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STABILITY CONDITION OF A PRIORITY QUEUEING SYSTEM

AMINA ANGELIKA BOUCHENTOUF AND HANANE SAKHI

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Abstract. The present paper is devoted to the study of a priority multiclass system stability, where we establish a sufficient condition for the stability of a queueing network composed of N-units, $N \geq 3$ and N^2 classes (N classes at each unit); that is the stability under all work conserving disciplines with the priority discipline.

1. Introduction

Queueing systems constitute a central tool in modelling and performance analysis of computer systems, communication systems, manufacturing systems and logistic systems.
An important tool for studying the stability of a queueing network is its corresponding fluid network, which is a continuous analog of the queueing network. An elegant theorem proposed by Rybko and Stolyar [10] then extended by Dai [4] states that a queueing network is stable if its corresponding fluid limit model or fluid network model is stable. This motivates the study of the stability of fluid networks.
Dai and Vande Vate [6, 7] characterized the global stability region of two-station fluid networks, in [5] the authors extended the methods used in [1, 3, 6, 7] to networks with three stations.
In this paper, we establish the global condition stability of a priority multiclass queueing networks composed of N units $N \geq 3$. using the peacewise linear Lyapunov function approch.
Determining the global stability region of such system is very difficult under bad disciplines, even if the traffic intensity at each unit of the network is less that one. In fact we show that, with the usual traffic conditions

$$\rho = \left(\alpha \sum_{k:\sigma(k)=i} m_k < 1 \right), \text{ for } i = 1, 2, ..., N,$$

the following additional condition:

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$$\alpha m_1 + \sum_{k:\sigma(k)=1} \frac{m_k}{m_{k-1}} \leq 1$$

is sufficient to ensure global stability.

2. Model Description and Notations

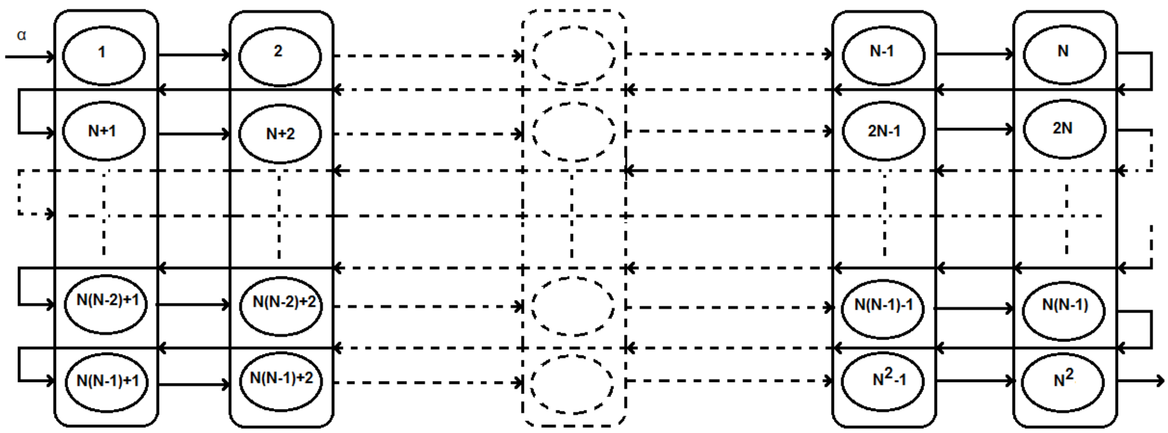

Figure 1. Queueing system with N units and N^2 classes

We consider a model of a priority queueing system composed of N units with N^2 classes, at each unit a server may serve N classes of customers. Fluid comes from outside to the network at rate α per unit of time, requiring a service of mean m_1, then it aligns the second queue asking for a service of mean m_2, after that it removes to the third unit where it will be served another time, then to the fourth one, the fifth one..., until the Nth unit, where it will be served with mean m_N, afterward it returns to unit 1 and is again served by each unit. After being processed by the Nth units N^2 times it leaves definitively the system. The specific priority discipline under consideration is as follows: If the order of the station i, $i = \overline{1, N}$ is odd, the class k which belongs to this station has a priority over the above class which is the class $k - N$, and if the order of the station $i, i = \overline{1, N}$ is even, the class k which belongs to this station, has a priority over the bellow class which is the class $k + N$. Each fluid residing in buffer k, $k = 1, 2, ..., N^2$ is called class k fluid.
Let $Q_k(t)$ denote the fluid level in buffer k at time t, and $T_k(t)$, the cumulative time vector $\sigma(k)$ devotes the class k in the interval $[0, t]$.
We denote by $U_i(t) = t - \sum_{k:\sigma(k)=i} T_k(t)$ the cumulative idle time at unit i, $i = 1, 2, ...N$ in the interval $[0, t]$. The buffer levels $Q(.) = (Q_k(.))_{1\leq k\leq N^2}$ and the allocations $T(.) = (T_k(.))_{1\leq k\leq N^2}$ must satisfy:

$$Q_k(t) = Q_k(0) + \mu_{k-1}T_{k-1}(t) - \mu_k T_k(t), \ \ t \geq 0, \ \text{ for } \ k = 1, ..., N^2, \tag{2.1}$$

$$Q_k(t) \geq 0, \ \ t \geq 0, \ \text{ for } \ k = 1, ..., N^2, \tag{2.2}$$

$$T_k(.) \text{ is nondecreasing}, \ \ k = 1, ..., N^2, \tag{2.3}$$

$$U_i(.) \text{ is nondecreasing, } \ i = 1, ..., N, \tag{2.4}$$

where, $\mu_k = 1/m_k$ is the service rate for class k, $k = 1, 2, ..., N^2$, $\mu_0 = \alpha$ is the exogenous arrival rate and $T_0(t) = t$ models the exogenous arrival process, $\mu_k T_k(t)$ is the amount of fluid to have departed buffer k by time t.
Any solution $(Q(.), T(.))$ to (2.1)-(2.4) is a fluid solution. A fluid solution $(Q(.), T(.))$ satisfying:

$$\int_0^\infty Z_i(t) dU_i(t) = 0, \ \ i = 1, ..., N, \tag{2.5}$$

where

$$Z_i(t) = \sum_{k:\sigma(k)=i} Q_k(t), \ \ i = 1, ..., N, \tag{2.6}$$

is said to be non-idling or work conserving. Equations (2.1)-(2.5) define the fluid network under non-idling disciplines.
For any fluid solution $(Q(.), T(.))$, $Q(.)$ is differentiable for almost all t in $(0, \infty)$; t is regular point for fluid solution $(Q(.), T(.))$ if $T(.)$ is differentiable at t. For a differentiable function (at t) $h : [0, \infty) \to \mathbb{R}$, $\dot{h}(t)$ is the derivative of h at t. Notice that (2.5) is equivalent to the condition "$Z_i(t) > 0$ implies $\dot{U}_i(t) = 0$," for each regular point t.
It was shown in [3] that each fluid limit is a fluid solution satisfying (2.1)-(2.5). One of the particularly of non-idling discipline are the static buffer priority disciplines, which means that the server can only work on lower priority classes at a unit when the requirements of higher priority classes are satisfied. Each unit in our network serves N classes. Let π_i denote the high priority class at unit i under the static buffer priority discipline π. So, the specific priority discipline under consideration is as follows: If the order of the station i, $i = \overline{1, N}$ is odd, the class k which belongs to this station has a priority over the above class which the class $k - N$, and if the order of the station $i, i = \overline{1, N}$ is even, the class k which belongs to this station, has a priority over the bellow class which the class $k + N$. For instance $\pi_{\{13,9,5,1,2,6,10,14,15,11,7,3,4,8,12,16\}}$ denotes the static buffer priority discipline that gives in station 1 higher priority to class 13 over class 9, higher priority to class 9 over class 5 and higher priority to 5 over 1, in station 2 it gives higher priority to class 2 over 6, higher priority to 6 over 10 and higher priority to 10 over 14, and in station 3 it gives higher priority to 15 over 11, higher priority to 11 over 7 and higher priority to 7 over 3, and finally in station 4 it gives higher priority to 4 over 8, higher priority to 8 over 12 and higher priority 12 over 16.

With this notation, our fluid network under the static buffer priority π requires the additional equations:

$$\dot{T}_{\pi(i)}(t) = 1 \ \text{ if } \ Q_{\pi(i)}(t) > 0, \ \ i = 1, ..., N, \tag{2.7}$$

for each regular point t of $T(.)$. Any solution $(Q(.), T(.))$ to (2.1)-(2.5) and (2.7) is a fluid solution under the discipline π.

