Mathematical Sciences And Applications E-Notes
Volume 1 No. 2 pp. 143–149 (2013) ©MSAEN

COUNTING THE GENERATOR MATRICES OF $\mathbb{Z}_2\mathbb{Z}_8$-CODES

IRFAN SIAP, ISMAIL AYDOGDU

(Communicated by Murat TOSUN)

Abstract. In this paper, we count the number of matrices whose rows generate different $\mathbb{Z}_2\mathbb{Z}_8$ additive codes. This is a natural generalization of the well known Gaussian numbers that count the number of matrices whose rows generate vector spaces with particular dimension over finite fields. Due to this similarity we name this numbers as Mixed Generalized Gaussian Numbers (MGN). By specialization of MGN formula the well known formula for the number of binary codes and the number of codes over $\mathbb{Z}_8$, and for additive $\mathbb{Z}_2\mathbb{Z}_4$ codes are easily derived. Also, we conclude by some properties and examples of the MGN numbers that provide a good source for new number sequences that are not listed in The On-Line Encyclopedia of Integer Sequences.

1. Introduction

Let $\mathbb{Z}_m$ be the ring of integers modulo m. $\mathbb{Z}_m^n$ will denote the the set of cartesian product of n copies of $\mathbb{Z}_m$. Any nonempty subset C of $\mathbb{Z}_m^n$ is called a code and a subgroup of a $\mathbb{Z}_m^n$ is called a linear code of length n. For the special cases $m = 2$ and $m = 4$, the codes are called binary and quaternary codes respectively. Most of the work and applications in digital communications is done on binary linear codes. However, due to the relations between the algebraic structures via some special maps which are referred to as Gray maps, the images of codes over non binary rings provide structural binary codes. In this context, such a work is introduced by Hammons et al. [10] and since then, the study on codes over various rings has been of quite interest on Algebraic Coding Theory. One of such a successful attempt is the study of codes which are group isomorphic to additive subgroups of the group $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ where α and β positive integers. It is clear that in $\mathbb{Z}_2^\alpha \times \mathbb{Z}_4^\beta$ if β does not exist then the subgroups give linear binary codes, or if α does not exist then the subgroups give linear quaternary codes. So this is a generalization of the well known families of (binary and quaternary) codes and are known as additive codes. Additive codes were originally defined by Delsarte in 1973 in the context of association schemes [8, 9]. Such abelian groups also appear in the work by Puyol

Date: Received: February 20, 2012; Accepted: August 14, 2013.
2000 *Mathematics Subject Classification.* 94B05, 94B60.
Key words and phrases. Gaussian numbers, Mixed Generalized Gaussian Numbers, New Sequences.

at el. in [12]. Also, codes defined over two different alphabets which are binary and ternary fields and called mixed codes are studied by Brouwer at el. in [7].

The basic and introductory concepts on $\mathbb{Z}_2\mathbb{Z}_4-$codes are presented in [2, 5, 6]. An additive code C over $\mathbb{Z}_2\times\mathbb{Z}_4$ which is a subgroup of $\mathbb{Z}_2^{\alpha}\times\mathbb{Z}_4^{\beta}$ is group isomorphic to $\mathbb{Z}_2^{k_0}\times\mathbb{Z}_2^{k_1}\times\mathbb{Z}_4^{k_2}$. Here, k_0 represents the number of generators of the subgroup C of order 2 that are contributed through the binary ($\mathbb{Z}_2$) part, k_1 represents the number of generators of the subgroup C of order 4 that are contributed through the quaternary ($\mathbb{Z}_4$) part and k_2 represents the number of generators of the subgroup C of order 2 that are contributed through the quaternary ($\mathbb{Z}_4$) part. Thus, this leads to the following fact that is proved in [6]: an additive $\mathbb{Z}_2\mathbb{Z}_4$ code of type $(\alpha,\beta;k_0,k_1,k_2)$ is equivalent to an additive code generated by the following matrix ([6])

$$G=\left[\begin{array}{cc|ccc} I_{k_0} & \bar{A}_{01} & 0 & 0 & 2T_{02}\\ 0 & S_1 & I_{k_1} & A_{01} & A_{02}\\ 0 & 0 & 0 & 2I_{k_2} & 2A_{12}\end{array}\right], \tag{1.1}$$

(P.S. The artificial vertical line only helps to distinguish between the binary and and non binary (quaternary) parts.)

