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TRANSMUTED EXPONENTIATED EXPONENTIAL DISTRIBUTION

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Abstract. In this article, we generalize the exponentiated exponential distribution using the quadratic rank transmutation map studied by Shaw et al. [6] to develop a transmuted exponentiated exponential distribution. The properties of this distribution are derived and the estimation of the model parameters is discussed. An application to real data set are finally presented for illustration.

1. Introduction

The quality of the procedures used in a statistical analysis depends heavily on the assumed probability model or distributions. Because of this, considerable effort has been expended in the development of large classes of standard probability distributions along with relevant statistical methodologies. However, there still remain many important problems where the real data does not follow any of the classical or standard probability models.

The exponentiated exponential distribution has been introduced by Ahuja and Nash [1] and further studied by Gupta and Kundu [4, 5].

The pdf of exponentiated exponential distribution, also known as the generalized exponential distribution, is defined as

$$g(x,\alpha,\beta) = \alpha\beta\Big(1 - e^{-\beta x}\Big)^{\alpha-1} e^{-\beta x}, \quad x > 0, \alpha > 0, \beta > 0, \tag{1.1}$$

and its cdf is

$$G(x) = (1 - e^{-\beta x})^{\alpha}, \quad x > 0, \alpha > 0, \beta > 0. \tag{1.2}$$

In this article we use transmutation map approach suggested by Shaw et al. [6] to define a new model which generalizes the exponentiated exponential model.

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We will call the generalized distribution as the transmuted exponentiated exponential distribution. According to the Quadratic Rank Transmutation Map(QRTM), approach the cumulative distribution function(cdf) satisfy the relationship

$$F(x) = (1+\lambda)G(x) - \lambda G^2(x), |\lambda| \leq 1 \tag{1.3}$$

where $G(x)$ is the cdf of the base distribution.

Observe that at $\lambda = 0$, we have the distribution of the base random variable. Aryal et al. [2] studied the transmuted Gumbel distribution and it has been observed that transmuted Gumbel distribution can be used to model climate data. In the present study we will provide mathematical formulations of the transmuted exponentiated exponential distribution and also some of its properties.

2. Transmuted Exponentiated Exponential Distribution

Definition 2.1. A random variable X is said to have the transmuted exponentiated exponential distribution with parameter α, β and λ if its probability density is defined as:

$$f(x) = \alpha\beta\Big(1 - e^{-\beta x}\Big)^{\alpha-1} e^{-\beta x}\Big[1 + \lambda - 2\lambda(1 - e^{-\beta x})^{\alpha}\Big], \tag{2.1}$$

and its cdf is

$$F(x) = \Big[1 - e^{-\beta x}\Big]^{\alpha} \cdot \Big[1 + \lambda - \lambda(1 - e^{-\beta x})^{\alpha}\Big]. \tag{2.2}$$

Note that the transmuted exponentiated exponential distribution is an extended model to analyze more complex data. The exponentiated exponential distribution is clearly a special case for $\lambda = 0$. Figure 1 illustrates some of the possible shapes of the pdf of a transmuted exponentiated exponential distribution for selected values of the parameters λ, α and for $\beta = 1$.

3. Moments and Quantiles

Now let us consider the different moments of the transmuted exponentiated exponential distribution. Suppose X denote the transmuted exponentiated exponential distribution random variable with parameter α, β and λ, then

$$\begin{aligned} E(X^k) &= \alpha\beta \int_0^{\infty} x^k \Big(1 - e^{-\beta x}\Big)^{\alpha-1} e^{-\beta x}\Big[1 + \lambda - 2\lambda(1 - e^{-\beta x})^{\alpha}\Big] \\ &= \alpha\beta(1+\lambda)\int_0^{\infty} x^k \Big(1 - e^{-\beta x}\Big)^{\alpha-1} e^{-\beta x} - 2\alpha\beta\lambda \int_0^{\infty} x^k \Big(1 - e^{-\beta x}\Big)^{2\alpha-1} e^{-\beta x}. \end{aligned}$$

