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THE EFFECT OF DRAZIN INVERSE IN SOLVING SINGULAR DUAL FUZZY LINEAR SYSTEMS

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Abstract. The dual fuzzy linear system $A\tilde{x} = B\tilde{x} + \tilde{y}$ where A, B are both crisp matrices, $\tilde{x}$ and $\tilde{y}$ are fuzzy vectors is called a singular dual fuzzy linear system while the coefficients matrix of the extended linear system of it be a crisp singular matrix. In this paper, on the solving the singular dual fuzzy linear system using Drazin inverse is discussed.

1. Introduction

A matrix has an inverse only if it is square, and even then only if it is nonsingular or, in other words, if its columns (or rows) are linearly independent. In recent years, needs have been felt in numerous areas of applied mathematics for some kind of partial inverse of a matrix that is singular or even rectangular. Any square matrix, even a singular matrix, has a unique inverse. This inverse is called the Drazin inverse, because it was first studied by Drazin (though in the more general context of rings and semigroups without specific reference to matrices). The spectral properties of the Drazin inverse of a square matrix have been studied by Cline and Greville; not all of them will be mentioned in [5]. In [11] some results on the index of matrix and Drazin inverse, are given.

The concept of fuzzy numbers and fuzzy arithmetic operations were first introduced by Zadeh [1]. One of the major applications using fuzzy number arithmetic is treating linear systems their parameters are all or partially represented by fuzzy numbers [2]. The linear system of equations

$$A\tilde{x} = \tilde{b}, \tag{1.1}$$

where $A \in R^{(m\times n)}$ is a crisp matrix and the right-hand side is a fuzzy vector is called a fuzzy linear system. Friedman et al. [8] use an embedding method for solving an special case of (1.1) when A is a nonsingular matrix. They introduce new notions for solution of a fuzzy linear system (1.1) and transform the fuzzy linear system to a $2m \times 2n$ crisp linear system. Conditions for the existence of a unique fuzzy solution to linear system are derived and a numerical procedure for calculating the solution

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is designed in [8]. On the inconsistent fuzzy matrix equations and its fuzzy least squares solutions is discussed in [14]. In [4] the original fuzzy linear system with a nonsingular matrix A is replaced by two $n \times n$ crisp linear system. A method for finding an inner estimation of the solution set of a fuzzy linear system is given in [12]. Fuzzy linear system of equations, play a major role in several applications in various area such as engineering, physics and economics. An application of dual linear system of equations in Leontieff input-output analysis is explained [13].

The fuzzy linear system $A\tilde{X} = B\tilde{X} + \tilde{Y}$ where A, B are both crisp matrix, $\tilde{X}$ and $\tilde{Y}$ are fuzzy vectors is called a dual fuzzy linear system of equations. The existence of a solution of duality fuzzy linear equation system is investigated and two necessary and sufficient conditions for the solution existence are given [10]. According to fuzzy arithmetic, dual fuzzy linear system can not be replaced by a fuzzy linear system. A numerical method for finding minimal solution of a general duality fuzzy linear system based on pseudo-inverse calculation, is given by Abbasbandy et al [1].

The aim of this paper, is present a method for solving singular dual fuzzy linear systems using Drazin inverse. Some preliminaries are studied in section 2. In section 3, the singular dual fuzzy linear systems is introduced and the effect of Drazin inverse in solving consistent or inconsistent singular dual fuzzy linear systems is investigated. Then we give numerical examples to illustrate previous sections in section 5. Section 6 ends the paper with the conclusions and suggestions remarks.

2. Preliminaries

In this section, the following definitions and basic results on the index of matrix, Drazin inverse and fuzzy linear systems are reviewed. We refer the readers to [3,7,9,15] for more details.

Definition 2.1. The index of matrix $A \in C^{n\times n}$ is the dimension of largest Jordan block corresponding to the zero eigenvalue of A and is denoted by $ind(A)$.

