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THE PRODUCT OF SHAPE FIBRATIONS

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Abstract. The following fact is shown: Let $p\colon E \to B$, $p'\colon E' \to B'$ be maps of compact Hausdorff spaces. Then $p \times p'\colon E \times E' \to B \times B'$ is a shape fibration if and only if p and p' are shape fibrations.Also the following fact on resolutions is shown:

Let $\mathbf{q} = (q_\lambda)\colon E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ and $\mathbf{r} = (r_\mu)\colon B \to \mathbf{B} = (B_\mu, r_{\mu\mu'}, M)$ are morphisms of **pro-Cpt** such that $\mathbf{E}$ and $\mathbf{B}$ are compact ANR-systems. Then $\mathbf{q} \times \mathbf{r} = (q_\lambda \times r_\mu)\colon E \times B \to \mathbf{E} \times \mathbf{B} = (E_\lambda \times B_\mu, q_{\lambda\lambda'} \times r_{\mu\mu'}, \Lambda \times M)$ is a resolution of $E \times B$ if and only if $\mathbf{q}$ and $\mathbf{r}$ are resolutions of E and B, respectively. (Theorem 1).

1. Introduction

The notion of shape fibration for maps between metric compacta was introduced by S. Mardešić and T. B. Rushing in [5] and [9] In [5] S. Mardešić has extended this notion to maps of arbitrary topological spaces. The author has established some further properties of shape fibrations in the noncompact case (see e.g. [1],[2],[3],[4]).

In this paper we give another proof of the following fact: if $p\colon E \to B$, $p'\colon E' \to B'$ are maps, where E, E', B, B' are compact Hausdorff spaces, then $p\times p'\colon E\times E' \to B \times B'$ is a shape fibration if and only if p and p' are shape fibrations.

Our proof is designed so that if Proposition 3 bellow holds for some conditions (weaker than compactness) on space E then the above statement on product of shape fibrations remains true also in the case when E, E' satisfy such conditions. Thus, answer in the

Question: Which conditions (weaker than compactness) must satisfy spaces E, E' so that for compact Hausdorff spaces B, B' holds true: $p\times p'\colon E\times E' \to B\times B'$ is shape fibration if and only if $p\colon E \to B$ and $p'\colon E' \to B'$ are shape fibrations ?
is equivalent to the answer in the following

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Question: Which conditions (weaker than compactness) must satisfy space E so that for compact Hausdorff spaces B holds true the Proposition 3 bellow ?

2. Preliminaries

By a map $p\colon E \to B$ we mean a continuous function between topological spaces. If $p, q\colon E \to B$ are maps and U is a covering of B we say that p and q are $\mathcal{U}$- *near maps*, and we write$(p, q) \leq \mathcal{U}$, provided for each $x \in E$ there is a $U \in \mathcal{U}$ such that $p(x), q(x) \in U$.

If $\mathcal{U}$ and $\mathcal{V}$ are two coverings of a space E we say that $\mathcal{U}$ *refines* $\mathcal{V}$, and we write $\mathcal{U} \succcurlyeq V$, if for every $U \in \mathcal{U}$ there is a $V \in \mathcal{V}$ such that $U \subseteq V$.

If U is a covering of a space E and $A \subseteq E$ then a *star* of A with respect to U is the set $St(A,\ U) = \bigcup\{U \in\ U : U \cap A \neq \emptyset\}$.

A *normal covering* of a space E is an open covering U which admits a locally finite partition of unity subordinated to U. It is well known that every open covering of a paracompact space is normal (see e.g [10, Corollary 1,p.325]). Consequently, every open covering of a compact space (or polyhedron, ANR-space) is normal.

By $\mathbf{pro}-\mathbf{Top}$ we denote the procategory of topological spaces whose objects are inverse systems of topological spaces and whose morphisms are equivalent classes of maps of such systems; $\mathbf{pro}-\mathbf{Cpt}$ denotes the procategory of compact Hausdorff spaces whose objects are inverse systems of compact Hausdoorff spaces and whose morphisms are equivalent classes of maps of such systems. (More on procategories see [7] or [10]).

Watanabe in [14] (see also [15, Theorem (3.3)] or [6, Theorem 1]) has proved the following fact:

Proposition 1. A morphism $\mathbf{q} = (q_\lambda)\colon E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda)$ of $\mathbf{pro}-\mathbf{Top}$ is a resolution of a topological space E if and only if $\mathbf{q}$ satisfies the following two conditions :

$(B1)$ For every $\lambda \in \Lambda$ and every normal covering U_λ of E_λ there is a $\lambda' \geq \lambda$ such that $q_{\lambda\lambda'}(E_{\lambda'}) \subseteq St(q_\lambda(E),\ U_\lambda)$.

$(B2)$ For every normal covering U of E there is a $\lambda \in \Lambda$ and a normal covering U_λ of E_λ such that $q_\lambda^{-1}(\ U_\lambda) \succcurlyeq\ U$. ■

A *level resolution* of a map $p : E \to B$ is a triple $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ consisting of resolutions $\mathbf{q} = (q_\lambda)\colon E \to \mathbf{E} = (E_\lambda, q_{\lambda\lambda'}, \Lambda), \mathbf{r} = (r_\lambda)\colon B \to \mathbf{B} = (B_\lambda, r_{\lambda\lambda'}, \Lambda)$ of spaces E and B, respectively, and of a level map of inverse systems $\mathbf{p} = (p_\lambda)\colon \mathbf{E} \to \mathbf{B}$ such that $\mathbf{pq} = \mathbf{r}p$, i.e. $p_\lambda q_\lambda = r_\lambda p$ for every $\lambda \in \Lambda$. If all E_λ's and B_λ's are polyhedrons (ANR's) then $\mathbf{q}\colon E \to \mathbf{E}, \mathbf{r}\colon B \to \mathbf{B}$ and $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ are called *polyhedral* (ANR)-*resolutions* of E, B and p, respectively .