Now since $0 < e^{-\beta x} < 1$, for $\beta > 0$ and $x > 0$, therefore by using the series representation of

$$(1 - e^{-\beta x})^{\alpha-1} = \sum_{i=0}^{\infty} (-1)^i c(\alpha-1, i) e^{-i\beta x},$$

where $c(\alpha - 1, i) = \frac{(\alpha-1)\cdots(\alpha-i)}{i!}$, we obtain

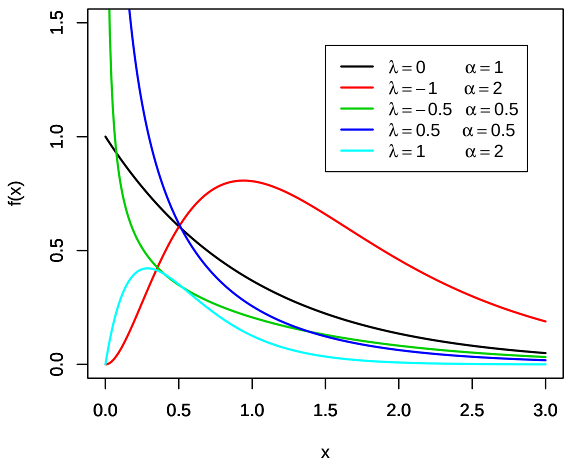

FIGURE 1. The pdf's of various transmuted exponentiated exponential distribution for $\beta = 1$.

$$E(X^k) = \frac{\alpha(1+\lambda)\Gamma(k+1)}{\beta^k} \sum_{i=0}^{\infty} (-1)^i c(\alpha-1, i) \frac{1}{(i+1)^{k+1}} - \frac{2\alpha\lambda\Gamma(k+1)}{\beta^k} \sum_{i=0}^{\infty} (-1)^i c(2\alpha-1, i) \frac{1}{(i+1)^{k+1}}. \tag{3.1}$$

Since (3.1) is a convergent series for any $k \geq 0$, therefore all the moments exist and for integer values of α. Therefore putting $k = 1$, we obtain the mean as

$$E(X) = \frac{\alpha(1+\lambda)}{\beta} \sum_{i=0}^{\infty} (-1)^i c(\alpha-1, i) \frac{1}{(i+1)^2} - \frac{2\alpha\lambda}{\beta} \sum_{i=0}^{\infty} (-1)^i c(2\alpha-1, i) \frac{1}{(i+1)^2}, \tag{3.2}$$

and putting $k = 2$ we obtain the second moment as

$$E(X^2) = \frac{2\alpha(1+\lambda)}{\beta^2} \sum_{i=0}^{\infty} (-1)^i c(\alpha-1, i) \frac{1}{(i+1)^3} - \frac{4\alpha\lambda}{\beta^2} \sum_{i=0}^{\infty} (-1)^i c(2\alpha-1, i) \frac{1}{(i+1)^3}.$$

It is also possible to express the moment generating function in terms of the gamma function, which in turn can be used to obtain different moments. The moment generating function of X, say $M(t)$, for $0 < t < \beta$ can be written as

(3.3)

$$M(t) = E(e^{tX}) = \alpha\beta \int_0^\infty \Big(1 - e^{-\beta x}\Big)^{\alpha-1} e^{(t-\beta)x} \Big[1 + \lambda - 2\lambda(1 - e^{-\beta x})^\alpha\Big]$$

$$= \alpha\beta(1+\lambda) \int_0^\infty \Big(1 - e^{-\beta x}\Big)^{\alpha-1} e^{(t-\beta)x} - 2\alpha\beta\lambda \int_0^\infty \Big(1 - e^{-\beta x}\Big)^{2\alpha-1} e^{(t-\beta)x}.$$

Making the substitution $y = e^{-\beta x}$, (3.3) reduces to

$$M(t) = \alpha(1+\lambda) \int_0^1 (1-y)^{\alpha-1} y^{-\frac{t}{\beta}} dy - 2\alpha\lambda \int_0^1 (1-y)^{2\alpha-1} y^{-\frac{t}{\beta}} dy \tag{3.4}$$

$$= (1+\lambda)\frac{\Gamma(\alpha+1)\Gamma(1-\frac{t}{\beta})}{\Gamma(\alpha-\frac{t}{\beta}+1)} - 2\alpha\lambda\frac{\Gamma(2\alpha)\Gamma(1-\frac{t}{\beta})}{\Gamma(2\alpha-\frac{t}{\beta}+1)}.$$