Definition 2.1. • The fluid network is globally stable if there exists a time $\sigma > 0$ such that for each non-idling fluid solution $(Q(.), T(.))$ satisfying (2.1)-(2.5) with $|Q(0)| = 1$, $Q(t) = 0$ for all $t \geq \sigma$.

• The fluid network under a static buffer priority discipline π is stable if there exists a time $\sigma > 0$ such that for each fluid solution $(Q(.), T(.))$ satisfying (2.1)-(2.5) and (2.7) with $|Q(0)| = 1$, $Q(t) = 0$ for all $t \geq \sigma$.
• A fluid solution $(Q(.), T(.))$ is unstable if there is no $\sigma > 0$ such that $Q(t) = 0$ for all $t \geq \sigma$.
• For a given $\alpha > 0$, the global stability region of the fluid network is the set of positive service times $m = (m_k)$ for which the fluid network is globally stable.
• For a given $\alpha > 0$ and a static buffer priority discipline π, the stability region of the fluid network under the discipline is the set of positive services times $m = (m_k)$ for which the fluid network under the discipline is stable.

We say that the usual condition is satisfied if the traffic intensity for each unit is less than 1 i.e.,

$$\rho = \alpha \sum_{k:\sigma(k)=i} m_k < 1, \text{ for each } \quad i = 1, ..., N. \tag{2.8}$$

At that moment, we say that the usual traffic conditions are satisfied.

Now, we define the following system of linear constraints, which is related to a piecewise linear Lyapunov function for our N-unit fluid network:

$$\alpha \sum_{k:\sigma(k)=i} x_k < x_k \mu_k, \quad i = 1, ..., N, \tag{2.9}$$

$$\left(\sum_{k:\sigma(k)=1} x_k \right) - x_1 \leq \sum_{k:\sigma(k)=N} x_k \tag{2.10}$$

$$\left(\sum_{k:\sigma(k)=i} x_k \right) - x_i \leq \sum_{k:\sigma(k)=i-1} x_k - x_i \quad i = 2, ..., N, \tag{2.11}$$

$$\sum_{k:\sigma(k)=i} x_k \leq \sum_{k:\sigma(k)=i-1} x_k \quad i = 2, ..., N. \tag{2.12}$$

The system of linear constraints (2.9)-(2.12) derived from our piecewise linear Lyapunov function provides conditions sufficient to ensure the global stability of the system.

3. Main result

In this section we characterize the global stability region using the piecewise linear Lyapunov functions. Given $x = (x_k) > 0$ and a fluid solution $Q(.)$, let

$$h_i(x, Q(t)) = \sum_{k:\sigma(k)=i} x_k Q_k^+(t), \quad i = 1, ..., N,$$

where $Q_k^+(t) = \sum_{l=1}^{k} Q_l(t)$, and $h(x, Q(t)) = \max\{h_i(x, Q(t))\}, \; i = 1, ..., N$. So, $h(Q(t))$ is a convex, piecewise linear function of $Q(t) = (Q_k(t))$.
The piecewise linear function h is said to be Lyapunov function for the global

stability of the fluid model if there exist $\varepsilon > 0$ such that for each non-idling fluid solution $(Q(.), T(.))$ satisfying (2.1)-(2.5),

$$\frac{dh(Q(t))}{dt} \leq -\varepsilon \tag{3.1}$$

for each time $t > 0$ that is regular for $T(.)$ and $h(Q(t))$ with $|Q(t)| > 0$.
Let $m > 0$ be a service time vector for which there is a piecewise Lyapunov function h satisfying (3.1). It follows from Lemma 2.2 of Dai and Weiss [6] that $h(Q(t)) = 0$ for all $t \geq h(Q(0))/\varepsilon$, or $Q(t) = 0$ for all $t \geq h(Q(0))/\varepsilon$.

The next lemma suggests a way in which to construct piecewise linear Lyapunov functions, it was introduced by Botvich and Zamyatin [2] for a two-station network. It was independently generalized by Dai and Weiss [8], Down and Meyn [9] and [5]. Using that result we get

Lemma 3.1. *Suppose there exists $x = (x_k) > 0$, $t_0 \geq 0$ and $\varepsilon > 0$ such that for each non-idling fluid solution $(Q(.), T(.))$ and each regular point $t > t_0$ of $T(.)$, the following hold for each $i = 1, 2, ..., N$*

$$\frac{dh_i(x, Q(t))}{dt} \leq -\varepsilon \ \textit{ whenever } \ Z_i(t) > 0, \tag{3.2}$$

$$h_i(x, Q(t)) \leq \max\{h_j(x, Q(t)) : j \in \{1, ..., N\}, j \neq i\} \ \textit{ whenever } \ Z_i(t) = 0, \tag{3.3}$$

$$\max\{h_j(Q(t)) : j \in \{1, ..., N\}, j \neq i\} \leq h_i(Q(t)) \ \textit{ whenever } \ \sum_{j \neq i} Z_j(t) = 0. \tag{3.4}$$

Then h is a piecewise linear Lyapunov function.