It is a well known fact that every topological space and every map of topological spaces admit a polyhedral (ANR) resolutions ([5, Theorems 10, 11,12,13]). Without loss of generality we can assume that these resolutions are level resolutions (see [1, Lemma 4.6 and Remark 4.7]). Also it is known that compact spaces and maps of such spaces admit compact polyhedral (ANR) level resolutions (see the proof of Theorem 3.2 and Corollary 3.5 in [3]).

Since every open covering of a compact Hausdorff space is a normal covering and every open covering of such a space admits a finite subcovering which refines it, if in the proof of Theorem 11 of [5] we let Γ be the set of all finite open coverings of B, we obtain the following result

Proposition 2. Every map $p\colon E \to B$ of topological space E to a compact Hausdorf space B admits a polyhedral (ANR) resolution $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ with $\mathbf{r}\colon B \to \mathbf{B}$ in $\mathbf{pro} - \mathbf{Cpt}$.

By Lemma 4.6 and Remark 4.7 of [1] in the above Proposition, without loss of generality, we can assume that such a resolution of a map p is a level resolution.

For further information on resolutions of spaces and maps see [5],[6], [10],[1],[2],[3], [11],[14],[15]. A level map $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ is said to have **the approximate homotopy lifting property** (abbreviated the *AHLP*) with respect to a class of spaces X provided for each $\lambda \in \Lambda$ and for any two normal coverings U, V of E_λ and B_λ respectively, there is a $\lambda' \geq \lambda$ and there is a normal covering V' of $B_{\lambda'}$ with the following property: whenever one has maps $h\colon X \to E_{\lambda'}$ and $H\colon X \times I \to B_{\lambda'}$, $X \in X, I = [0,1]$, such that $(p_{\lambda'}h, H_0) \leq V'$ then there is a homotopy $\widetilde{H}\colon X \times I \to E_\lambda$ such that

$$(q_{\lambda\lambda'}h, \widetilde{H}_0) \leq U \qquad \text{and} \qquad (p_\lambda \widetilde{H}, r_{\lambda\lambda'}H) \leq V.$$

λ' and V' are called a *lifting index* and *lifting mesh*, respectively, for λ, U, and V with respect to $\mathbf{p}$ ([2, Definition 4.2]).

A map of topological spaces $p\colon E \to B$ is called a **shape fibration** provided there exists an ANR level resolution $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ of p such that the level map $\mathbf{p}\colon \mathbf{E} \to \mathbf{B}$ has the $AHLP$ with respect to the class of all topological spaces.

(In original definition of shape fibration given in [5] for $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is required to be an approximate polyhedral resolution. But, since every ANR is an approximative polyhedron, without loss of generality we can require for $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ to be an ANR resolution. Also, by [1] we can assume for $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ to be a level resolution).

From [5], Theorem 4, it follows that whenever $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ is an ANR resolution of a shape fibration $p\colon E \to B$ then $\mathbf{p}$ has the $AHLP$ with respect to the class of all topological spaces.

Since we will deal with paracompact (ANR) spaces, all open coverings are normal.

3. Some auxiliary facts

In this section we will establish some facts which we will need in the sequel.

From [12, Lemma 2, p.375], immediately it follows the following

Proposition 3. Let E and B be compact Hausdorff spaces. Then for every normal covering U of $E \times B$ there are a normal covering V of E and an open covering W of B such that $V \times W = \{V \times W : V \in V,\ W \in W\}$ is a normal covering of $E \times B$ which refines U.

Proof. By Lemma 2 of [12] there is a normal covering V of E such that every $V \in V$ admits an open (finite) covering W_V of B such that the stacked covering $\{V \times W_V : V \in V\}$ refines U. Since E is compact, without loss of generality we can assume that V is a finite covering (otherwise we replace V with finite covering which refines it). Let $V = \{V_1, V_2, \dots, V_n\}$ and $W_{V_i} = \{W_1, W_2, \dots, W_{n_i}\}$, $i \in \{1, 2, \dots, n\}$. Now we put

$$W = W_{V_1} \wedge W_{V_2} \wedge \cdots \wedge W_{V_n} = \left\{ \bigcap_{i=1}^{n} W_i \mid (W_1, W_2, \dots W_n) \in \prod_{i=1}^{n} W_{V_i} \right\}.$$

$\mathcal{W}$ is a normal (open) covering of B such that $\mathcal{V}\times\mathcal{W}\succcurlyeq\mathcal{U}$. Indeed, since $V_i\times\mathcal{W}_{V_i}\succcurlyeq\mathcal{U}$ for $i\in\{1,2,\dots,n\}$ we conclude that for every $V_i\in\mathcal{V}$ and every $\bigcap_{i=1}^n W_i\in\mathcal{W}$ there is an $U\in\mathcal{U}$ such that $V_i\times\bigcap_{i=1}^n W_i\subseteq V_i\times W_i\subseteq U$. ■

