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A NEW SEQUENCE OF FUNCTIONS INVOLVING ${}_{p_j}F_{q_j}$

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Abstract. A remarkably large number of operational techniques have drawn the attention of several researchers in the study of sequences of functions and polynomials. Very recently, Agarwal and Chand gave a interesting new sequence of functions involving the ${}_pF_q$ .Using the same method, in this paper, we present a new sequence of functions involving product of the ${}_pF_q$. Some generating relations and finite summation formula of the sequence presented here are also considered. In the last, we use Matlab (R2012a) for each parameter of our main sequence, which gives the eccentric characteristics in the area of sequences of functions or class of polynomials.

1. Introduction

The idea of representing the processes of calculus, differentiation, and integration, as operators, is called an operational technique, which is also known as an operational calculus. Many operational techniques involve various special functions have found some significant applications in various sub-fields of applicable mathematical analysis. Many applications of operational techniques can be found in the mathematical analysis, solving a polynomial equations and differential equations. Since last four decades, a number of workers like Chak[2], Gould and Hopper [6], Chatterjea[5], Singh[16], Srivastava and Singh[19], Mittal[8, 9, 10], Chandal[3, 4], Srivastava[14], Joshi and Parjapat[7], Patil and Thakare[11] and Srivastava and Singh[18] have made deep research of the properties, applications and different extensions of the various operational techniques.

The key element of the operational technique is to consider differentiation as an operator $D = \frac{d}{dx}$ acting on functions. Linear differential equations can then be recast in the form of an operator valued function F(D) of the operator D acting on an unknown function which equals a known function. Solutions are then obtained by making the inverse operator of F acting on the known function.

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Indeed, a remarkably large number of sequences of functions involving a variety of special functions have been developed by many authors (see, for example, [18]; for a very recent work, see also [1, 16]). Here we aim at presenting a new sequence of functions involving a product of the ${}_pF_q$ by using operational techniques, which are expressed in terms of the Gauss hypergeometric function. Some generating relations and finite summation formulae are also obtained.

For our purpose, we begin by recalling some known functions and earlier works. In 1971, by Mittal [8] gave the Rodrigues formula for the generalized Lagurre polynomials defined by

$$T_{kn}^{(\alpha)}(x) = \frac{1}{n!} x^{-\alpha} \exp\left(p_k(x)\right) D^n \left[x^{\alpha+n} \exp\left(-p_k(x)\right)\right], \tag{1.1}$$

where $p_k(x)$ is a polynomial in x of degree k.

Mittal [9] also proved the following relation for (1.1) given by

$$T_{kn}^{(\alpha+s-1)}(x) = \frac{1}{n!} x^{-\alpha-n} \exp\left(p_k(x)\right) T_s^n \left[x^{\alpha} \exp\left(-p_k(x)\right)\right], \tag{1.2}$$

where s is a constant and $T_s \equiv x(s + xD)$.

In this sequel, in 1979, Srivastava and Singh [18] studied a sequence of functions $V_n^{(\alpha)}(x; a, k, s)$ defined by

$$V_n^{(\alpha)}(x; a, k, s) = \frac{x^{-\alpha}}{n!} \exp\left\{p_k(x)\right\} \theta^n \left[x^{\alpha} \exp\left\{-p_k(x)\right\}\right] \tag{1.3}$$

By using the operator $\theta \equiv x^a(s + xD)$, where s is constant, and $p_k(x)$ is a polynomial in x of degree k.

Here, a new sequence of function $\left\{V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x; a, k_1, ..., k_r, y_1, ..., y_r, s)\right\}_{n=0}^{\infty}$ is introduced as follows:

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x; a, k_1, ..., k_r, y_1, ..., y_r, s) := \frac{1}{n!} x^{-\alpha} \prod_{j=1}^{r} \times \tag{1.4}$$

$$ {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ y_j P_{k_j}(x)\right] (T_x^{a,s})^n \left\{x^{\alpha} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ -y_j P_{k_j}(x)\right]\right\},$$

where $T_x^{a,s} \equiv x^a(s + xD), D \equiv \dfrac{d}{dx}$, a and s are constants, $\beta \geq 0$, k_j is a finite and non-negative integer, $y_j \in R$, $P_{k_j}(x)$ are polynomials in x of degree k_j, where $j = 1, 2, ..., r$. ${}_{p_j}F_{q_j}$ is a special case of the generalized hypergeometric functions of one variable. For the sake of completeness, we recall the ${}_pF_q$.