Proposition 3.1. *If there exists $x = (x_k) > 0$ satisfying the linear constraints (2.9)-(2.12), then there exists $\varepsilon > 0$ such that (3.2)-(3.4) hold and hence, h is a piecewise linear Lyapunov function.*

Proof. Let $t_0 = 0$ and let $x = (x_k) > 0$ satisfying the linear constraints (2.9)-(2.12), define $\varepsilon > 0$ to be the minimum of the following terms $x_k\mu_k - \alpha\left(\sum_{k:\sigma(k)=i} x_k\right)$, $i = 1, ..., N$. Remark that the amount of fluids in buffers 1 through k is

$$Q_k^+(t) = Q_k^+(0) + \alpha t - \mu_k T_k(t).$$

Thus

$$h_1(x, Q(t)) = h_1(0) + \alpha t\left(\sum_{k:\sigma(k)=1} x_k\right) - \sum_{k:\sigma(k)=1} x_k\mu_k T_k(t)$$

and

$$\frac{dh_1(Q(t))}{dt} = \alpha(\sum_{k:\sigma(k)=1} x_k) - \sum_{k:\sigma(k)=1} x_k\mu_k \dot{T}_k(t)$$

If $Z_1(t) > 0$, it follows from (3.3), $\sum_{k:\sigma(k)=1} \dot{T}_k(t) = 1$, thus $\dot{h}_1(t) < -\varepsilon$.
We follow similar analysis for $i = \overline{2, N}$.

Next, we establish (3.3).

• When $Z_1(t) = 0$, equation (2.10) ensures that $h_1(Q(t)) < h_N(Q(t))$.
• When $Z_i(t) = 0, \;\; i = \overline{2, N}$, equation (2.11) ensures that $h_i(Q(t)) < h_{i-1}(Q(t))$.

Finally, we establish (3.4).

When $\sum_{j \neq i} Z_j(t) = 0, \;\; j = \overline{1, N}$, equations (2.11)-(2.12) ensures that $h_j(Q(t)) < h_i(Q(t))$. □

The following result establishes a sufficient conditions to ensure global stability for our network.

Proposition 3.2. *If*

$$\alpha m_1 + \sum_{k:\sigma(k)=1} \frac{m_k}{m_{k-1}} \leq 1, \tag{3.5}$$

$$\alpha \sum_{k:\sigma(k)=i} m_k < 1, \;\; i = 2, ..., N, \tag{3.6}$$

the fluid network is globally stable.

Proof. Let $(Q(.), T(.))$ be a non-idling fluid solution with $|Q(0)| = 1$. Let

$$f_1(t) = \sum_{k:\sigma(k)=1} \sum_{j=1}^{k} m_k Q_j(t)$$

be the total workload at the first unit at time t.

From (2.1) we get

$$f_1(t) = f_1(0) + \alpha t \left(\sum_{k:\sigma(k)=1} m_k \right) - \sum_{k:\sigma(k)=1} T_k.$$

For each regular t with $Z_1(t) > 0$, by (2.5), $\dot{f}_1(t) = \left(\alpha \sum_{k:\sigma(k)=1} m_k \right) - 1$. Since $\alpha \sum_{k:\sigma(k)=1} m_k < 1$, there is positive t_0 with

$$t_0 \leq \frac{f_1(0)}{1 - \alpha \sum_{k:\sigma(k)=1} m_k} \leq \frac{\sum_{k:\sigma(k)=1} k m_k}{1 - \alpha \sum_{k:\sigma(k)=1} m_k}$$

such that $Z_1(t_0) = 0$. Assume that (3.5) holds. We next show $Z_1(t) = 0$ for $t \geq t_0$. To see this, let

$$f_2(t) = \sum_{k:\sigma(k)=1} m_k Q_k(t)$$

be the immediate workload at unit 1. Then, from (2.1)-(2.4),

$$f_2(t) = f_2(0) + \alpha m_1 t - \sum_{k:\sigma(k)=1} T_k(t) + \sum_{k:\sigma(k)=1} m_k \mu_{k-1} T_{k-1}(t)$$

and, for any regular t with $f_2(t) > 0$,

$$\dot{f}_2(t) = \alpha m_1 + \sum_{k:\sigma(k)=1} \frac{m_k}{m_{k-1}} - 1 \leq 0.$$