The following propositions are easily proved:

Proposition 4. If $\mathcal{U}$, $\mathcal{U}'$ are coverings of E, $\mathcal{V}$, $\mathcal{V}'$ coverings of B and $\mathcal{U}\succcurlyeq\mathcal{U}'$, $\mathcal{V}\succcurlyeq\mathcal{V}'$ then $\mathcal{U}\times\mathcal{V}\succcurlyeq\mathcal{U}'\times\mathcal{V}'$. ■

Proposition 5. Let $\mathcal{U}$ be a covering of E, $\mathcal{V}$ a covering of B, $P\subseteq E$ and $Q\subseteq B$. Then $St(P,\mathcal{U})\times St(Q,\mathcal{V})=St(P\times Q,\mathcal{U}\times\mathcal{V})$. ■

Proposition 6. Let $\mathcal{U}$ and $\mathcal{V}$ be coverings of a topological space E and $P\subseteq E$. If $\mathcal{U}\succcurlyeq\mathcal{V}$ then $St(P,\mathcal{U})\subseteq St(P,\mathcal{V})$. ■

Theorem 1. Let $\mathbf{q}=(q_\lambda)\colon E\to\mathbf{E}=(E_\lambda,q_{\lambda\lambda'},\Lambda)$ and $\mathbf{r}=(r_\mu)\colon B\to\mathbf{B}=(B_\mu,r_{\mu\mu'},M)$ are morphisms of **pro-Cpt** such that $\mathbf{E}$ and $\mathbf{B}$ are compact ANR-systems. Then $\mathbf{q}\times\mathbf{r}=(q_\lambda\times r_\mu)\colon E\times B\to\mathbf{E}\times\mathbf{B}=(E_\lambda\times B_\mu,q_{\lambda\lambda'}\times r_{\mu\mu'},\Lambda\times M)$ is a resolution of $E\times B$ if and only if $\mathbf{q}$ and $\mathbf{r}$ are resolutions of E and B, respectively.

Proof. First of all we note that the index set $\Lambda\times M$ is ordered in this way:

$$(\lambda,\mu)\le(\lambda',\mu')\iff\lambda\le\lambda'\quad\text{and}\quad\mu\le\mu'$$

Suppose that $\mathbf{q}\times\mathbf{r}$ is a resolution and we show that $\mathbf{q}$ and $\mathbf{r}$ are resolutions. By Proposition 1, it suffices to show that $\mathbf{q}$ and $\mathbf{r}$ satisfy conditions $(B1)$ and $(B2)$.

Condition $(B1)$ *for* $\mathbf{q}\colon E\to\mathbf{E}$. Let $\lambda\in\Lambda$ and let $\mathcal{U}_\lambda$ be an open covering of E_λ. Let $pr_{1\lambda}\colon E_\lambda\times B_\mu\to E_\lambda$ be the projection on the first factor. Then $pr_{1\lambda}^{-1}(\mathcal{U}_\lambda)=\{U\times B_\mu:U\in\mathcal{U}_\lambda\}=\mathcal{U}_\lambda\times\{B_\mu\}$ is an open covering of $E_\lambda\times B_\mu$. By $(B1)$ for $\mathbf{q}\times\mathbf{r}$ there is a $(\lambda',\mu')\ge(\lambda,\mu)$ such that

$$(q_{\lambda\lambda'}\times r_{\mu\mu'})(E_{\lambda'}\times B_{\mu'})\subseteq St\left((q_\lambda\times r_\mu)(E\times B),\ \mathcal{U}_\lambda\times\{B_\mu\}\right)$$

i.e

$$q_{\lambda\lambda'}(E_{\lambda'})\times r_{\mu\mu'}(B_{\mu'})\subseteq St(q_\lambda(E),\mathcal{U}_\lambda)\times St(r_\mu(B),\{B_\mu\})=St(q_\lambda(E),\mathcal{U}_\lambda)\times B_\mu.$$

Consequently,

$$q_{\lambda\lambda'}(E_{\lambda'})\subseteq St(q_\lambda(E),\mathcal{U}_\lambda),$$

and, thus, $\mathbf{q}$ satisfies $(B1)$.

Similarly it is shown that $\mathbf{r}\colon B\to\mathbf{B}$ satisfies $(B1)$.

Condition $(B2)$ *for* $\mathbf{q}\colon E\to\mathbf{E}$. Let $\mathcal{U}$ be a normal covering of E and $pr_1\colon E\times B\to E$ the projection on the first factor. Then $pr_1^{-1}(\mathcal{U})=\{U\times B:U\in\mathcal{U}\}=\mathcal{U}\times\{B\}$ is a normal covering of $E\times B$. By $(B2)$ for $\mathbf{q}\times\mathbf{r}$ there are a $(\lambda,\mu)\in\Lambda\times M$ and an open covering (normal) $\mathcal{U}'$ of $E_\lambda\times B_\mu$ such that $(q_\lambda\times r_\mu)^{-1}(\mathcal{U}')\succcurlyeq\mathcal{U}\times\{B\}$. Since $pr_{1\lambda}\colon E_\lambda\times B_\mu\to E_\lambda$ is an open surjective map we conclude that $\mathcal{U}_\lambda=pr_{1\lambda}(\mathcal{U}')=\{pr_{1\lambda}(U'):U'\in\mathcal{U}'\}$ is an open covering of E_λ. Since $q_\lambda pr_1=pr_{1\lambda}(q_\lambda\times r_\mu)$ we have that