A generalized hypergeometric function ${}_pF_q$ is defined and represented as follows (see [15, Section 1.5]):

$$ {}_pF_q\left[\begin{array}{c}(a_p)\,;\\(b_q)\,;\end{array}\; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_n}{\prod_{j=1}^{q}(b_j)_n}\frac{z^n}{n!}, \tag{1.5}$$

where $(\lambda)_n$ is the Pochhammer symbol defined (for $\lambda \in \mathbb{C}$) by (see [15, p. 2 and p. 4-6]):

$$\begin{aligned}(\lambda)_n :&= \begin{cases} 1 & (n=0) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (n\in\mathbb{N}:=\{1,\,2,\,3,\,\ldots\}) \end{cases} \\ &= \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} \quad (\lambda\in\mathbb{C}\setminus\mathbb{Z}_0^-)\end{aligned} \tag{1.6}$$

and $\mathbb{Z}_0^-$ denotes the set of non-positive integers. Note that the function ${}_pF_q$ converges if $p \le q; p = q+1$ and $|z| < 1$.

Some generating relations and finite summation formula of class of polynomials or sequences of functions have been obtained by using the properties of the differential operators. The operators $T_x^{a,s} \equiv x^a\,(s+xD)\,\left(D\equiv\dfrac{d}{dx}\right)$ is based on the work of Mittal [10], Patil and Thakare [11], Srivastava and Singh [18].

Some useful operational techniques are given below:

$$\exp\left(tT_x^{a,s}\right)\left(x^{\beta}f\left(x\right)\right) = x^{\beta}\left(1-ax^at\right)^{-\left(\frac{\beta+s}{a}\right)} f\left(x\left(1-ax^at\right)^{-1/a}\right), \tag{1.7}$$

$$\exp\left(tT_x^{a,s}\right)\left(x^{\alpha-an}f\left(x\right)\right) = x^{\alpha}\left(1+at\right)^{-1+\left(\frac{\alpha+s}{a}\right)} f\left(x\left(1+at\right)^{1/a}\right), \tag{1.8}$$

$$\left(T_x^{a,s}\right)^n\left(xuv\right) = x\sum_{m=0}^{\infty}\binom{n}{m}\left(T_x^{a,s}\right)^{n-m}(v)\left(T_x^{a,1}\right)^m(u), \tag{1.9}$$

$$\begin{aligned}&(1+xD)\,(1+a+xD)\,(1+2a+xD)\times \\ &(1+3a+xD)\,...\,(1+(m-1)a+xD)\,x^{\beta-1} = a^m\left(\frac{\beta}{a}\right)_m x^{\beta-1}\end{aligned} \tag{1.10}$$

and

$$(1-at)^{\frac{-\alpha}{a}} = (1-at)^{\frac{-\beta}{a}}\sum_{m=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m\frac{(at)^m}{m!}. \tag{1.11}$$

2. Generating Relations

First generating relation:

$$\sum_{n=0}^{\infty} V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)\,x^{-an}t^n \tag{2.1}$$

$$= (1-at)^{-\left(\frac{\alpha+s}{a}\right)} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right] \times$$

$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; -y_j P_{k_j}\left(x(1-at)^{-1/a}\right)\right]$$

Second generating relation:

$$\sum_{n=0}^{\infty} V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha-an)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)\,x^{-an}t^n \tag{2.2}$$

$$= (1+at)^{-1+\left(\frac{\alpha+s}{a}\right)} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right] \times$$

$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; -y_j P_{k_j}\left(x(1+at)^{1/a}\right)\right]$$

Third generating relation:

$$\sum_{n=0}^{\infty} \binom{m+n}{m} V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)\,x^{-an}t^n \tag{2.3}$$

$$= \frac{(1-at)^{-\left(\frac{\alpha+s}{a}\right)} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; y_j P_{k_j}\left(x(1-at)^{-1/a}\right)\right]} \times$$

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}\left(x(1-at)^{-1/a};a,k_1,\ldots,k_r,y_1,\ldots,y_r,s\right)$$

Proof of the first generating relation.