Similar to the discussions above, if C is a $\mathbb{Z}_2\mathbb{Z}_8-$ additive code of type $(\alpha,\beta;k_0,k_1,k_2,k_3)$, then in [1] it is proven that C is equivalent to a code generated by the following matrix ([1])

$$G=\left[\begin{array}{cc|cccc} I_{k_0} & \bar{A}_{01} & 0 & 0 & 0 & 4T_{03}\\ 0 & S_1 & I_{k_1} & A_{01} & A_{02} & A_{03}\\ 0 & S_2 & 0 & 2I_{k_2} & 2A_{12} & 2A_{13}\\ 0 & 0 & 0 & 0 & 4I_{k_3} & 4A_{23}\end{array}\right]. \tag{1.2}$$

Here k_0 represents the number of order 2 generators that are contributed through the binary part, and respectively, k_1, k_2 and k_3 represent the number of order 8, 4 and 2 generators 2 that are contributed through the $\mathbb{Z}_8$ part. Note that the order 2 elements from the $\mathbb{Z}_8$ part have the first α components all zero. This remark will play a crucial role in the main counting theorem in the next section.

We present some facts regarding the duality of this codes which is introduced in [1]. The inner product of two elements $u,v\in\mathbb{Z}_2^{\alpha}\times\mathbb{Z}_8^{\beta}$ is defined as $\langle u,v\rangle=$

$4\left(\sum_{i=1}^{\alpha}u_iv_i\right)+\sum_{j=\alpha+1}^{\alpha+\beta}u_jv_j\in\mathbb{Z}_8$. The additive dual code of C, denoted by $C^{\perp}$, is then defined as

$$C^{\perp}=\left\{v\in\mathbb{Z}_2^{\alpha}\times\mathbb{Z}_8^{\beta}|\langle u,v\rangle=0\text{ for all }u\in C\right\}.$$

It is easy to check that $C^{\perp}$ is a subgroup of $\mathbb{Z}_2^{\alpha}\times\mathbb{Z}_8^{\beta}$, so $C^{\perp}$ is a $\mathbb{Z}_2\mathbb{Z}_8-$additive code too.

Let C be a $\mathbb{Z}_2\mathbb{Z}_8-$additive code of type $(\alpha,\beta;k_0,k_1,k_2,k_3)$ with canonical generator matrix (1.2). Then, the parity-check matrix of C which is the generator matrix of its dual is

$$\begin{bmatrix} -\bar{A}_{01}^t & I_{\alpha-k_0} & & -2S_2^t & 0 & 0\\ -T_{03}^t & 0 & P & -A_{13}^t+A_{23}^tA_{12}^t & -A_{23}^t & I_{\beta-k_1-k_2-k_3}\\ 0 & 0 & & -2A_{12}^t & 2I_{k_3} & 0\\ 0 & 0 & & 4I_{k_2} & 0 & 0\end{bmatrix},$$

where $P = \begin{bmatrix} -4S_1^t + 2S_2^t A_{01}^t \\ -A_{03}^t + A_{13}^t A_{01}^t + A_{23}^t A_{02}^t - A_{23}^t A_{12}^t A_{01}^t \\ -2A_{02}^t + 2A_{12}^t A_{01}^t \\ -4A_{01}^t \end{bmatrix}$.

If C is an $\mathbb{Z}_2\mathbb{Z}_8-$ additive code of type $(\alpha, \beta; k_0, k_1, k_2, k_3)$, then $C^{\perp}$ is an $\mathbb{Z}_2\mathbb{Z}_8-$ additive code of type $(\alpha, \beta; \alpha - k_0, \beta - k_1 - k_2 - k_3, k_3, k_2)$.

2. Mixed Generalized Gaussian Numbers

In this section, we present the main theorem of this paper that gives a direct computation of the number of matrices that generate different (not necessarily equivalent) additive codes. First, we present a very moderate example in order to illustrate the problem. Even in this example, getting the exact number is not an easy problem. As the size of the matrix gets larger the difficulty of counting these matrices becomes a very difficult problem. After stating and proving the main theorem we revisit this example and solve the problem directly.

As mentioned in introduction the counting problem is originated from the study of the number of the subspaces generated by the rows of matrices over finite fields. Recently, there has been some generalizations of these concept on the number of generating matrices of particular types over the ring $\mathbb{Z}_m$ [13], over Galois rings [13] and over rings $F_p + uF_p$. All these generalizations are done for codes over a single alphabet. The main Theorem 2.1 presents a further generalization to the work done in [13] in a different direction.