$$pr_1^{-1}q_\lambda^{-1}(\mathcal{U}_\lambda)=(q_\lambda\times r_\mu)^{-1}pr_{1\lambda}^{-1}(\mathcal{U}_\lambda)=(q_\lambda\times r_\mu)^{-1}pr_{1\lambda}^{-1}pr_{1\lambda}(\mathcal{U}')\succcurlyeq$$
$$\succcurlyeq(q_\lambda\times r_\mu)^{-1}(\mathcal{U}')\succcurlyeq pr_1^{-1}(\mathcal{U}),$$

from which it follows that

$$pr_1pr_1^{-1}q_\lambda^{-1}(\mathcal{U}_\lambda)\succcurlyeq pr_1pr_1^{-1}(\mathcal{U}).$$

Since pr_1 is a surjective map we conclude that $q_\lambda^{-1}(\mathcal{U}_\lambda)\succcurlyeq\mathcal{U}$, which means that $\mathbf{q}$ satisfies $(B2)$.

Similarly it is shown that $\mathbf{r}\colon B \to \mathbf{B}$ satisfies $(B2)$.

Conversely, suppose that $\mathbf{q}$ and $\mathbf{r}$ are resolutions and show that $\mathbf{q}\times\mathbf{r}\colon E\times B \to \mathbf{E}\times\mathbf{B}$ is a resolution. By Proposition 1, it is sufficient to show that $\mathbf{q}\times\mathbf{r}$ satisfies conditions $(B1)$ and $(B2)$.

Condition $(B1)$ *for* $\mathbf{q}\times\mathbf{r}$. Let $(\lambda,\mu)\in\Lambda\times M$ and let $\mathcal{W}$ be any open (normal) covering of $E_\lambda\times B_\mu$. By Proposition 3, there are open coverings $\mathcal{U}$ of E_λ and $\mathcal{V}$ of B_μ such that $\mathcal{U}\times\mathcal{V}\succcurlyeq\mathcal{W}$. By $(B1)$ for $\mathbf{q}$ and $\mathbf{r}$ there are indices $\lambda'\geq\lambda$ and $\mu\geq\mu'$ such that $q_{\lambda\lambda'}(E_{\lambda'})\subseteq St(q_\lambda(E),\mathcal{U})$ and $r_{\mu\mu'}(B_{\mu'})\subseteq St(r_\mu(B),\mathcal{V})$. Then $(\lambda',\mu')\geq(\lambda,\mu)$ and, by Propositions 4 and 5, we obtain that

$$(q_{\lambda\lambda'}\times r_{\mu\mu'})(E_{\lambda'}\times B_{\mu'}) = q_{\lambda\lambda'}(E_{\lambda'})\times r_{\mu\mu'}(B_{\mu'})\subseteq St(q_\lambda(E),\mathcal{U})\times St(r_\mu(B),\mathcal{V}) =$$
$$= St(q_\lambda(E)\times r_\mu(B),\mathcal{U}\times\mathcal{V}) = St((q_\lambda\times r_\mu)(E\times B),\mathcal{U}\times\mathcal{V})\subseteq St((q_\lambda\times r_\mu)(E\times B),\mathcal{W}),$$

which means that $\mathbf{q}\times\mathbf{r}$ satisfies $(B1)$.

Condition $(B2)$ *for* $\mathbf{q}\times\mathbf{r}$. Let $\mathcal{W}$ be a normal covering of $E\times B$. By Proposition 3 there are a normal covering $\mathcal{U}$ of E and an open covering $\mathcal{V}$ of B such that $\mathcal{U}\times\mathcal{V}\succcurlyeq\mathcal{W}$. Since $\mathbf{q}\colon E\to\mathbf{E}$ and $\mathbf{r}\colon B\to\mathbf{B}$, as resolutions, have property $(B2)$ there are indices $\lambda\in\Lambda,\mu\in M$ and open coverings $\mathcal{U}_\lambda$ of E_λ and $\mathcal{V}_\mu$ of B_μ such that $q_\lambda^{-1}(\mathcal{U}_\lambda)\succcurlyeq\mathcal{U}$ and $r_\mu^{-1}(\mathcal{V}_\mu)\succcurlyeq\mathcal{V}$. Then $\mathcal{U}_\lambda\times\mathcal{V}_\mu$ is an open covering of $E_\lambda\times B_\mu$ and, by Proposition 4, holds

$$(q_\lambda\times r_\mu)^{-1}(\mathcal{U}_\lambda\times\mathcal{V}_\mu) = q_\lambda^{-1}(\mathcal{U}_\lambda)\times r_\mu^{-1}(\mathcal{V}_\mu)\succcurlyeq\mathcal{U}\times\mathcal{V}\succcurlyeq\mathcal{W},$$