Definition 2.2. Let $A \in C^{n\times n}$, with $ind(A) = k$. The matrix X of order n is the Drazin inverse of A ,denoted by A^D , if X satisfies the following conditions

$$AX = XA, XAX = X, A^k XA = A^k.$$

When $ind(A) = 1$, A^D is called the group inverse of A , and denoted by A_g .

Theorem 2.1. [5, 6] *Let $A \in C^{n\times n}$, with $ind(A) = k$, $rank(A^k) = r$. We may assume that the Jordan normal form of A has the form as follows*

$$A = P \begin{pmatrix} D & 0 \\ 0 & N \end{pmatrix} P^{-1},$$

where P is a nonsingular matrix, D is a nonsingular matrix of order r , and N is a nilpotent matrix that $N^k = \bar{o}$. Then we can write the Drazin inverse of A in the form

$$A^D = P \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix} P^{-1}.$$

When $ind(A) = 1$, it is obvious that $N = \bar{o}$.

Theorem 2.2. [5] *$A^D b$ is a solution of*

$$Ax = b, k = ind(A), \tag{2.1}$$

if and only if $b \in R(A^k)$, and $A^D b$ is an unique solution of (2.1) provided that $x \in R(A^k)$.

Definition 2.3. [1] A fuzzy number u in parametric form is a pair $(\bar{u}(r), \underline{u}(r))$ of functions $\bar{u}(r), \underline{u}(r), 0 \leq r \leq 1$, which satisfy the following requirements
1. $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0, 1]$.
2. $\bar{u}(r)$ is a bounded left continuous non-increasing function over $[0, 1]$.
3. $\underline{u}(r) \leq \bar{u}(r), 0 \leq r \leq 1$.

Definition 2.4. For arbitrary fuzzy numbers $\tilde{x} = (\underline{x}(r), \bar{x}(r))$, $\tilde{y} = (\underline{y}(r), \bar{y}(r))$ and $k \in R$, we may define the addition and the scalar multiplication of fuzzy numbers as
1. $\tilde{x} + \tilde{y} = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,
2. $k \times \tilde{x} = \begin{cases} (k\underline{x}, k\bar{x}) & k \geq 0 \\ (k\bar{x}, k\underline{x}) & k < 0 \end{cases}$

Definition 2.5. [1] The fuzzy linear system

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_m \end{pmatrix} = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \vdots \\ \tilde{x}_n \end{pmatrix} + \begin{pmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_m \end{pmatrix}, \tag{2.2}$$

where $A = (a_{ij}), B = (b_{ij}); 1 \leq i \leq m,\ 1 \leq j \leq n$ are both crisp matrix, $\tilde{x}$ and $\tilde{y}$ are fuzzy vectors is called a dual fuzzy linear system of equations. The dual fuzzy linear systems (2.2) can be extended into a crisp linear system as follows

$$\begin{pmatrix} s_{11} & \cdots & s_{1n} \\ \vdots & \ddots & \vdots \\ s_{2m1} & \cdots & s_{2m2n} \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{pmatrix} = \begin{pmatrix} m_{11} & \cdots & m_{1n} \\ \vdots & \ddots & \vdots \\ m_{2m1} & \cdots & m_{2m2n} \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{pmatrix} + \begin{pmatrix} \underline{b}_1 \\ \vdots \\ \underline{b}_n \\ -\bar{b}_1 \\ \vdots \\ -\bar{b}_m \end{pmatrix},$$

where s_{ij} are determined as follows :

$$a_{ij} \geq 0 \Rightarrow s_{ij} = a_{ij},\ s_{i+n,j+n} = a_{ij},$$
$$a_{ij} < 0 \Rightarrow s_{i,j+n} = -a_{ij},\ s_{i+n,j} = -a_{ij},$$

and

$$b_{ij} \geq 0 \Rightarrow m_{ij} = b_{ij},\ m_{i+n,j+n} = b_{ij},$$
$$b_{ij} < 0 \Rightarrow m_{i,j+n} = -b_{ij},\ m_{i+n,j} = -b_{ij},$$

while all the remaining s_{ij}, m_{ij} are taken zero. Using matrix notation we get

