

## AN APPROXIMATION PROPERTY OF THE GENERALIZED JAIN'S OPERATORS OF TWO VARIABLES

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ABSTRACT. The purpose of this work is to introduce a new class of double positive linear operators which depend on a parameter  $\beta$ . For these operators we proved a Korovkin type theorem and we presented some associated convergence properties.

### 1. INTRODUCTION

After a survey of the literature we observed that most of the approximation operators preserve the test functions  $e_i = x^i$ , ( $i = 0, 1$ ). For example the operators given in [1], the Bernstein polynomials [2] and the Szász-Mirakjan type operators [13] preserve these test functions. Different issues arise in the case when  $i = 2$ . In the two-dimensional case, the test functions are  $f_{0,0}(x, y) = 1$ ,  $f_{1,0}(x, y) = x$ ,  $f_{0,1}(x, y) = y$  and  $f_{2,0}(x, y) + f_{0,2}(x, y) = x^2 + y^2$ . For example the Bernstein polynomials of two variables or the Szász-Mirakjan type operators of two variables preserve these test functions. Results related to this subject can be found in [5, 6, 8, 12, 14] and references therein.

In this paper we study the generalized Jain's operators of two variables and their convergence properties. Our paper is structured as follows. In Section 2 we present the form of the operators and prove a Korovkin-type theorem while in Section 3 we give a new version of our operators and we show their convergence properties by means of A-statistical approximation.

In order to achieve the mentioned results we need some background results like the convergence of double sequences, the A-statistical convergence for double sequences and the regularity of a four dimensional matrix also known as Robinson-Hamilton regularity, or, briefly, RH-regularity.

To this end, we first mention that  $C(D)$  denotes the space of all continuous functions on  $D$ , where  $D$  represents a compact subset of the real two dimensional space.

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We first recall that a double sequence  $x = (x_{m,n})$  is said to be convergent in the Pringheim's sense if for every  $\varepsilon > 0$  there exists  $N \in \mathbb{N}$ , such that  $|x_{m,n} - L| < \varepsilon$ , whenever  $m, n > N$ .  $L$  is called the Pringheim's limit of  $x$  and it is denoted by  $P - \lim x = L$ . More simply we refer at such an  $x$  as " $P$ -convergent" (see [10]).

On the other hand, we say that a double sequence is bounded if there exists a number  $b$  such that  $|x_{m,n}| < b$  for all  $(m, n) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ . As a remark, we mention that a convergent double sequence is not necessarily bounded.

Recall now that a four dimensional matrix  $A = (a_{j,k,m,n})$  is said to be RH-regular [7], [11] if it maps every bounded  $P$ -convergent sequence into a  $P$ -convergent sequence with the same limit.

Note that in the case of double sequences, a convergent double sequence is not necessarily bounded. According to [7] and [11], a four dimensional matrix  $A = (a_{j,k,m,n})$  is RH-regular if and only if

- (i)  $P - \lim_{j,k \rightarrow \infty} a_{j,k,m,n} = 0$  for each  $m$  and  $n$ ;
- (ii)  $P - \lim_{j,k \rightarrow \infty} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{j,k,m,n} = 1$ ;
- (iii)  $P - \lim_{j,k \rightarrow \infty} \sum_{m=1}^{\infty} |a_{j,k,m,n}| = 0$  for each  $n \in \mathbb{N}$ ;
- (iv)  $P - \lim_{j,k \rightarrow \infty} \sum_{n=1}^{\infty} |a_{j,k,m,n}| = 0$  for each  $m \in \mathbb{N}$ ;
- (v)  $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{j,k,m,n}|$  is  $P -$  convergent;
- (vi) There exists finite positive integers  $A$  and  $B$  such that  $\sum_{m,n > B} |a_{j,k,m,n}| < A$  holds for every  $(j, k) \in \mathbb{N}^2$ .

Let  $A = (a_{j,k,m,n})$  be a nonnegative RH-regular summability method and let  $K \subset \mathbb{N}^2$ . Then  $A$ -density of  $K$  is given by (see for example [4]).

$$\delta_{(A)}^2(K) := P - \lim_{j,k \rightarrow \infty} \sum_{(m,n) \in K} a_{j,k,m,n}.$$

A real double sequence  $x = (x_{m,n})$  is said to be  $A$ -statistically convergent to  $L$  if, for every  $\varepsilon > 0$

$$(1.1) \quad \delta_{(A)}^2((m, n) \in \mathbb{N}^2 : |x_{m,n} - L| \geq \varepsilon) = 0,$$

where  $\delta_{(A)}^2(K)$  represents the  $A$ -density of  $K$ . In this case we write

$$st_A^2 - \lim x = L.$$

Note that a  $P$ -convergent sequence is  $A$ -statistically convergent to the same value but its converse is not always true. Moreover, an  $A$ -statistically convergent double sequence is not necessarily bounded.