which means that $\mathbf{q}\times\mathbf{r}$ satisfies $(B2)$. ■

Corollary 1. Let $\mathbf{q}\colon E\to\mathbf{E}=(E_\lambda,q_{\lambda\lambda'},\Lambda)$, $\mathbf{q}'\colon E'\to\mathbf{E}'=(E'_\mu,q'_{\mu\mu'},M)$, $\mathbf{p}=(p_\lambda)\colon\mathbf{E}\to\mathbf{B}$, $\mathbf{r}\colon B\to\mathbf{B}=(B_\lambda,r_{\lambda\lambda'},\Lambda)$, $\mathbf{r}'\colon B'\to\mathbf{B}'=(B'_\mu,r'_{\mu\mu'},M)$, $\mathbf{p}'=(p'_\mu)\colon\mathbf{E}'\to\mathbf{B}'$ be morphisms of $\mathbf{pro}-\mathbf{Cpt}$, such that $\mathbf{E},\mathbf{E}'$, $\mathbf{B},\mathbf{B}'$ are compact ANR-systems. Then $(\mathbf{q}\times\mathbf{q}',\mathbf{r}\times\mathbf{r}',\mathbf{p}\times\mathbf{p}')$ is a level resolution of $p\times p'\colon E\times E'\to B\times B'$ if and only if $(\mathbf{q},\mathbf{r},\mathbf{p})$ and $(\mathbf{q}',\mathbf{r}',\mathbf{p}')$ are resolutions of $p\colon E\to B$ and $p'\colon E'\to B'$, respectively.

Proof. Since $\mathbf{pq}=\mathbf{r}p$ and $\mathbf{p}'\mathbf{q}'=\mathbf{r}'p'$ if and only if $(\mathbf{p}\times\mathbf{p}')(\mathbf{q}\times\mathbf{q}')=(\mathbf{r}\times\mathbf{r}')(p\times p')$, the assertion of Corollary 1, it follows immediately from Theorem 1. ■

4. The main results

Theorem 2. Let E,E',B,B' be compact Hausdorff spaces, $(\mathbf{q}\times\mathbf{q}',\mathbf{r}\times\mathbf{r}',\mathbf{p}\times\mathbf{p}')$ be a level compact ANR (polyhedral)-resolution of $p\times p'\colon E\times E'\to B\times B'$ such that $(\mathbf{q},\mathbf{r},\mathbf{p})$, $(\mathbf{q}',\mathbf{r}',\mathbf{p}')$ are compact ANR (polyhedral)-resolutions of $p\colon E\to B$ and $p'\colon E'\to B'$, respectively. Then, $\mathbf{p}\times\mathbf{p}'\colon\mathbf{E}\times\mathbf{E}'\to\mathbf{B}\times\mathbf{B}'$ has the $AHLP$ with respect to the class of all topological spaces if and only if $\mathbf{p}\colon\mathbf{E}\to\mathbf{B}$ and $\mathbf{p}'\colon\mathbf{E}'\to\mathbf{B}'$ have the $AHLP$ with respect to the same class of spaces.

Proof. *Necessity.* Let $\mathbf{p}\times\mathbf{p}'\colon\mathbf{E}\times\mathbf{E}'=(E_\lambda\times E'_\mu,q_{\lambda\lambda'}\times q'_{\mu\mu'},\Lambda\times M)\to(B_\lambda\times B'_\mu,r_{\lambda\lambda'}\times r'_{\mu\mu'},\Lambda\times M)=\mathbf{B}\times\mathbf{B}'$ has the $AHLP$ with respect to the class of all topological spaces. We show that $\mathbf{p}\colon\mathbf{E}\to\mathbf{B}$ has the $AHLP$ with respect to that class of spaces.

Let $\lambda\in\Lambda$ and let $\mathcal{U}_\lambda$, $\mathcal{V}_\lambda$ be open coverings of E_λ and B_λ. Let $(\lambda,\mu)\in\Lambda\times M$ be any index with its first coordinate λ and let $pr_{1\lambda}\colon E_\lambda\times E'_\mu\to E_\lambda$ and $pr'_{1\lambda}\colon B_\lambda\times B'_\mu\to B_\lambda$ be projections on the first factor. Then $pr_{1\lambda}^{-1}(\mathcal{U}_\lambda)=\{U\times E'_\mu : U\in\mathcal{U}_\lambda\}$ and $pr_{1\lambda}^{'-1}(\mathcal{V}_\lambda)=\{V\times B'_\mu : Vin\,\mathcal{V}_\lambda\}$ are open coverings of $E_\lambda\times E'_\mu$ and $B_\lambda\times B'_\mu$, respectively. Let $(\lambda',\mu')\in\Lambda\times M, (\lambda',\mu')\geq(\lambda,\mu)$, be a lifting index and let an open covering $\mathcal{V}'$ of $B_{\lambda'}\times B'_{\mu'}$ be a lifting mesh for (λ,μ), $pr_{1\lambda}^{-1}(\mathcal{U}_\lambda)$ and $pr_{1\lambda}^{'-1}(\mathcal{V}_\lambda)$ with respect to $\mathbf{p}\times\mathbf{p}'$. We claim that $\lambda'\geq\lambda$ is a lifting index and that open covering $pr'_{1\lambda'}(\mathcal{V}')$ of $B_{\lambda'}$ is a lifting mesh for λ, $\mathcal{U}_\lambda$, $\mathcal{V}_\lambda$ with respect to $\mathbf{p}$.