Differentiating $\ln(M(t))$ and evaluating at $t = 0$, we get the mean and the variance of X as

$$E(X) = \frac{1+\lambda}{\beta}\Big(\psi(\alpha+1) - \psi(1)\Big) - \frac{2\alpha\lambda}{\beta}\Big(\psi(2\alpha+1) - \psi(1)\Big), \tag{3.5}$$

$$Var(X) = \frac{1+\lambda}{\beta^2}\Big(\psi^{'}(1) - \psi^{'}(\alpha+1)\Big) - \frac{2\alpha\lambda}{\beta^2}\Big(\psi^{'}(1) - \psi^{'}(2\alpha+1)\Big), \tag{3.6}$$

where $\psi(\cdot)$ is the digamma function and $\psi^{'}(\cdot)$ is its derivative. The higher central moments can be obtained in terms of the polygamma functions. The q^{th} quantile x_q of the transmuted exponentiated exponential distribution can be obtained from (2.2) as

$$x_q = \frac{1}{\beta}\Bigg\{ -\ln\Bigg[1 - \sqrt[\alpha]{\frac{1+\lambda-\sqrt{(1+\lambda)^2 - 4\lambda q}}{2\lambda}}\Bigg]\Bigg\}. \tag{3.7}$$

In particular, the distribution median is:

$$x_{0.5} = \frac{1}{\beta}\Bigg\{ -\ln\Bigg[1 - \sqrt[\alpha]{\frac{1+\lambda-\sqrt{1+\lambda^2}}{2\lambda}}\Bigg]\Bigg\}.$$

4. Random Number Generation and Parameter Estimation

Using the method of inversion we can generate random numbers from the transmuted exponentiated exponential distribution as

$$\Big[1 - e^{-\beta x}\Big]^\alpha \cdot \Big[1 + \lambda - \lambda(1 - e^{-\beta x})^\alpha\Big] = u,$$

where $u \sim U(0,1)$. After simple calculation this yields

$$x = \frac{1}{\beta}\Bigg\{ -\ln\Bigg[1 - \sqrt[\alpha]{\frac{1+\lambda-\sqrt{(1+\lambda)^2 - 4\lambda u}}{2\lambda}}\Bigg]\Bigg\}. \tag{4.1}$$

One can use (4.1) to generate random numbers when the parameters α, β and λ are known. The maximum likelihood estimates, MLE's, of the parameters that are inherent within the transmuted exponentiated exponential probability distribution function is given by the following:

$$L = (\alpha\beta)^n \prod_{i=1}^{n} \left(1 - e^{-\beta x_i}\right)^{\alpha-1} \cdot e^{-\beta x_i} \cdot \left[1 + \lambda - 2\lambda(1 - e^{-\beta x_i})^{\alpha}\right], \tag{4.2}$$

and

$$\ln L = n\ln(\alpha\beta) + (\alpha - 1)\sum_{i=1}^{n} \ln\left(1 - e^{-\beta x_i}\right) - \beta \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln\left[1 + \lambda - 2\lambda(1 - e^{-\beta x_i})^{\alpha}\right]. \tag{4.3}$$

Now setting

$$\frac{\partial \ln L}{\partial \alpha} = 0, \quad \frac{\partial \ln L}{\partial \beta} = 0, \quad and \quad \frac{\partial \ln L}{\partial \lambda} = 0,$$

we have

$$\frac{n}{\alpha} + \sum_{i=1}^{n} \ln\left(1 - e^{-\beta x_i}\right) - 2\lambda \sum_{i=1}^{n} \frac{(1 - e^{-\beta x_i})\ln(1 - e^{-\beta x_i})}{1 + \lambda - 2\lambda(1 - e^{-\beta x_i})} = 0,$$