$$SX = MX + Y(r),$$

where $S = (s_{ij}), 1 \leq i \leq 2m, 1 \leq j \leq 2n$ and $M = (m_{ij}), 1 \leq i \leq 2m, 1 \leq j \leq 2n$ and

$$X = \begin{pmatrix} \underline{x}_1 \\ \vdots \\ \underline{x}_n \\ -\bar{x}_1 \\ \vdots \\ -\bar{x}_n \end{pmatrix},\ Y(r) = \begin{pmatrix} \underline{y}_1 \\ \vdots \\ \underline{y}_n \\ -\bar{y}_1 \\ \vdots \\ -\bar{y}_m \end{pmatrix}, \tag{2.3}$$

Consequently,

$$(S-M)X = Y(r),\ k = ind(S-M), \tag{2.4}$$

is a crisp linear system of equations. The structure of $S-M$ implies that

$$S-M = \begin{pmatrix} B & C \\ C & B \end{pmatrix}.$$

Theorem 2.3. [14] *The dual fuzzy system $A\tilde{X} = B\tilde{X} + \tilde{Y}$ is called a consistent fuzzy linear system, if and only if the crisp system (2.4) be consistent system. i.e.,*

$$rank[S-M] = rank[S-M|Y(r)].$$

Definition 2.6. Let $X = \{\underline{x}_i(r), -\bar{x}_i(r)), 1 \le i \le n\}$ denote a solution of (2.4). The fuzzy number vector $U = \{\underline{u}_i(r), -\bar{u}_i(r)), 1 \le i \le n\}$ defined by

$$\underline{u}_i(r) = min\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\},$$
$$\bar{u}_i(r) = max\{\underline{x}_i(r), \bar{x}_i(r), \underline{x}_i(1), \bar{x}_i(1)\},$$

is called a fuzzy solution of $SX = Y$. If $(\underline{x}_i(r), \bar{x}_i(r)), 1 \le i \le n)$, are all fuzzy numbers and $\underline{x}_i(r) = \underline{u}_i(r), \bar{x}_i(r) = \bar{u}_i(r), 1 \le i \le n$, then U is called a strong fuzzy solution. Otherwise, U is a weak fuzzy solution.

3. Indicial Equations

In this section, singular dual fuzzy linear systems is introduced and an application of Drazin inverses in solving consistent or inconsistent singular dual fuzzy linear systems is explained.

Definition 3.1. The dual fuzzy linear system $A\tilde{x} = B\tilde{x} + \tilde{y}$ is called a singular dual fuzzy linear system (SDFLS) while the coefficients matrix of the extended linear system of it be a crisp singular matrix. On the other words, in the crisp linear system

$$(S-M)X = Y(r),\ k = ind(S-M), \tag{3.1}$$

the index of the matrix $(S-M)$ be nonzero.

Theorem 3.1. *The consistent crisp linear system (3.1) has a set of solutions and*

$$X = (S-M)^D Y(r),$$

is the element of this set if and only if $Y \in R(S-M)^k$, and $(S-M)^D Y(r)$ is an unique solution of (3.1) provided that $X \in R((S-M)^k)$.

Proof. The system (3.1) is consistent. i.e. $rank[S-M] = rank[S-M|Y]$. By theorem2.3 we know the linear system (3.1) has solution and from [8] the singular linear system (3.1) has a set of solutions. Same as proof of theorem2.3 in [6],

$$X = (S-M)^D Y(r)$$

is a member of set of solutions of the system if and only if $Y \in R(S-M)^k$.

Definition 3.2. Let

$$A\tilde{x} = B\tilde{x} + \tilde{y}$$

be a consistent or inconsistent singular dual fuzzy linear systems, and the extended system of it, is

$$(S-M)X = Y(r),$$

The consistent system

$$(S-M)^k(S-M)X=(S-M)^kY(r),$$

wherein $k=ind(S-M)$ is called indicial equations.