Example 2.1. Let C be a $\mathbb{Z}_2\mathbb{Z}_8-$ additive code of type $(2, 2; 1, 1, 1, 0)$ then all possible matrices are 36 matrices that generate different codes. Here $\alpha = 2, \beta = 2, k_0 = k_1 = k_2 = 1$, and $k_3 = 0$.

$\begin{bmatrix} 1 & g_{21} & 0 & 0 \\ 0 & g_{22} & 1 & g_{24} \\ 0 & g_{32} & 0 & 2 \end{bmatrix}$ There are 16 such matrices

$\begin{bmatrix} 1 & g_{21} & 0 & 0 \\ 0 & g_{22} & 0 & 1 \\ 0 & g_{32} & 2 & 0 \end{bmatrix}$ There are 8 such matrices.

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ g_{21} & 0 & 1 & g_{24} \\ g_{31} & 0 & 0 & 2 \end{bmatrix}$ There are 8 such matrices

$\begin{bmatrix} 0 & 1 & 0 & 0 \\ g_{21} & 0 & 0 & 1 \\ g_{31} & 0 & 2 & 0 \end{bmatrix}$ There are 4 such matrices,

where all unknown above are either 0 or 1. So altogether we have 36 generating matrices.

Theorem 2.1. *The number of $\mathbb{Z}_2\mathbb{Z}_8$ additive codes of the type (k_0, k_1, k_2, k_3) is equal to*

$$N_{2\times 8}(\alpha, \beta; k_0, k_1, k_2, k_3) = 2^{\delta} \begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \cdot \begin{bmatrix} \beta \\ k_1, k_2, k_3 \end{bmatrix}_2 \tag{2.1}$$

where $\delta = k_0(\beta - l) + k_1(\alpha - k_0 + 2(\beta - l) + k_3) + k_2((\beta - l) + (\alpha - k_0))$ *and* $l = k_1 + k_2 + k_3$.

Proof: In order to prove this theorem we count ordered generators for the group (code) of the type (k_0, k_1, k_2, k_3). First we count them by choosing them from the all space $\mathbb{Z}_2^{\alpha} \times \mathbb{Z}_8^{\beta}$ which gives say A and given a group (code) of type (k_0, k_1, k_2, k_3) then we choose them in within this group. So, if the number of groups of the type (k_0,k_1,k_2,k_3) is $N_{2\times 8}$, then $N_{2\times 8}=A/B$. First, we compute $A:$ In $\mathbb{Z}_2^\alpha\mathbb{Z}_8^\beta$ we can choose an element of order 2 that is contributed through the binary part in $(2^\alpha-1)\cdot 2^\beta$. Next, the second element with the same property can be choose in $(2^\alpha-2)\cdot 2^\beta$ ways, inductively the last element can be chosen in $(2^\alpha-2^{k_0-1})\cdot 2^\beta$. So, in total k_0 elements of order 2 that contribute through the binary part in all space can be chose in N_1 ways. Further, by reinterpreting this formula we get $N_1=\prod_{i=0}^{k_0-1}(2^\alpha-2^i)2^\beta=2^{k_0\beta}2^{\binom{k_0}{2}}\frac{[\alpha]_2!}{[\alpha-k_0]_2!}$.

Next, there are $(8^n-4^n)\cdot 2^\alpha$ ways to pick an element of order 8 contributed through the $\mathbb{Z}_8$ part. The second such element can be chosen in $(8^n-4^n\cdot 4)\cdot 2^\alpha$ ways excluding the linear combinations of the first chosen element of order 8. Inductively, we have N_2 choices for such elements. Again, we have $N_2=\prod_{i=0}^{k_1-1}(8^\beta-4^\beta\cdot 2^i)2^\alpha=2^{2\beta k_1+\alpha k_1}\prod_{i=0}^{k_1-1}(2^\beta-2^i)=2^{2\beta k_1+\alpha k_1+\binom{k_1}{2}}\frac{[\beta]_2!}{[\beta-k_1]_2!}$,