$$\frac{n}{\beta} + (\alpha - 1)\sum_{i=1}^{n} \frac{x_i e^{-\beta x_i}}{1 - e^{-\beta x_i}} - \sum_{i=1}^{n} x_i - 2\alpha\lambda \sum_{i=1}^{n} \frac{x_i e^{-\beta x_i}(1 - e^{-\beta x_i})^{\alpha-1}}{1 + \lambda - 2\lambda(1 - e^{-\beta x_i})^{\alpha}} = 0,$$

$$\sum_{i=1}^{n} \frac{1 - 2(1 - e^{-\beta x_i})^{\alpha}}{1 + \lambda - 2\lambda(1 - e^{-\beta x_i})^{\alpha}} = 0.$$

The maximum likelihood estimator $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda})'$ of $\theta = (\alpha, \beta, \lambda)'$ is obtained by solving this nonlinear system of equations. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton to numerically maximize the log-likelihood function given in (4.2). In order to compute the standard error and asymptotic confidence interval we use the usual large sample approximation in which the maximum likelihood estimators of θ can be treated as being approximately trivariate normal. Hence as $n \to \infty$ the asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta}, \hat{\lambda})$ is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\lambda} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \beta \\ \lambda \end{pmatrix}, \begin{pmatrix} \hat{V_{11}} & \hat{V_{12}} & \hat{V_{13}} \\ \hat{V_{21}} & \hat{V_{22}} & \hat{V_{23}} \\ \hat{V_{31}} & \hat{V_{32}} & \hat{V_{33}} \end{pmatrix}\right],$$

where, $V_{ij} = V_{ij|\theta=\hat{\theta}}$ and

$$\begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}^{-1},$$

is the approximate variance covariance matrix with its elements obtained from

$$A_{11} = -\frac{\partial^2 \ln L}{\partial \alpha^2}, \qquad A_{12} = -\frac{\partial^2 \ln L}{\partial \alpha \partial \beta},$$

$$A_{22} = -\frac{\partial^2 \ln L}{\partial \beta^2}, \qquad A_{23} = -\frac{\partial^2 \ln L}{\partial \beta \partial \lambda},$$

$$A_{33} = -\frac{\partial^2 \ln L}{\partial \gamma^2}, \qquad A_{13} = -\frac{\partial^2 \ln L}{\partial \alpha \partial \lambda}.$$

where $\ln L$ is the log-likelihood function given in (4.3). Approximate $100(1-\alpha)\%$ two sided confidence intervals for α, β and λ are, respectively, given by

$$\hat{\alpha} \pm z_{\alpha/2}\sqrt{\hat{V_{11}}}, \hat{\beta} \pm z_{\alpha/2}\sqrt{\hat{V_{22}}}, \text{ and } \hat{\lambda} \pm z_{\alpha/2}\sqrt{\hat{V_{33}}},$$

where z_α is the upper α−th percentiles of the standard normal distribution. Using R we can easily compute the Hessian matrix and its inverse and hence the values of the standard error and asymptotic confidence intervals.

We can compute the maximized unrestricted and restricted log - likelihoods to construct the likelihood ratio (LR) statistics for testing some transmuted exponentiated exponential sub-models. For example, we can use LR statistics to check whether the fitted transmuted exponentiated exponential distribution for a given data set is statistically "superior" to the fitted exponentiated exponential distribution. In any case, hypothesis tests of the type

$$H_0 : \Theta = \Theta_0 \text{ versus } H_0 : \Theta \neq \Theta_0$$

can be performed using LR statistics. In this case, the LR statistic for testing H_0 versus H_1 is

$$\omega = 2(L(\hat{\Theta}) - L(\hat{\Theta_0})),$$

where $\hat{\Theta}$ and $\hat{\Theta_0}$ are the MLEs under H_1 and H_0. The statistic ω is asymptotically(as $n \to \infty$) distributed as χ^2_k, where k is the dimension of the subset Ω of interest. The LR test rejects H_0 if $\omega > \xi_\gamma$, where ξ_γ denotes the upper $100\gamma\%$ point of the χ^2_k distribution.