We start from (1.4) and consider

$$\sum_{n=0}^{\infty} V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,r_1,\ldots,k_r,y_1,\ldots,y_r,s)\,t^n \tag{2.4}$$

$$= x^{-\alpha} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right] \times$$

$$\exp(tT_x^{a,s})\left\{x^{\alpha} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\; -y_j P_{k_j}(x)\right]\right\}$$

Using the operational technique (1.7), Equation (2.4) reduces to

$$\sum_{n=0}^{\infty} V_n^{(p_1,\dots,p_r,q_1,\dots,q_r,\alpha)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)\,t^n \tag{2.5}$$

$$= x^{-\alpha}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right] x^{\alpha}(1-ax^at)^{-\left(\frac{\alpha+s}{a}\right)}\times$$

$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ -y_jP_{k_j}\left(x(1-ax^at)^{-1/a}\right)\right]$$

$$= (1-ax^at)^{-\left(\frac{\alpha+s}{a}\right)}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]\times$$

$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ -y_jP_{k_j}\left(x(1-ax^at)^{-1/a}\right)\right],$$

which upon replacing t by tx^{-a}, yields (2.1).

Proof of the second generating relation.

Again from (1.4), we have

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(p_1,\dots,p_r,q_1,\dots,q_r,\alpha-an)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)\,t^n \tag{2.6}$$

$$= x^{-\alpha}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]\times$$

$$\exp(tT_x^{a,s})\left\{x^{\alpha-an}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ -y_jP_{k_j}(x)\right]\right\}.$$

Applying the operational technique (1.8), we get

$$\sum_{n=0}^{\infty} x^{-an} V_n^{(p_1,\dots,p_r,q_1,\dots,q_r,\alpha-an)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)\,t^n \tag{2.7}$$

$$= x^{-\alpha}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right] x^{\alpha}\,(1+at)^{\frac{\alpha+s}{a}-1}\times$$

$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ -y_jP_{k_j}\left(x(1+at)^{1/a}\right)\right]$$

$$= (1+at)^{\frac{\alpha+s}{a}-1}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]\times \tag{2.8}$$

$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\ (b_{q_j});\end{matrix}\ -y_jP_{k_j}\left(x(1+at)^{1/a}\right)\right].$$

This proves (2.2).

Proof of the third generating relation.

We can write (1.4) as

$$(T_x^{a,s})^n \left[x^\alpha \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} -y_jP_{k_j}(x)\right]\right] \tag{2.9}$$

$$= n!x^\alpha \frac{V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} y_jP_{k_j}(x)\right]}$$

or

$$\exp\left(t\left(T_x^{a,s}\right)\right)\left\{(T_x^{a,s})^n \left[x^\alpha \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} -y_jP_{k_j}(x)\right]\right]\right\} \tag{2.10}$$

$$= n!\exp\left(tT_x^{a,\alpha}\right)\left[x^\alpha \frac{V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} y_jP_{k_j}(x)\right]}\right]$$

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}\left(T_x^{a,s}\right)^{m+n}\left\{x^\alpha \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} -y_jP_{k_j}(x)\right]\right\} \tag{2.11}$$

$$= n!\exp\left(tT_x^{a,s}\right)\left\{x^\alpha \frac{V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} y_jP_{k_j}(x)\right]}\right\}.$$

Using the operational technique (1.7), Equation (2.11) can be written as:

$$\sum_{m=0}^{\infty}\frac{t^m}{m!}\left(T_x^{a,s}\right)^{m+n}\left[x^\alpha \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} -y_jP_{k_j}(x)\right]\right] \tag{2.12}$$

$$= n!x^\alpha\left(1-ax^a t\right)^{-\left(\frac{\alpha+s}{a}\right)} \frac{1}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} y_jP_{k_j}\left(x(1-ax^a t)^{-1/a}\right)\right]} \times$$

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}\left(x\left(1-ax^a t\right)^{-1/a};a,k_1,\ldots,k_r,y_1,\ldots,y_r,s\right)$$

which, upon using (2.9), gives

$$\text{(2.13)}\quad \sum_{m=0}^{\infty} \frac{t^m\,(m+n)!}{m!n!} x^{\alpha} \frac{V_{m+n}^{(p_1,\ldots,p_m,q_1,\ldots,q_m,\alpha)}\left(x;a,k_1,...,k_m,y_1,...,y_m,s\right)}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} -y_jP_{k_j}(x)\right]}$$

$$= x^{\alpha}\left(1-ax^at\right)^{-\left(\frac{\alpha+s}{a}\right)} \frac{1}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} -y_jP_{k_j}\left(x(1-ax^at)^{-1/a}\right)\right]}\times$$

$$V_n^{(p_1,\ldots,p_m,q_1,\ldots,q_m,\alpha)}\left(x\left(1-ax^at\right)^{-1/a};a,k_1,...,k_m,y_1,...,y_m,s\right).$$