Next, to choose elements of order 4 in the all space, first there are $(4^n-2^n)\cdot 2^{k_1}2^\alpha$ to pick elements of order 4 that are contributed through the $\mathbb{Z}_8$ part. Here, the elements of order 4 that are formed from the k_1 elements of order 8 by taking their 2 multiples need to be considered. Then, similarly as discussed above we have N_3 elements of order 4 in the all space to be chosen which is equal to $N_3=\prod_{i=0}^{k_2-1}(4^\beta-2^{\beta+k_1+i})2^\alpha=2^{(\beta+k_1)k_2+\alpha k_2}\prod_{i=0}^{k_2-1}(2^{\beta-k_1}-2^i)=2^{(\beta+k_1)k_2+\alpha k_2+\binom{k_2}{2}}\frac{[\beta-k_1]_2!}{[\beta-k_1-k_2]_2!}$. Next, to choose elements that are of order 2 and solely contributed through $\mathbb{Z}_8$ part. This imposes that the first α entries of such elements to be all zero. Thus, in order to pick such an element first we subtract order 2 elements that are obtained through already chosen k_1 and k_2 elements for the $\mathbb{Z}_8$ part. So, we have $(2^{k_0+k_1+k_2+k_3}-2^{k_0+k_1+k_2})$ choices. The next choice comes from $(2^{k_0+k_1+k_2+k_3}-2\cdot 2^{k_0+k_1+k_2})$ and inductively we reach N_4. $N_4=\prod_{i=0}^{k_3-1}(2^\beta-2^{k_2+k_1+i})=2^{(k_1+k_2)k_3}\prod_{i=0}^{k_3-1}(2^{\beta-k_1-k_2}-2^i)=2^{(k_1+k_2)k_3+\binom{k_3}{2}}\frac{[\beta-k_1-k_2]_2!}{[\beta-k_1-k_2-k_3]_2!}$. Hence, $A=N_1N_2N_3N_4$. Now, we argue in the similar way within the group of the type (k_0,k_1,k_2,k_3). In order to pick an element of order 2 that contributes through the binary part, we subtract all order 2 elements within the group from the ones that are not coming through the binary part, i.e $(2^{k_0+2k_1+k_2+k_3}-2\cdot 2^{k_0+2k_1+k_2})$ and inductively we reach at D_1. $D_1=\prod_{i=0}^{k_0-1}(2^{k_0+k_1+k_2+k_3}-2^{k_1+k_2+k_3+i})=2^{(k_1+k_2+k_3)k_0}\prod_{i=0}^{k_0-1}(2^{k_0}-2^i)=2^{(k_1+k_2+k_3)k_0+\binom{k_0}{2}}[k_0]_2!$. Next, to choose an element of order 8 within the group, we have $(8^{k_1}\cdot 2^{k_0+2k_2+k_3}-4^{k_1}\cdot 2^{k_0+2k_2++k_3})$ choices. Inductively, we have D_2 choices altogether. $D_2=\prod_{i=0}^{k_1-1}(8^{k_1}-4^{k_1}2^i)2^{k_0+2k_2+k_3}=2^{(2k_1+k_0+2k_2+k_3)k_1+\binom{k_1}{2}}[k_1]_2!$. Next, to choose an element of order 4 within the group, we have $(4^{k_2}-2^{k_2})\cdot 2^{k_0+2k_1+k_3}$ choices. Inductively, we have D_3 choices altogether where $D_3=\prod_{i=0}^{k_2-1}(4^{k_2}-2^{k_2+i})2^{k_0+2k_1+k_3}=2^{(2k_1+k_0+k_2+k_3)k_2+\binom{k_2}{2}}[k_2]_2!$. Finally, to choose an element of order 2 within the group, we have $(2^{k_1+k_2+k_3}-2^{k_1+k_2})$ choices. Inductively, we have D_4 choices altogether, i.e $D_4=\prod_{i=0}^{k_3-1}(2^{k_1+k_2+k_3}-2^{k_1+k_2+i})=2^{(k_1+k_2)k_3}\prod_{i=0}^{k_3-1}(2^{k_3}-2^i)=2^{(k_1+k_2)k_3+\binom{k_3}{2}}[k_3]_2!$. Thus, $B=D_1D_2D_3D_4$ and now we write A/B and relate this to Gauss multinomial coefficients as follows:

$$N=2^\delta\frac{[\alpha]_2!}{[k_0]_2![\alpha-k_0]_2!}\frac{[\beta]_2!}{[k_1]_2![\beta-k_1]_2!}\frac{[\beta-k_1]_2!}{[k_2]_2![\beta-k_1-k_2]_2!}\frac{[\beta-k_1-k_2]_2!}{[k_3]_2![\beta-k_1-k_2-k_3]_2!}$$

i.e.