5. Reliability Analysis

The reliability function $R(t)$, which is the probability of an item not failing prior to time t, is defined by $R(t) = 1 - F(t)$. The reliability function of a transmuted exponentiated exponential distribution is given by

$$R(t) = 1 - \left[1 - e^{-\beta x}\right]^{\alpha} \cdot \left[1 + \lambda - \lambda(1 - e^{-\beta x})^{\alpha}\right]. \tag{5.1}$$

The other characteristic of interest of a random variable is the hazard rate function defined by

$$h(t) = \frac{f(t)}{1 - F(t)},$$

which is an important quantity characterizing life phenomenon. It can be loosely interpreted as the conditional probability of failure, given it has survived to time t. The hazard rate function for a transmuted exponentiated exponential random variable is given by

$$h(t) = \frac{\alpha\beta\left(1 - e^{-\beta x}\right)^{\alpha-1} e^{-\beta x}\left[1 + \lambda - 2\lambda(1 - e^{-\beta x})^{\alpha}\right]}{1 - \left[1 - e^{-\beta x}\right]^{\alpha} \cdot \left[1 + \lambda - \lambda(1 - e^{-\beta x})^{\alpha}\right]}. \tag{5.2}$$

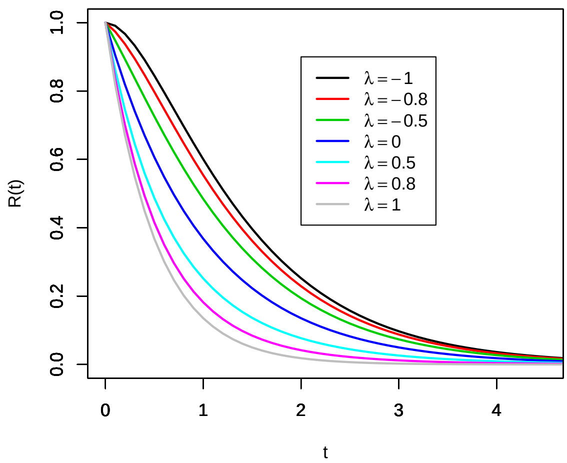

FIGURE 2. Reliability function of transmuted exponentiated exponential for $\alpha = 1$ and $\beta = 1$.

Figure 2 illustrates the reliability behavior of a transmuted exponentiated exponential distribution as the value of the parameter λ varies from -1 to 1. Figure 3 illustrates the behavior of the hazard rate function of a transmuted exponentiated exponential distribution.

6. ORDER STATISTICS

In statistics, the k^{th} order statistic of a statistical sample is equal to its k^{th}-smallest value. Together with rank statistics, order statistics are among the most fundamental tools in non-parametric statistics and inference. For a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is,

$$X_{(n)} = \max\{X_1, X_2, \dots, X_n\}.$$

The sample range is the difference between the maximum and minimum. It is clearly a function of the order statistics:

$$Range\{X_1, X_2, \dots, X_n\} = X_{(n)} - X_{(1)}.$$

We know that if $X_{(1)} \leq X_{(2)} \leq \dots \leq X_{(n)}$ denotes the order statistics of a random sample $X_1, X_2, \dots, X_n$ from a continuous population with cdf $F_X(x)$ and pdf $f_X(x)$ then the pdf of $X_{(j)}$ is given by

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1 - F_X(x)]^{n-j}, \tag{6.1}$$

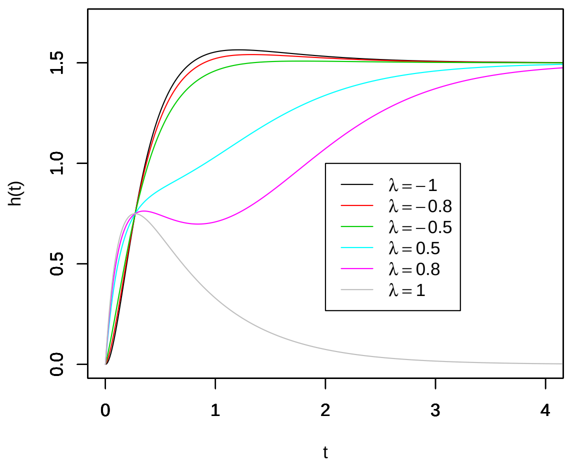

FIGURE 3. Hazard rate function of transmuted exponentiated exponential for $\alpha = 2$ and $\beta = \frac{3}{2}$.