## 2. THE GENERALIZED JAIN'S OPERATORS

The original operators given by Jain are of the form

$$(2.1) \quad (P_n^{[\beta]}f)(x) = \sum_{k=0}^{\infty} \omega_{\beta}(k, nx) \cdot f\left(\frac{k}{n}\right), \quad f \in C[0, \infty),$$

where

$$(2.2) \quad \omega_{\beta}(k, \alpha) = \alpha(\alpha + k\beta)^{k-1} e^{-(\alpha+k\beta)} / k! ; \quad k \in \mathbb{N}_0.$$

represents the Poisson-type distribution. We note that relation (2.2) was first considered by Consul and Jain in [3].

We have the following lemma.

**Lemma 2.1.** ([9]) *Let  $0 < \alpha < \infty$ ,  $|\beta| < 1$  and let the Poisson distribution given by (2.2). Then*

$$(2.3) \quad \sum_{k=0}^{\infty} \omega_{\beta}(k, \alpha) = 1.$$

Next, we present a new class of operators namely a generalization of Jain's operators on the nodes  $\left(\frac{k_1 + \alpha_1}{m + \gamma_1}, \frac{k_2 + \alpha_2}{m + \gamma_2}\right)$ .

$$(2.4) \quad [^{\beta}] \mathcal{J}_{m,n}^{\alpha,\gamma} = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \omega_{\beta}^1(k_1, mx) \omega_{\beta}^2(k_2, ny) f\left(\frac{k_1 + \alpha_1}{m + \gamma_1}, \frac{k_2 + \alpha_2}{n + \gamma_2}\right)$$

with  $(x, y) \in D$ ,  $f \in C(D)$  and  $\alpha = (\alpha_1, \alpha_2)$ ,  $\gamma = (\gamma_1, \gamma_2)$ .

*Remark 2.1.* If we take  $\beta = 0$  and  $\alpha = \gamma = 0$  in (2.4), we get the double Szász-Mirakjan operators.

**Lemma 2.2.** *Let  $(x, y) \in C(D)$  and let  $f_{00}(x, y) = 1$ ,  $f_{10}(x, y) = x$ ,  $f_{01}(x, y) = y$ ,  $f_{20}(x, y) = x^2$ ,  $f_{02}(x, y) = y^2$ . Then for the operators described in relation (2.4) we have:*

$$(2.5) \quad [^{\beta}] \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,0}; x, y) = 1;$$

$$(2.6) \quad [^{\beta}] \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{1,0}; x, y) = \frac{mx}{(m + \gamma_1)(1 - \beta)} + \frac{\alpha_1}{m + \gamma_1};$$

$$(2.7) \quad [^{\beta}] \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,1}; x, y) = \frac{ny}{(n + \gamma_2)(1 - \beta)} + \frac{\alpha_2}{n + \gamma_2};$$

$$(2.8) \quad [^{\beta}] \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{2,0} + f_{0,2}; x, y) = \frac{mx^2}{(m + \gamma_1)^2(1 - \beta)^2} \\ + \frac{mx}{(m + \gamma_1)^2} \left[ \frac{1}{(1 - \beta^3)} + \frac{2\alpha_1}{1 - \beta} \right] \\ + \frac{\alpha_1^2}{(m + \gamma_1)^2} + \frac{ny^2}{(n + \gamma_2)^2(1 - \beta)^2} \\ + \frac{ny}{(n + \gamma_2)^2} \left[ \frac{1}{(1 - \beta^3)} + \frac{2\alpha_2}{1 - \beta} \right] + \frac{\alpha_2^2}{(n + \gamma_2)^2}.$$

**Proof.** It is clear from (2.3) that

$$^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,0}; x, y) = 1.$$

$$\begin{aligned} ^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{1,0}; x, y) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \omega_{\beta}^1(k_1, mx) \omega_{\beta}^2(k_2, ny) \frac{k_1 + \alpha_1}{m + \gamma_1} \\ &= \sum_{k_1=0}^{\infty} \omega_{\beta}^1(k_1, mx) \frac{k_1}{m + \gamma_1} + \frac{\alpha_1}{m + \gamma_1} \sum_{k_1=0}^{\infty} \omega_{\beta}^1(k_1, mx). \\ &= \frac{mx}{(m + \gamma_1)(1 - \beta)} + \frac{\alpha_1}{m + \gamma_1}. \end{aligned}$$

