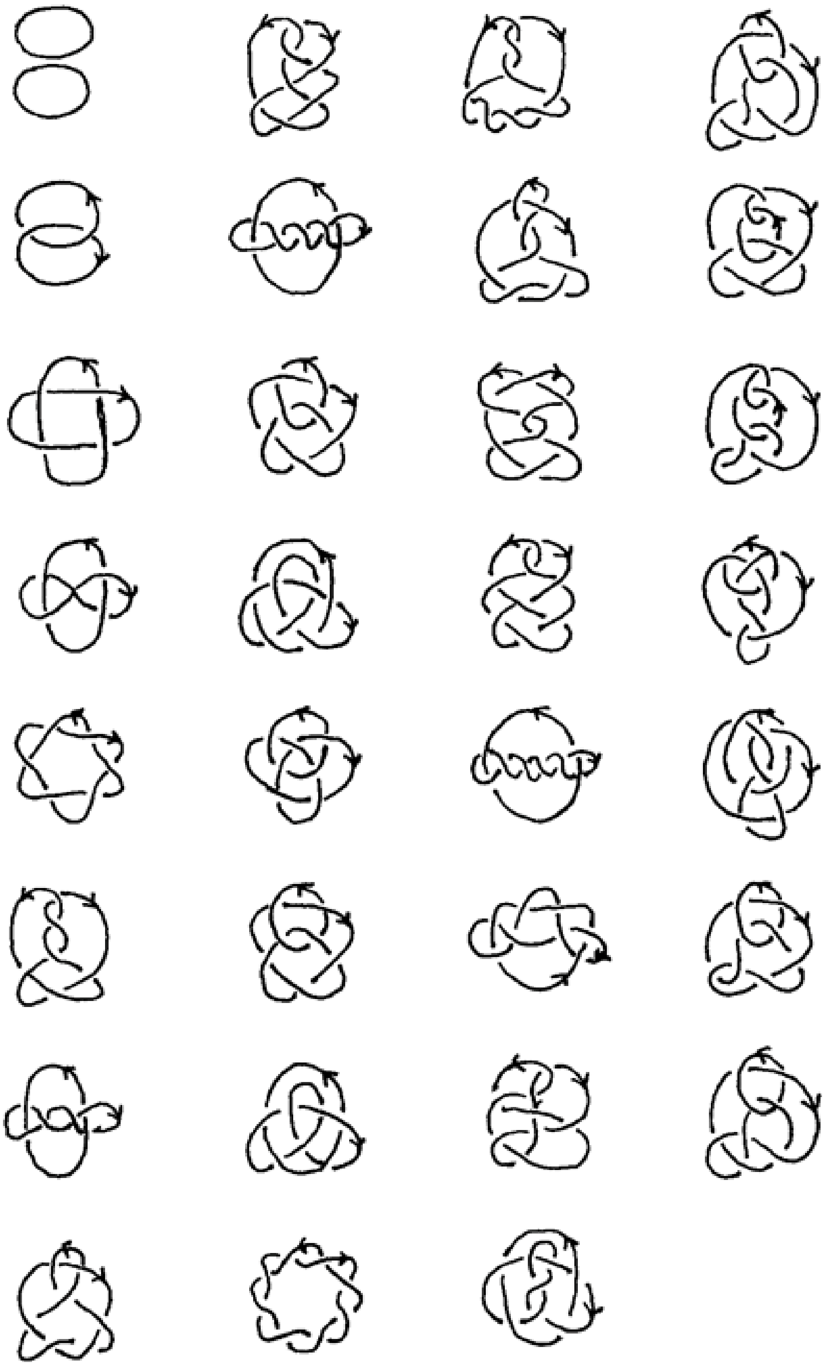
[8] Kauffman, L.H., Knot and physics, Worıd Scientific, (1991), (second edition 1993).
[9] Tait, P. G., On Knots I,II,III., Scientific Papers Vol. I, *Cambridge University Press*, London, (1898), 273-347.
[10] Kirkman, T.P., The enumeration, description and construction of knots with fewer than 10 crossings, *Trans.R.Soc. Edinb.*, 32, (1865), 281-309.[9]
[11] Little, C.N., Non-alternate $\mu-$ knots, Trans.R.Soc. Edinb., 35, (1889), 663-664.
[12] Murasugi, K., Knot theory and its applications, translated by Kurpito, B., Birkhause, *Boston*, (1996).
[13] Rolfsen, D., Knot and Links, Mathematics Lectures Series No. 7 *Publish or Perish Press*, (1976).
[14] Kauffman, L.H., On knots, *Princeton University Pres*, Princeton, New Jersey, (1987).
[15] www.knotplot.com

4. APPENDIX I. KNOT PROJECTIONS $0_1 - 9_{49}$





5. APPENDIX II. LINK PROJECTIONS $0_1^2 - 8_{16}^2$



Sakarya University-TURKEY
E-mail address: `ialtintas@sakarya.edu.tr`