for $j = 1, 2, \ldots, n$. The pdf of the j^{th} order statistic for transmuted exponentiated exponential distribution is given by

$$f_{X_{(j)}}(x) = \frac{\alpha\beta n!}{(j-1)!(n-j)!} e^{-\beta x}\left(1 - e^{-\beta x}\right)^{\alpha j - 1}\left[1 + \lambda - 2\lambda(1 - e^{-\beta x})^{\alpha}\right]^{j}$$

$$\times\left[1 - \left(1 - e^{-\beta x}\right)^{\alpha}\cdot\left(1 + \lambda - \lambda(1 - e^{-\beta x})^{\alpha}\right)\right]^{n-j}. \tag{6.2}$$

Therefore, the pdf of the largest order statistic $X_{(n)}$ is given by

$$f_{X_{(n)}}(x) = \alpha\beta n e^{-\beta x}\left(1 - e^{-\beta x}\right)^{\alpha n - 1}\left[1 + \lambda - 2\lambda(1 - e^{-\beta x})^{\alpha}\right]^{n},$$

and the pdf of the smallest order statistic $X_{(1)}$ is given by

$$f_{X_{(1)}}(x) = \alpha\beta n e^{-\beta x}\left(1 - e^{-\beta x}\right)^{\alpha - 1}\left[1 + \lambda - 2\lambda(1 - e^{-\beta x})^{\alpha}\right]$$

$$\times\left[1 - \left(1 - e^{-\beta x}\right)^{\alpha}\cdot\left(1 + \lambda - \lambda(1 - e^{-\beta x})^{\alpha}\right)\right]^{n-1}.$$

7. Data Analysis

In this section we fit the transmuted exponentiated exponential distribution model to one real data set. The data set is obtained from Smith and Naylor [7]. The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data set is given in table 1.

0.55	0.93	1.25	1.36	1.49	1.52	1.58
1.61	1.64	1.68	1.73	1.81	2.0	0.74
1.04	1.27	1.39	1.49	1.53	1.59	1.61
1.66	1.68	1.76	1.82	2.01	0.77	1.11
1.28	1.42	1.50	1.54	1.60	1.62	1.66
1.69	1.76	1.84	2.24	0.81	1.13	1.29
1.48	1.5	1.55	1.61	1.62	1.66	1.70
1.77	1.84	0.84	1.24	1.30	1.48	1.51
1.55	1.61	1.63	1.67	1.70	1.78	1.89

TABLE 1. The strengths of 1.5 cm glass fibres

Distri.	Param.Estim.	-2LL	AIC	BIC	KS
Trans.	$\hat{\alpha} = 31.153$	56.951	61.951	69.380	0.21
EExp.	$\hat{\beta} = 2.909$				
	$\hat{\lambda} = -0.695$				
EExpo.	$\hat{\alpha} = 31.348$	62.766	66.766	71.053	0.23
	$\hat{\beta} = 2.611$				

TABLE 2. Criteria for Comparison

In order to compare the distributions, we consider some criterion like $-2\log(L)$, AIC (Akaike Information Criterion), BIC(Bayesian information criterion) and KS(Kolmogorow Smirnow)test statistics of the empirical distribution function for the real data set.

$$KS = \max_{1\leq i\leq n}\left(F(X_i) - \frac{i-1}{n}, \frac{i}{n} - F(X_i)\right)$$

$$AIC = 2k - 2\log(L),$$

and

$$BIC = k\log(n) - 2\log L,$$

where $F(X_i)$ is cdf, k is the number of parameters in the statistical model, n the sample size and L is the maximized value of the likelihood function for the estimated model. The best distribution correspond to lower $-2\log(L)$,AIC, BIC and KS values.

From table 2, it can be seen that the transmuted exponentiated exponential distribution fits the subject data better than the exponentiated exponential distribution.