$$N = 2^{\delta}\begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \cdot \begin{bmatrix} \beta \\ k_1 \end{bmatrix}_2 \cdot \begin{bmatrix} \beta - k_1 \\ k_2 \end{bmatrix}_2 \cdot \begin{bmatrix} \beta - k_1 - k_2 \\ k_3 \end{bmatrix}_2 \tag{2.2}$$

and further, we have

$$N = 2^{\delta}\begin{bmatrix} \alpha \\ k_0 \end{bmatrix}_2 \cdot \begin{bmatrix} \beta \\ k_1, k_2, k_3 \end{bmatrix}_2 \tag{2.3}$$

where $\delta = k_0(\beta - l) + k_1(\alpha - k_0 + 2(\beta - l) + k_3) + k_2((\beta - l) + (\alpha - k_0)$ and $l = k_1 + k_2 + k_3$. □

Example 2.2. (An application of the main theorem) In Example 2.1, by explicitly working out the cases, we computed the $N_{2\times 8}(2,2;1,1,1,0)$ which is alternatively given in a direct way by Theorem 2.1:

$N_1(2,2;1,1,1,0) = 12$, $N_2 = 192$, $N_3 = 32$, N_4 does not exist so we skip this term. $D_1 = 4$, $D_2 = 32$, $D_3 = 16$, and we disregard D_4. Hence,

$$N_{2\times 8}(2,2;1,1,1,0) = \frac{N_1 N_2 N_3}{D_1 D_2 D_3} = \frac{73728}{2048} = 36.$$

The next corollary shows that the number of distinct linear codes over $\mathbb{Z}_8$ ([14]) can be obtained via the main Theorem 2.1.

Corollary 2.1. *Let $N_8(n;k_1,k_2,k_3)$ be the number of distinct linear codes of type (k_1,k_2,k_3) over $\mathbb{Z}_8$. If, $r = n - k_1 + k_2 + k_3$, then,*

$$N_8(n,k_1,k_2,k_3) = 2^{-r} \cdot N_{2\times 8}(1,n;1,k_1,k_2).$$

In the sequel we relate the q binomial and multinomial coefficients with MGN. First, the following corollary shows that the number of distinct binary linear codes over $\mathbb{Z}_2$ ([11]) of length n can be obtained via the main Theorem 2.1, by simple observation we get the formula for the 2-binomial coefficients (Gaussian Numbers over $\mathbb{Z}_2$).

Corollary 2.2. $\begin{bmatrix} n \\ k \end{bmatrix}_2 = N_{2\times 8}(n,1;k,0,0,1).$

Now, we can also give the number of distinct codes of a code having dual code parameters of a code of type $(\alpha,\beta;k_0,k_1,k_2,k_3)$ which is equal to

$$N_{2\times 8}(\alpha,\beta;\alpha - k_0, \beta - l, k_3, k_2) = 2^{\overline{\delta}}\begin{bmatrix} \alpha \\ \alpha - k_0 \end{bmatrix}_2 \cdot \begin{bmatrix} \beta \\ \beta - l, k_3, k_2 \end{bmatrix}_2 \tag{2.4}$$

where $l = \sum_{i=0}^{2} k_i$ and $\overline{\delta} = k_1(\alpha - k_0) + (\beta - l)(k_0 + 2k_1 + k_2) + k_3(k_1 + k_0)$.

The following lemma states a condition for the number of codes that equal to the number their duals can be shown by applying the definitions carefully:

Lemma 2.1. *If $\alpha k_2 = k_0(k_2 + k_3)$, then*

$$N_{2\times 8}(\alpha,\beta;\alpha - k_0, \beta - l, k_3, k_2) = N_{2\times 8}(\alpha,\beta;\alpha - k_0, \beta - l, k_3, k_2). \tag{2.5}$$

As special cases to the previous Lemma , we have the following two corollaries:

Corollary 2.3. *If C is an additive $\mathbb{Z}_2\mathbb{Z}_8$-code of type $(r,s;k_0,k_1,0,0)$, then the number of such codes is equal to the number of codes with parameters of its dual.*

Corollary 2.4. *The number of additive $\mathbb{Z}_2\mathbb{Z}_8$-codes that are of type $(r, s; k_0, 0, k_2, s-k_2)$ and their duals are equal.*

Moreover, if we let $N_{2\times 4}(\alpha, \beta; k_0, k_1, k_2)$ denote the number of $\mathbb{Z}_2\mathbb{Z}_4$ distinct additive codes of type $(\alpha, \beta; k_0, k_1, k_2)$. Now, by making use of the observation and facts mentioned above we easily obtain the following corollary that gives a formula for the number of $\mathbb{Z}_2\mathbb{Z}_4$ additive codes of the type $(k_0; k_1, k_2)$ by making use of Theorem 2.1.