Thus f_2 is non-increasing. Since $f_2(t_0) = 0$, we have $Z_1(t) = 0$ for $t \geq t_0$.
Now, it remains to show that $\sum_{i=2}^{N} Z_i(t) = 0$ for each time $t \geq t_1 \geq t_0$ and that the network is globally stable. So, we consider times $t \geq t_0$ and specialize the lemma 3.1 to the case where $Z_1(t) = 0$ and $\dot{Q}_1(t) = ... = \dot{Q}_{N(N-1)+1}(t) = 0$. First, observe that since $Z_1(t) = 0$ for $t \geq t_0$, (3.2), (3.4) are satisfied for $i = 1$, Then, recalling that (3.4) implies (3.3) in our network, we see that we are left with the conditions:

$$\text{(3.7)} \qquad \frac{dh_i(x, Q(t))}{dt} \leq -\varepsilon \ \text{ whenever } \ Z_i(t) > 0, \ \ i = 2, ..., N$$

$$\text{(3.8)} \qquad \begin{array}{l} h_k(Q(t)) = max\{h_i(Q(t)), h_k(Q(t)), i \in \{1, ..., N\}\backslash\{k, j\}, \ \ k = 2, ..., N\} \\ \leq h_j(Q(t)), \ \ j \in \{1, ..., N\}\backslash\{k, i\} \ \text{ whenever } \ \sum_{l=1}^{N\backslash\{j\}} Z_l = 0. \end{array}$$

After that, by Lemma 3.1, we can easily show that (3.7)-(3.8) and hence (3.2)-(3.4) hold if there exists $(x_2, x_3, ..., x_{N^2-1}, x_{N^2}) > 0$ satisfying

$$\text{(3.9)} \qquad \alpha \sum_{k:\sigma(k)=i} x_k < \mu_k x_k, \ \ i = 2, ..., N$$

$$\text{(3.10)} \qquad \left(\sum_{k:\sigma(k)=i} x_k \right) - x_i < \sum_{k:\sigma(k)=N} x_k, \ \ i = 2, ..., N-1$$

$$\text{(3.11)} \qquad \sum_{k:\sigma(k)=i} x_k < \sum_{k:\sigma(k)=i-1} x_k, \ \ i = 2, ..., N$$

$$\text{(3.12)} \qquad \left(\sum_{k:\sigma(k)=i} x_k \right) - x_i < \sum_{k:\sigma(k)=i-1} x_k, \ \ i = 3, ..., N$$

$$\text{(3.13)} \qquad \left(\sum_{k:\sigma(k)=i} x_k \right) - x_i < \sum_{k:\sigma(k)=i-1} x_k - x_i, \ \ i = 3, ..., N$$

Then, given $(x_1, x_2, ..., x_{N^2}) > 0$, let

$$y_i^k = \frac{x_k}{\displaystyle\sum_{k:\sigma(k)=i} x_k}, i = 1, ..., N$$

And $(x_1, ..., x_{N^2}) > 0$ satisfies (2.9)-(2.12) iff $(y_1, ..., y_N, x_{N+1}, ..., x_{N^2}) > 0$ satisfies

$$\text{(3.14)} \qquad \alpha m_k < y_i^k, \ \ k = 1, ..., N^2, \ \ i = 1, ..., N.$$

So, there exists $x > 0$ satisfying (3.9)-(3.13) if and only if the usual traffic condition (3.6) at units $2, 3, ..., N$ hold. Therefore, the proposition follows from Lemma 3.1. □

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Department of Mathematics, Djillali LIABES University of Sidi Bel Abbes, BP. 89, Sidi Bel Abbes 22000, Algeria.

E-mail address: `bouchentouf_amina@yahoo.fr`

Department of Mathematics, Sciences and Technology University of Oran: Mohamed Boudiaf, USTOMB BP. 1505 EL-M'NAOUAR-ORAN, Algeria.

E-mail address: `sakhi.hanane@yahoo.fr`