The LR statistics to test the hypotheses $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$: $\omega = 5.81594 > 3.841 = \chi_1^2(\alpha = 0.05)$, so we reject the null hypothesis.

8. CONCLUSION

In this article, we propose a new model so-called the transmuted exponentiated exponential distribution to extend the exponentiated exponential distribution in the analysis of data with real support. An obvious reason for generalizing a

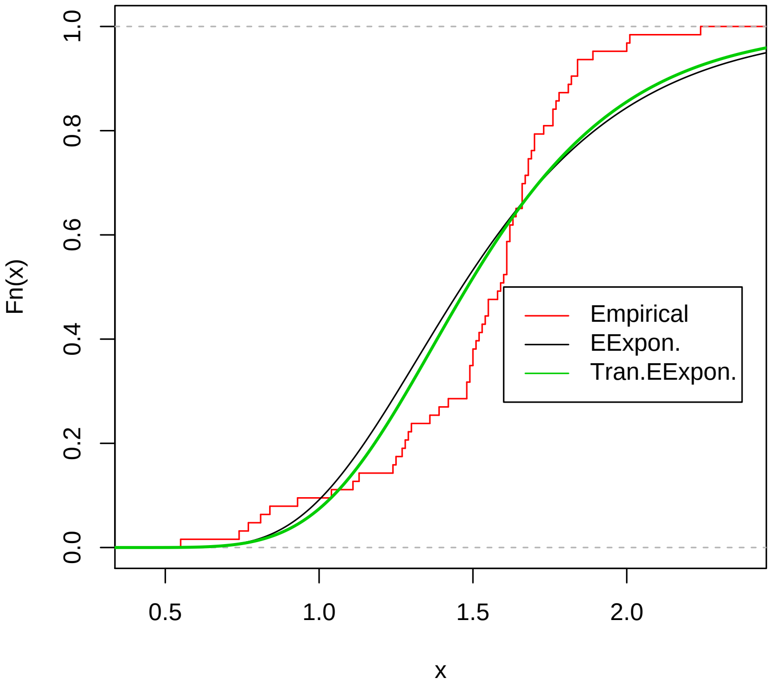

FIGURE 4. Empirical, fitted exponentiated exponential and transmuted exponentiated exponential cdf of the strengths of 1.5 cm glass fibres data

standard distribution is because the generalized form provides greater flexibility in modeling real data. We derive expansions for the expectation,variance, moments and the moment generating function. The estimation of parameters is approached by the method of maximum likelihood. We consider the likelihood ratio statistic to compare the model with its baseline model. An application of the transmuted exponentiated exponential distribution to real data show that the new distribution can be used quite effectively to provide better fits than the exponentiated exponential distribution.

References

[1] Ahuja, J.C. and Nash, S.W., The generalized Gompertz-Verhulst family of distributions. *Sankhya,* (1967) 29, 141-161.

[2] Aryal G.R. and Tsokos Ch., On the transmuted extreme value distribution with application. *Nonlinear Analysis: Theory, Methods and Applications,* (2009) 71: 1401 1407.

[3] Aryal G.R., Tsokos Ch. P., Transmuted Weibull Distribution: A Generalization of the Weibull probability distribution,*Europian journal of pure and applied mathemtics* (2011) Vol. 4, No. 2, 89- 102.

[4] Gupta, R.D. and Kundu, D., Generalized exponential distributions, *Australian and New Zealand Journal of Statistics,* (1999) vol. 41, 173 - 188,
[5] Gupta R. D., Kundu D., Exponentiated Exponential Family: An Alternative to Gamma and Weibull Distributions, *Biometrical Journal* (2001) 43 1, 117 130 .
[6] Shaw, W. T., Buckley, I. R. The alchemy of probability distributions: beyond Gram-Charlier expansions, and a skew-kurtotic-normal distribution from a rank transmutation map. arXiv preprint arXiv:0901.0434.(2009)
[7] Smith, R. L. and Naylor, J. C., A comparison of maximum likelihood and Bayesian estimators for the three-parameter Weibull distribution, *Applied Statistics,* (1987) 36, pp. 358-369.

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