Indeed, let X be arbitrary topological space and $h\colon X\to E_{\lambda'}$, $H\colon X\times I\to B_{\lambda'}$ be maps such that

$$(p_{\lambda'}h, H_0)\leq pr'_{1\lambda'}(\mathcal{V}'). \tag{1}$$

Let $e=(e_{\lambda'},e'_{\mu'})\in E_{\lambda'}\times E'_{\mu'}$ be a fixed point and let $b=(b_{\lambda'},b'_{\mu'})\in B_{\lambda'}\times B'_{\mu'}$ be such a point that $b_{\lambda'}=p_{\lambda'}(e_{\lambda'})$, $b'_{\mu'}=p'_{\mu'}(e'_{\mu'})$. Then $(p_{\lambda'}\times p'_{\mu'})(e)=b$. Now we put $E^*_{\lambda'}=E_{\lambda'}\times\{e'_{\mu'}\}$ and $B^*_{\lambda'}=B_{\lambda'}\times\{b'_{\mu'}\}$. Let $s_{\lambda'}\colon E_{\lambda'}\to E^*_{\lambda'}$ and $s'_{\lambda'}\colon B_{\lambda'}\to B^*_{\lambda'}$ be maps given by $s_{\lambda'}(x)=(x,e'_{\mu'})$ for every $x\in E_{\lambda'}$ and $s'_{\lambda'}(x)=(x,b'_{\mu'})$ for every $x\in B_{\lambda'}$. $s_{\lambda'}$ and $s'_{\lambda'}$ are homeomorphisms such that

$$s_{\lambda'}^{-1}=pr_{1\lambda'}|_{E^*_{\lambda'}},\qquad s_{\lambda'}^{'-1}=pr'_{1\lambda'}|_{B^*_{\lambda'}}. \tag{2}$$

It can easily be shown that

$$(p_{\lambda'}\times p'_{\mu'})s_{\lambda'}=s'_{\lambda'}p_{\lambda'}. \tag{3}$$

From (1), it follows that for each $x\in X$ there is a $V\in\mathcal{V}'$ such that $p_{\lambda'}h(x),H_0(x)\in pr'_{1\lambda'}(V)$, and thus,

$$pr_{1\lambda'}^{'-1}p_{\lambda'}h(x),\ pr_{1\lambda'}^{'-1}H_0(x)\subseteq V.$$

From this, by (2), it follows that

$$s'_{\lambda'}p_{\lambda'}h(x),\ s'_{\lambda'}H_0(x)\in V\cap B^*_{\lambda'}\subseteq V.$$

Now, by (3), we have that

$$(p_{\lambda'}\times p'_{\mu'})s_{\lambda'}h(x),\ s'_{\lambda'}H_0(x)\in V\qquad\text{i.e.}\qquad\big((p_{\lambda'}\times p'_{\mu'})s_{\lambda'}h, s'_{\lambda'}H_0\big)\leq\mathcal{V}'.$$

Since (λ',μ') is the lifting index and $\mathcal{V}'$ is the lifting mesh for (λ,μ), $pr_{1\lambda}^{-1}(\mathcal{U}_\lambda)$, $pr_{1\lambda}^{'-1}(\mathcal{V}_\lambda)$ with respect to $\mathbf{p}\times\mathbf{p}'$, we conclude that there exists a map $\widetilde{H}\colon X\times I\to E_\lambda\times E'_\mu$ such that

$$\Big((q_{\lambda\lambda'}\times q'_{\mu\mu'})s_{\lambda'}h,\ \widetilde{H}_0\Big)\leq pr_{1\lambda}^{-1}(\mathcal{U}_\lambda)$$

and

$$\Big((p_\lambda\times p'_\mu)\widetilde{H},\ (r_{\lambda\lambda'}\times r'_{\mu\mu'})s'_{\lambda'}H\Big)\leq pr_{1\lambda}^{'-1}(\mathcal{V}_\lambda).$$

Then we have

$$\Big(pr_{1\lambda}(q_{\lambda\lambda'}\times q'_{\mu\mu'})s_{\lambda'}h, pr_{1\lambda}\widetilde{H}_0\Big)\leq\mathcal{U}_\lambda$$

and

$$\Big(pr'_{1\lambda}(p_\lambda\times p'_\mu)\widetilde{H},\ pr'_{1\lambda}(r_{\lambda\lambda'}\times r'_{\mu\mu'})s'_{\lambda'}H\Big)\leq\mathcal{V}_\lambda.$$

Since

$$pr_{1\lambda}(q_{\lambda\lambda'}\times q'_{\mu\mu'})=q_{\lambda\lambda'}pr_{1\lambda'}$$
$$pr'_{1\lambda}(p_\lambda\times p'_\mu)=p_\lambda pr_{1\lambda}$$
$$pr'_{1\lambda}(r_{\lambda\lambda'}\times r'_{\mu\mu'})=r_{\lambda\lambda'}pr'_{1\lambda'}$$

we conclude that

$$\Big(q_{\lambda\lambda'}pr_{1\lambda'}s_{\lambda'}h,\ pr_{1\lambda}\widetilde{H}_0\Big)\leq\mathcal{U}_\lambda$$

and

$$\Big(p_\lambda pr_{1\lambda}\widetilde{H},\ r_{\lambda\lambda'}pr'_{1\lambda'}s'_{\lambda'}H\Big)\leq\mathcal{V}_\lambda.$$

Now, since $pr_{1\lambda'}s_{\lambda'} = 1_{E_{\lambda'}}$ and $pr'_{1\lambda'}s'_{\lambda'} = 1_{B_{\lambda'}}$, we obtain that

$$\left(q_{\lambda\lambda'}h,\ pr_{1\lambda}\widetilde{H}_0\right) \leq\ U \qquad \text{and} \qquad (p_\lambda pr_{1\lambda}\widetilde{H},\ r_{\lambda\lambda'}H) \leq\ V,$$

which means that $\mathbf{p}$ has the $AHLP$ with respect to the class of all topological spaces.