Corollary 3.1. *According to [11] and properties of the Drazin inverse , in order to obtain the Drazin inverse the projection method solves consistent or inconsistent singular linear system*

$$(S-M)X=Y(r),\ \ k=ind(S-M),$$

through solving the consistent singular linear system In order to obtain to obtain the Drazin inverse for solving inconsistent singular linear system

$$((S-M)^k(S-M))X=(S-M)^kY(r).$$

The algorithm for solving singular dual fuzzy linear systems using Drazin inverse is as follows,

Algorithm 3.1 Find solution of SDFLS using Drazin inverse (General Method)

Step 1 :	Form the crisp linear system of $(S-M)X=Y(r)$.
Step 2 :	Compute $rank[S-M]$ and $rank[S-M\|Y(r)]$.
Step 3 :	**If** $rank[S-M]=rank[S-M\|Y(r)]$ **then** SFLS has a set of solutions and $X=(S-M)^DY(r)$ is a solution of $(S-M)X=Y(r)$. **else** compute $k=ind(S-M)$, in order to obtain to obtain the Drazin inverse for solving inconsistent singular linear system $(S-M)^k(S-M)X=(S-M)^kY(r)$, then $X=((S-M)^k(S-M))^D(S-M)^kY(r)$.
Step 4 :	**Stop**

4. Generalized Method

In this section, Asady's method [4] is extended and on the consistent singular dual fuzzy linear systems, is performed.

Generalized Asady's method. Let (3.1) be the extended linear system of the consistent singular duzl fuzzy linear system $A\tilde{x}=B\tilde{x}+\tilde{y}$. By noting the structure of $(S-M)$ and (2.3) we obtain the following linear system

$$\begin{cases} B(\underline{x})+C(-\bar{x}) &= \underline{y}(r), \\ C(\underline{x})+B(-\bar{x}) &= -\bar{y}(r), \end{cases} \tag{4.1}$$

which is a crisp function linear system. If (3.1) be consistent, by adding and then subtracting the part of Equation (4.1), we obtain

$$\begin{cases} (B+C)(\bar{x}-\underline{x}) &= \bar{y}(r)-\underline{y}(r), \\ (B-C)(\bar{x}+\underline{x}) &= \bar{y}(r)+\underline{y}(r), \end{cases}$$

We can get

$$\begin{cases} E(\bar{x}-\underline{x}) &= \bar{y}(r)-\underline{y}(r), \\ A(\bar{x}+\underline{x}) &= \bar{y}(r)+\underline{y}(r), \end{cases}$$

Wherein , $E = B + C$ and $A = B - C$

$$\begin{cases} E\sigma &= \bar{y}(r) - \underline{y}(r), \\ A\delta &= \bar{y}(r) + \underline{y}(r), \end{cases}$$

where $\sigma = \bar{x} + \underline{x}$, $\delta = \bar{x} - \underline{x}$. By adding and subtracting the two solutions of above systems we obtain

$$\bar{x} = \frac{\sigma + \delta}{2}, \ \underline{x} = \frac{\delta - \sigma}{2}.$$

Theorem 4.1. *A solution of the the extended system (4.1) is*

$$\begin{cases} \sigma &= E^D(\bar{y}(r) - \underline{y}(r)), \\ \delta &= A^D(\bar{y}(r) + \underline{y}(r)), \end{cases}$$

where $\sigma = \bar{x} + \underline{x}$, $\delta = \bar{x} - \underline{x}$, *if and only if* $(\bar{y}(r) - \underline{y}(r) \in R(E^k)$ *and* $(\bar{y}(r) + \underline{y}(r) \in R(A^k)$

Proof. Same as proof of theorem2.3 in [6].

Corollary 4.1. *Let*

$$(S - M)X = Y(r), \ k = ind(S - M),$$

be a inconsistent singular linear system. In this case, by corollary3.4 we perform generalized Asady's method, on the consistent singular linear system

$$(S - M)^k(S - M)X = (S - M)^k Y(r).$$

Comparative number of generalized Asady's method and general method for solving singular dual fuzzy linear system is given in table 4.1.