By analogy,

$$^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,1}; x, y) = \frac{ny}{(n + \gamma_2)(1 - \beta)} + \frac{\alpha_2}{n + \gamma_2}.$$

$$\begin{aligned} ^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{2,0}; x, y) &= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \omega_{\beta}^1(k_1, mx) \omega_{\beta}^2(k_2, ny) \left( \frac{k_1 + \alpha_1}{m + \gamma_1} \right)^2 \\ &= \sum_{k_1=0}^{\infty} \omega_{\beta}^1(k_1, mx) \frac{k_1^2}{(m + \gamma_1)^2} + \frac{2\alpha_1}{(m + \gamma_1)^2} \sum_{k_1=0}^{\infty} \omega_{\beta}^1(k_1, mx) k_1 \\ &\quad + \frac{\alpha_1^2}{(m + \gamma_1)^2} \sum_{k_1=0}^{\infty} \omega_{\beta}^1(k_1, mx) \\ &= \frac{mx}{(m + \gamma_1)^2} \left[ \frac{mx}{(1 - \beta)^2} + \frac{1}{(1 - \beta)^3} \right] + \frac{2\alpha_1 \cdot mx}{(m + \gamma_1)^2(1 - \beta)} \\ &\quad + \frac{\alpha_1^2}{(m + \gamma_1)^2} \\ &= \frac{(mx)^2}{(m + \gamma_1)^2(1 - \beta)^2} + \frac{mx}{(m + \gamma_1)^2} \left[ \frac{1}{(1 - \beta)^3} + \frac{2\alpha_1}{1 - \beta} \right] \\ &\quad + \frac{\alpha_1^2}{(m + \gamma_1)^2}. \end{aligned}$$

In a similar manner we get

$$^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f_{0,2}; x, y) = \frac{(ny)^2}{(n + \gamma_2)^2(1 - \beta)^2} + \frac{ny}{(n + \gamma_2)^2} \left[ \frac{1}{(1 - \beta)^3} + \frac{2\alpha_2}{1 - \beta} \right] + \frac{\alpha_2^2}{(n + \gamma_2)^2}.$$

Now we can state and prove the following result.

**Theorem 2.1.** *Let  $f \in C(D)$  and  $\beta_n \rightarrow 0$  when  $n \rightarrow \infty$ . Then the sequence  $^{[\beta_n]} \{ \mathcal{J}_{m,n}^{\alpha,\gamma}(f; x, y) \}$  converges uniformly to  $f(x, y)$  on  $K \subset D$ , where  $K = [0, A] \times [0, A]$ ,  $0 < A < \infty$ , that is*

$$\lim_{m,n \rightarrow \infty} \left\| ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f; x, y) - f(x, y) \right\|_K = 0.$$

**Proof.** The proof of the above theorem relies on arguments of Korovkin type, i.e. if we take into account that the equalities (2.5)–(2.8) hold then by the Korovkin Theorem we can conclude that

$$\left\| {}^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f; x, y) - f(x, y) \right\|_K = 0.$$

□

### 3. CONVERGENCE PROPERTIES

In this section we present a modified version of the operators defined by relation (2.4) and we show that these new operators satisfy an  $A$ -statistical convergence theorem but they do not satisfy Theorem 2.1.

To this end we present the following theorem.

**Theorem 3.1.** [4] *Let  $A = (a_{j,k,m,n})$  be a nonnegative RH-regular summability matrix method. Let  $\{L_{m,n}\}$  be a double sequence of positive linear operators acting from  $C(D)$  into itself. Then, for all  $f \in C(D)$ ,*

$$st_A^2 - \lim_{m,n} \|L_{m,n}f - f\|_{C(D)} = 0$$

if and only if

$$(3.1) \quad st_A^2 - \lim_{m,n} \|L_{m,n}f_i - f_i\|_{C(D)} = 0, \quad (i = 0, 1, 2, 3),$$

where  $f_0(x, y) = 1$ ,  $f_1(x, y) = x$ ,  $f_2(x, y) = y$  and  $f_3(x, y) = x^2 + y^2$ .