Similarly it is shown that $\mathbf{p}'$ has the $AHLP$.

Sufficiency. Let $\mathbf{p}$ and $\mathbf{p}'$ have the $AHLP$ with respect to the class of all topological spaces. We show that $\mathbf{p}\times\mathbf{p}'$ has the $AHLP$ with respect to the same class.

Let $(\lambda,\mu)\in\Lambda\times M$ and $U,\ V$ be open coverings of $E_\lambda\times E'_\mu$ and $B_\lambda\times B'_\mu$, respectively. By Proposition 3 there are open coverings $U_\lambda,\ U_\mu,\ V_\lambda,\ V_\mu$, of $E_\lambda, E'_\mu, B_\lambda$ and B'_μ, respectively, such that $U_\lambda\times\ U_\mu \succcurlyeq U$ and $V_\lambda\times\ V_\mu\succcurlyeq\ V$.

Let $\lambda'\geq\lambda$ be a lifting index and let an open covering $V_{\lambda'}$ of $B_{\lambda'}$ be a lifting mesh for $\lambda,\ U_\lambda,\ V_\lambda$ with respect to $\mathbf{p}$. Similarly, let $\mu\geq\mu'$ be a lifting index and let an open covering $V_{\mu'}$ of $B'_{\mu'}$ be a lifting mesh for $\mu,\ U_\mu,\ V_\mu$ with respect to $\mathbf{p}'$.

We claim that $(\lambda',\mu')\geq(\lambda,\mu)$ is a lifting index and an open covering $V_{\lambda'}\times V_{\mu'}=\{U\times V : U\in\ V_{\lambda'}, V\in\ V_{\mu'}\}$ of $B_{\lambda'}\times B'_{\mu'}$ is a lifting mesh for $(\lambda,\mu),\ U,\ V$ with respect to $\mathbf{p}\times\mathbf{p}'$.

Indeed, let X be an arbitrary topological space and let $h\colon X\to E_{\lambda'}\times E'_{\mu'},\ H\colon X\times I\to B_{\lambda'}\times B'_{\mu'}$ be maps such that

$$\left((p_{\lambda'}\times p'_{\mu'})h,\ H_0\right)\leq\ V_{\lambda'}\times\ V_{\mu'}. \tag{4}$$

Let $pr_{1\lambda'}\colon E_{\lambda'}\times E'_{\mu'}\to E_{\lambda'}$ and $pr'_{1\lambda'}\colon B_{\lambda'}\times B'_{\mu'}\to B_{\lambda'}$ be projections on the first factor and $h'=pr_{1\lambda'}h\colon X\to E_{\lambda'},\ H'=pr'_{1\lambda'}H\colon X\times I\to B_{\lambda'}$. Since $p_{\lambda'}pr_{1\lambda'}=pr'_{1\lambda'}(p_{\lambda'}\times p'_{\mu'})$, from (4), it follows that

$$(p_{\lambda'}pr_{1\lambda'}h,\ pr'_{1\lambda'}H_0)\leq pr'_{1\lambda'}(\ V_{\lambda'}\times\ V_{\mu'})=\ V_{\lambda'},$$

i.e

$$(p_{\lambda'}h',\ H'_0)\leq\ V_{\lambda'}. \tag{5}$$

Since λ' is the lifting index and $V_{\lambda'}$ is a lifting mesh for $\lambda,\ U_\lambda,\ V_\lambda$ with respect to $\mathbf{p}$, from (5), it follows that there is a homotopy $\widetilde{H'}\colon X\times I\to E_\lambda$ such that

$$(q_{\lambda\lambda'}h',\ \widetilde{H'_0})\leq\ U_\lambda \tag{6}$$

and

$$(p_\lambda\widetilde{H'},\ r_{\lambda\lambda'}H')\leq\ V_\lambda. \tag{7}$$

Similarly, let $pr_{2\mu'}\colon E_{\lambda'}\times E'_{\mu'}\to E'_{\mu'}$ and $pr'_{2\mu'}\colon B_{\lambda'}\times B'_{\mu'}\to B'_{\mu'}$ be projections on the second factor and $h''=pr_{2\mu'}h\colon X\to E'_{\mu'},\ H''=pr'_{2\mu'}H\colon X\times I\to B'_{\mu'}$. Then from (4), it follows that

$$\left(pr'_{2\mu'}(p_{\lambda'}\times p'_{\mu'})h,\ pr'_{2\mu'}H_0\right)\leq pr'_{2\mu'}(\ V_{\lambda'}\times\ V_{\mu'})=\ V_{\mu'}.$$

Since $pr'_{2\mu'}(p_{\lambda'}\times p'_{\mu'})=p'_{\mu'}pr_{2\mu'}$ we obtain that $(p'_{\mu'}pr_{2\mu'}h,\ pr'_{2\mu'}H_0)\leq\ V_{\mu'}$, i.e.