Corollary 2.5.

$$N_{2\times 4}(\alpha, \beta; k_0, k_1, k_2) = N_{2\times 8}(\alpha, \beta; k_0, 0, k_2, k_3).$$

3. Some Properties And New Number Sequences

The main theorem produced a formula that also enjoys some properties by its own such as classical and Gaussian binomials do. Here, we present some properties and also some new number sequences that are not recorded yet in the literature [16].

We list some further properties and skip the proof since it can be shown by applying the definitions carefully.

Lemma 3.1. (1) *Let r, s, k, l, m, t be non negative integers. Also let $m \le r$ and $s = k + l$. Then,*

$$N_{2\times 8}(r, s; m, k, l, 0) = N_{2\times 8}(r, s; m, l, k, 0).$$

Let $r, s \in \mathbb{Z}^+$.

(2) $\frac{N_{2\times 8}(r+1,s;1,1,1,0)}{N_{2\times 8}(r,s;1,1,1,0)} = 4\frac{(2^{r+1}-1)}{(2^r-1)}$.

(3) $N_{2\times 8}(r, s; r, s, 0, 0) = 1, N_{2\times 8}(r, s; r, 0, s, 0) = 1,$ *and* $N_{2\times 8}(r, s; r, 0, 0, s) = 1$.

(4) $N_{2\times 8}(1, r; 1, 1, 1, 0) = 2^{4r-9}(2^r - 2)(2^r - 1)$,

$$N_{2\times 8}(\alpha + 1, r; 1, 1, 1, 0) = 4N_{2\times 8}(\alpha, r; 1, 1, 1, 0) + (2^r - 1) \cdot (2^{r-1} - 1) \cdot 2^{3\alpha+4(r-2)}$$

where $\alpha \ge 1, r \ge 2$.

(5) $N_{2\times 8}(\alpha, \beta; \alpha, k_1, k_2, k_3) = N_{2\times 8}(1, \beta; 1, k_1, k_2, k_3)$ *for all* $\alpha \ge 1$.

Besides many sequences that we run into in this research we would like to mention a few of them. $N_{2\times 8}(1, k; 1, 1, 1, 1)$ where $k \ge 3$ and the sequence with its first three entries is $\{42, 10080, 1666560, 239984640, \ldots\}$. This also a new sequence which is not listed in [16]. We present some new sequences that are not recorded in Sloane's "The On-Line Encyclopedia of Integer Sequences (OEIS)" ("http://oeis.org/" accessed on March 8th, 2013) in Table (3).

4. Conclusion

In this work we established a formula that gives the number of distinct additive $\mathbb{Z}_2\mathbb{Z}_8$-codes. By specializing the parameters in this formula, we easily obtain the number of distinct codes over the ring $\mathbb{Z}_8$ and $\mathbb{Z}_2\mathbb{Z}_4$-codes. Further, some properties of this formula that is defined by the authors as Mixed Gaussian numbers are studied and some new number sequences are presented. Since Mixed Gaussian numbers are generalizations of Gaussian numbers we we believe that there further properties that waits to be explored.

The Sequences	Status
$N_{2\times 8}(\alpha, r; 1, 1, 1, 0); \qquad 1 \leq \alpha \leq 8, 2 \leq r \leq 4$	New
$N_{2\times 8}(r, 2k; r, k, 0, k) = \{6, 560, 714240, 13158776832, \ldots\}$ for $k \geq 1$ and $r \geq 1$	New
$N_{2\times 8}(r+1, 2; r, 1, 1, 0) = \{36, 84, 180, 372, 756, ...\}$; $r \geq 1$	New
$N_{2\times 8}(r+1, 3; r, 1, 1, 1) = \{504, 1176, 2520, 5208, 10584, ...\}$; $r \geq 1$	New
$N_{2\times 8}(r+1, 2r+1; r, 0, r, r) = \{504, 486080, 1360627200, ...\}$; $r \geq 1$	New
$N_{2\times 8}(r+2, 2r+1; r, 0, 1, r) = \{2352, 9721600, 449914060800, ...\}$; $r \geq 1$	New
$N_{2\times 8}(r, r+2; 2, 0, 1, r) = \{840, 52080, 2187360, ...\}$; $r \geq 2$	New
$N_{2\times 8}(r, 2k; r, k, k, 0) = N_{2\times 8}(r, 2k; r, 0, k, k) = \{3, 35, 1395, 200787, \ldots\}, r, k \geq 1$	(*)