We may observe that the generalized Jain's operators given by relation (2.4) satisfy Theorem 3.1. We can prove this by taking into account Lemma 2.2 and the fact that the generalized Jain's operators are positive linear operators.

Next, based on the definition (2.4) of the generalized Jain's operators we construct a new class of operators which satisfy Theorem 3.1 but they do not satisfy the Korovkin-type theorem, Theorem 2.1.

Let  $A = C(1, 1)$  which is the double Cesàro matrix and let  $(\eta_{m,n})$  be defined as

$$(3.2) \quad \eta_{m,n} = \begin{cases} 1, & \text{if } m \text{ and } n \text{ are squares;} \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$(3.3) \quad st_{C(1,1)}^2 - \lim \eta_{m,n} = 0.$$

Now, using (2.4) and relation (3) we define the following positive linear operators on the interval  $[0, \infty)$

$$(3.4) \quad \mathcal{N}_{m,n}(f; x, y) = (1 + \eta_{m,n})^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}(f; x, y).$$

Using Theorem 3.1 and relation (3.3) we have

$$(3.5) \quad st_{C(1,1)}^2 - \lim \| \mathcal{N}_{m,n}f - f \|_{C([0,A])} = 0, \quad 0 < A < \infty.$$

Since  $\eta_{m,n}$  is not  $P$ -convergent we conclude that the Theorem 2.1 does not work for the operators defined by (3.4).

Further on we refer at the rate of convergence of the generalized Jain's operators. Before we move forward we need to mention that  $C_B(D)$  stands for the space of all continuous and bounded functions on  $D$ . Also for a function  $f \in C_B(D)$  the modulus of continuity of  $f$  is given by

$$\omega(f; \delta) = \sup \left\{ |f(u, v) - f(x, y)| : \sqrt{(u-x)^2 + (v-y)^2} < \delta, (u, v), (x, y) \in D \right\}.$$

Then, for any  $\delta > 0$  and for each  $(x, y) \in D$  we have

$$|f(u, v) - f(x, y)| \leq \left( \frac{\sqrt{(u-x)^2 + (v-y)^2}}{\delta} + 1 \right) \omega(f; \delta).$$

In addition, after some simple calculations for any sequence  $\{L_{m,n}\}$  of positive linear operators on  $C_B(D)$  and for any function  $f \in C_B(D)$  we can write

$$\begin{aligned} (3.6) \quad |L_{m,n}(f; x, y) - f(x, y)| &\leq \omega(f; \delta) \left\{ 1 + \frac{1}{\delta^2} L_{m,n} \left( (u-x)^2 + (v-y)^2; x, y \right) \right. \\ &\quad \left. + |L_{m,n}(f_{0,0}; x, y) - f_{0,0}(x, y)| \right\} \\ &\quad + |f(x, y)| \cdot |L_{m,n}(f_{0,0}; x, y) - f_{0,0}(x, y)|. \end{aligned}$$

**Theorem 3.2.** Let  $^{[\beta]} \mathcal{J}_{m,n}^{\alpha,\gamma}$  be the operators defined in (2.4). In additon we take  $\beta = \beta_n$  and  $\alpha = \alpha_n, \gamma = \gamma_n$  with the properties

$$(3.7) \quad \beta_n \rightarrow 0, n \rightarrow \infty$$

and,

$$(3.8) \quad \alpha_n \rightarrow 0, n \rightarrow \infty, \gamma_n \rightarrow 0, n \rightarrow \infty,$$

respectively. Then we have

$$(3.9) \quad \left| ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n}(f; x, y) - f(x, y) \right| \leq \omega(f; \delta) \left\{ 1 + \frac{1}{\delta^2} (2x^2 + 2y^2) \right\}.$$

**Proof.** After some simple calculations we find that

$$\begin{aligned} ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} \left( (u-x)^2 + (v-y)^2; x, y \right) &= ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} (f_{2,0}; x, y) \\ &\quad - 2x \cdot ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} (f_{1,0}; x, y) \\ &\quad + x^2 \cdot ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} (f_{0,0}; x, y) \\ &\quad + ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} (f_{0,2}; x, y) \\ &\quad - 2y \cdot ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} (f_{0,1}; x, y) \\ &\quad + y^2 \cdot ^{[\beta_n]} \mathcal{J}_{m,n}^{\alpha_n, \gamma_n} (f_{0,0}; x, y). \end{aligned}$$

Now taking into account (2.5)–(2.8) and the fact that (3.7) and (3.8) hold, we obtain that (3.9) holds true.  $\square$

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