$$(p'_{\mu'}h'',\ H''_0)\leq\ V_{\mu'}. \tag{8}$$

Since μ' is the lifting index and $V_{\mu'}$ is the lifting mesh for $\mu,\ U_\mu,\ V_\mu$ with respect to $\mathbf{p}'$, from (8), we conclude that there is a homotopy $\widetilde{H''}\colon X\times I\to E'_\mu$ such that

$$(q'_{\mu\mu'}h'',\ \widetilde{H''_0}) \leq\ U_\mu \tag{9}$$

$$(p'_\mu \widetilde{H''},\ r'_{\mu\mu'}H'') \leq\ V_\mu. \tag{10}$$

Let $\widetilde{H} = \widetilde{H'} \triangle \widetilde{H''} \colon X \times I \to E_\lambda \times E'_\mu$ be a map given by

$$\widetilde{H}(x,t) = \left(\widetilde{H'}(x,t),\ \widetilde{H''}(x,t)\right), \quad \forall\ (x,t) \in X \times I. \tag{11}$$

Note that for every $x \in X$ holds

$$h(x) = (pr_{1\lambda'}h(x),\ pr_{2\mu'}h(x)) = (h'(x),\ h''(x)) = (h\triangle h'')(x). \tag{12}$$

Similarly,

$$H(x,t) = (H'\triangle H'')(x,t) = (H'(x,t), H''(x,t)), \quad \forall\ (x,t) \in X \times I. \tag{13}$$

Now, from (6) and (9), it follows that

$$\Big((q_{\lambda\lambda'} \times q'_{\mu\mu'})h,\ \widetilde{H_0}\Big) \leq\ U_\lambda \times\ U_\mu. \tag{14}$$

Similarly, from (7) and (10), it follows that

$$\Big((p_\lambda \times p'_\mu)\widetilde{H},\ (r_{\lambda\lambda'} \times r'_{\mu\mu'})H\Big) \leq\ V_\lambda \times\ V_\mu. \tag{15}$$

Since $U_\lambda \times\ U_\mu \succcurlyeq\ U$ and $V_\lambda \times\ V_\mu \succcurlyeq\ V$, from (14) and (15), it follows that

$$\Big((q_{\lambda\lambda'} \times q'_{\mu\mu'})h,\ \widetilde{H_0}\Big) \leq\ U \qquad \text{and} \qquad \Big((p_\lambda \times p'_\mu)\widetilde{H},\ (r_{\lambda\lambda'} \times r'_{\mu\mu'})H\Big) \leq\ V,$$

which means that $\mathbf{p} \times \mathbf{p'}$ has the $AHLP$ with respect to the class of all topological spaces. ■

Now we are able to state and to prove the main theorem of this paper.

Theorem 3. Let $p \colon E \to B,\ p' \colon E' \to B'$ be maps of compact Hausdorff spaces. Then, $p \times p' \colon E \times E' \to B \times B'$ is a shape fibration if and only if p and p' are shape fibrations.

Proof. *Necessity.* Let $p \times p'$ be a shape fibration. We show that p and p' are shape fibrations. Let $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ and $(\mathbf{q'}, \mathbf{r'}, \mathbf{p'})$ be ANR level resolutions of p and p', respectively, with $\mathbf{r} \colon B \to \mathbf{B}$ and $\mathbf{r'} \colon B' \to \mathbf{B'}$ in $\mathbf{pro} - \mathbf{Cpt}$. Such resolutions exist by Proposition 2. Then, by Corollary 1, $(\mathbf{q} \times \mathbf{q'}, \mathbf{r} \times \mathbf{r'}, \mathbf{p} \times \mathbf{p'})$ is an ANR level resolution of $p \times p'$. Since $p \times p'$ is a shape fibration we may assume that the level map $\mathbf{p} \times \mathbf{p'}$ has the $AHLP$ with respect to the class of all topological spaces. From Theorem 2, it follows that $\mathbf{p}$ and $\mathbf{p'}$ have the $AHLP$ with respect to the class of all topological spaces. This means that p and p' are shape fibrations.

Sufficiency. Let p and p' be shape fibrations. Then, there are ANR level resolutions $(\mathbf{q}, \mathbf{r}, \mathbf{p})$ and $(\mathbf{q'}, \mathbf{r'}, \mathbf{p'})$ of p and p', respectively, such that $\mathbf{p}$ and $\mathbf{p'}$ have the $AHLP$ with respect to the class of all topological spaces. By Corollary 1 and Theorem 2 $(\mathbf{q} \times \mathbf{q'}, \mathbf{r} \times \mathbf{r'}, \mathbf{p} \times \mathbf{p'})$ is an ANR level resolution of $p \times p'$ such that $\mathbf{p} \times \mathbf{p'}$ has the $AHLP$ with respect to the class of all topological spaces. Consequently, $p \times p'$ is a shape fibration. ■

From Theorem 3, by induction, it follows the following

Corollary 2. Let $p_n \colon E_n \to B_n$ be a map of a compact Hausdorff spaces for each $n = 1, 2, 3, \dots$, and let

$$p = \prod_{n=1}^{\infty} p_n \colon \prod_{n=1}^{\infty} E_n \to \prod_{n=1}^{\infty} B_n$$

be the product of maps p_n. Then, p is a shape fibration if and only if p_n is a shape fibration for each $n = 1, 2, \dots,$. ■

Question: Does Theorem 3 remains true if E and E' are arbitrary topological spaces (not necessary compact Hausdorff) ?

Its worth to be noticed that all the above preparation (all propositions and theorems) for the proof of Theorem 3 are designed so that the answer in the above question will be affirmative if the statement of Proposition 3 remains true when E and E' are arbitrary topological spaces.

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