Table 4.1 Comparative number of proposed methods

	Multiplication	Summation
General method	$4n^2$	$4n^2$
Generalized Asady's method	$2n^2$	$2n^2$

5. Numerical Results

We now give the following examples to explain the present results.

Example 5.1. Consider the following singular dual fuzzy linear systems

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} -2, -1 - r \\ 2 + r, 3 \end{pmatrix}, \tag{5.1}$$

The extended 4×4 crisp linear system is

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{pmatrix} + \begin{pmatrix} -2 \\ 2 + r \\ 1 + r \\ -3 \end{pmatrix}. \tag{5.2}$$

Since $rank[S - M] \neq rank[S - M|Y]$, then the system (5.1) is inconsistent. Indicial equations of (5.2) is

$$\begin{pmatrix} -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{pmatrix} = \begin{pmatrix} -2 + 2r \\ -2 + 2r \\ -2 + 2r \\ -2 + 2r \end{pmatrix}, \tag{5.3}$$

that is a consistent system.

$$\begin{pmatrix} -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{pmatrix}^D = \begin{pmatrix} -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} \\ -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} \\ -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} \\ -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} & -\frac{1}{32} \end{pmatrix}.$$

Therefore the solution of (5.3) is

$$X = \begin{pmatrix} \frac{1}{4} - \frac{1}{4}r \\ \frac{1}{4} - \frac{1}{4}r \\ \frac{1}{4} - \frac{1}{4}r \\ \frac{1}{4} - \frac{1}{4}r \end{pmatrix}.$$

We can get the weak fuzzy solution by definition2.6.

Example 5.2. Consider the following singular dual fuzzy linear systems

$$\begin{pmatrix} 2 & -2 \\ 4 & -4 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ -6 & 6 \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{pmatrix} + \begin{pmatrix} r, r-2 \\ 2r, 2r-4 \end{pmatrix}. \tag{5.4}$$

The extended 4×4 crisp linear system is

$$\begin{pmatrix} 2 & 0 & 0 & 2 \\ 4 & 0 & 0 & 4 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 4 & 0 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 3 & 3 & 0 \\ 0 & 6 & 6 & 0 \\ 3 & 0 & 0 & 3 \\ 6 & 0 & 0 & 6 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ -\bar{x}_1 \\ -\bar{x}_2 \end{pmatrix} + \begin{pmatrix} r \\ 2r \\ r-2 \\ 2r-4 \end{pmatrix}.$$

Since $rank[S-M] = rank[S-M|Y]$, then the system (5.4) is consistent. By proposed method, we have

$$\begin{cases} E\sigma & = \bar{y} - \underline{y}, \\ A\delta & = \bar{y} + \underline{y}, \end{cases}$$

then by theorem4.1

$$E^D = \begin{pmatrix} -\frac{1}{9} & -\frac{1}{9} \\ -\frac{2}{9} & -\frac{2}{9} \end{pmatrix}, \sigma = \begin{pmatrix} -\frac{2}{3} + \frac{2}{3}r \\ -\frac{4}{3} + \frac{4}{3}r \end{pmatrix},$$

and

$$A^D = \begin{pmatrix} \frac{1}{5} & -\frac{1}{5} \\ \frac{2}{5} & -\frac{2}{5} \end{pmatrix}, \delta = \begin{pmatrix} -\frac{2}{5} \\ -\frac{4}{5} \end{pmatrix}.$$

Therefore

$$\bar{x} = \begin{pmatrix} \frac{1}{3}r - \frac{8}{15} \\ \frac{2}{3}r - \frac{16}{15} \end{pmatrix}, \underline{x} = \begin{pmatrix} -\frac{1}{3}r + \frac{2}{15} \\ -\frac{2}{3}r + \frac{4}{15} \end{pmatrix},$$

is a solution of the system $(S-M)X = Y$ and we can get this solution by general method.

6. Conclusions and Suggestions

In this paper, the consistent singular dual fuzzy linear system is replaced by two $n \times n$ crisp linear system. We can solve the singular dual fuzzy linear systems using Drazin inverse. Investigating the singular fuzzy Leontieff input-output analysis is suggested.

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