Table 1. This table is partial list of the results. (*) exists in([16]) coded by A006098.

References

[1] Aydogdu, I. and Siap, I., The Structure of $\mathbb{Z}_2\mathbb{Z}_{2^s}$ Additive Codes: Bounds on the minimum distance, *Applied Mathematics & Information Sciences (AMIS)*, **7**, 6, 2271-2278 (2013).

[2] Bilal, M., Borges, J., Dougherty, S., Fernandez, C., Optimal Codes over $\mathbb{Z}_2\mathbb{Z}_4$ *In libro de acts VII Jornadas de Matematica Discreta i Algoritmica*, Castro Urdiales (Spain), 131-139, (2010).

[3] Bilal, M., Borges, J., Dougherty, S., Fernandez, C., Extensions of $\mathbb{Z}_2\mathbb{Z}_4$ -additive self-dual codes preserving their properties, *IEEE International Symposium on Information Theory* , 3101- 3105 (2012).

[4] Bona, M., Combinatorics of permutations,Discrete Mathematics and Its Applications, *Chapman and Hall/CRC*, (2004).

[5] Borges, J., Fernandez, C., Pujol, J., Rifa, J. and Villanueva, M., On $\mathbb{Z}_2\mathbb{Z}_4$ -linear codes and duality, *V Jornadas de Matematica Discreta i Algoritmica*, Soria (Spain), Jul. 11-14, 171-177, (2006).

[6] Borges, J., Fernandez-Cordoba, C., Pujol, J., Rifa, J. and Villanueva, M., $\mathbb{Z}_2\mathbb{Z}_4$-linear codes: generator matrices and duality, *Designs, Codes and Cryptography*, **54** (2), 167-179, (2010).

[7] Brouwer, A.E., Hamalainen,H.O., Ostergard, P.R.J., Sloane, N.J.A., Bounds on Mixed Binary/Ternary Codes, *IEEE Transactions on Information Theory* **44** (1): 140-161 (1998)

[8] Delsarte, P., An algebraic approach to the association schemes of coding theory, *Philips Research Rep.Supp.*, **10,** vi+97 (1973).

[9] Delsarte, P., Levenshtein, V.:Association schemes and coding theory, *IEEE Trans.Inform.Theory*, **44** (6) 2477-2504 (1998).

[10] Hammons, A.R., Kumar, V., Calderbank, A.R., Sloane, N.J.A., Solé, P., The $\mathbb{Z}_4$-linearity of Kerdock, Preparata, Goethals, and related codes, *IEEE Trans. Inform. Theory*, **40** 301-319 (1994).

[11] MacWilliams, F.J., and Sloane, N.J.A., The Theory of Error-Correcting Codes, North-Holland: New York, NY, (1977).

[12] Pujol J., Rifa J., Translation invariant propelinear codes, *IEEE Trans. Inform. Theory* **43** 590-598 (1997).

[13] Salturk E., Siap I., On Generalized Gaussian Numbers, *Albanian Journal of Mathematics*, **6**, 2 87-102 (2012).

[14] Salturk E., Siap I., Generalized Gaussian Numbers Related to Linear Codes over Galois Rings,*European Journal of Pure and Applied Mathematics* **5** 250-259 (2012).

[15] Salturk E., Siap I., Generalized Gaussian Numbers and Some New Sequences, Physica Macedonica, accepted **6**, (2013).

[16] Sloane, N., The On-Line Encyclopedia of Integer Sequences (OEIS), (”*http://oeis.org/*” accessed on March 8th, 2013).

Yildiz Technical University, Faculty of Arts and Science, Department of Mathematics, Istanbul-Turkey

E-mail address: `isiap@yildiz.edu.tr,iaydogdu@yildiz.edu.tr`