Therefore, we have

$$\text{(2.14)}\quad \sum_{m=0}^{\infty}\binom{m+n}{n} V_{m+n}^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}\left(x;a,k_1,...,k_r,y_1,...,y_r,s\right)t^m\times$$

$$=\left(1-ax^at\right)^{-\left(\frac{\alpha+s}{a}\right)} \frac{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} y_jP_{k_j}(x)\right]}{\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array} y_jP_{k_j}\left(x(1-ax^at)^{-1/a}\right)\right]}\times$$

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}\left(x\left(1-ax^at\right)^{-1/a};a,k_1,...,k_r,y_1,...,y_r,s\right).$$

Which, upon replacing t by tx^{-a}, proves the result (2.3).

Remark 2.1. If we give some suitable parametric replacement in (2.1), (2.2) and (2.3) respectively, then we can arrive at the known results (see [2, 3, 4, 5, 6, 7, 8, 9, 11, 13, 14, 16, 17]).

3. Finite Summation Formulas

First finite summation formula.

$$\text{(3.1)}\quad V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}\left(x;a,k_1,...,k_r,y_1,...,y_r,s\right)$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!}\left(ax^a\right)^m\left(\frac{\alpha}{a}\right)_m V_{n-m}^{(p_1,\ldots,p_r,q_1,\ldots,q_r,0)}\left(x;a,k_1,...,k_r,y_1,...,y_r,s\right).$$

Second finite summation formula.

$$\text{(3.2)}\quad V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}\left(x;a,k_1,...,k_r,y_1,...,y_r,s\right)$$

$$=\sum_{m=0}^{\infty}\frac{1}{m!}\left(ax^a\right)^m\left(\frac{\alpha-\beta}{a}\right)_m V_{n-m}^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\beta)}\left(x;a,k_1,...,k_r,y_1,...,y_r,s\right).$$

Proof of the first finite summation formula.

From Equation (1.4), we have

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s) \tag{3.3}$$
$$= \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]\times$$
$$(T_x^{a,s})^n\left\{xx^{\alpha-1}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ -y_jP_{k_j}(x)\right]\right\}.$$

Using the operational technique (1.9), we have

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s) \tag{3.4}$$
$$= \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]x\sum_{m=0}^{\infty}\binom{n}{m}(T_x^{a,s})^{n-m}\times$$
$$\left\{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ -y_jP_{k_j}(x)\right]\right\}(T_x^{a,1})^m\left(x^{\alpha-1}\right)$$

$$= \frac{1}{n!}x^{-\alpha}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]x\sum_{m=0}^{\infty}\frac{n!}{m!\,(n-m)!}x^{a(n-m)}\times$$
$$[(s+xD)(s+a+xD)(s+2a+xD)\ldots(s+(n-m-1)a+xD)]$$
$$\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ -y_jP_{k_j}(x)\right]x^{am}\times$$
$$[(1+xD)(1+a+xD)(1+2a+xD)\ldots(1+(m-1)a+xD)]\left(x^{\alpha-1}\right).$$

Using the result (1.9), we have

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s) \tag{3.5}$$
$$= \prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]\sum_{m=0}^{n}\frac{1}{m!\,(n-m)!}x^{an}\times$$
$$\prod_{i=0}^{n-m-1}(s+ia+xD)\left\{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ -y_jP_{k_j}(x)\right]\right\}a^m\left(\frac{\alpha}{a}\right)_m.$$

Put $\alpha=0$ and replacing n by $n-m$ in (3.3), we get

$$V_{n-m}^{(p_1,\ldots,p_r,q_1,\ldots,q_r,0)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s) \tag{3.6}$$
$$= \frac{1}{(n-m)!}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ y_jP_{k_j}(x)\right]\times$$
$$(T_x^{a,s})^{n-m}\left\{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\ -y_jP_{k_j}(x)\right]\right\}.$$

$$\Rightarrow \frac{1}{(n-m)!}\left(T_x^{a,s}\right)^{n-m}\left\{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ -y_jP_{k_j}(x)\right]\right\} \tag{3.7}$$

$$= \frac{V_{n-m}^{(p_1,\dots,p_r,q_1,\dots,q_r,0)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)}{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ y_jP_{k_j}(x)\right]}.$$

This gives

$$\frac{1}{(n-m)!}\prod_{i=0}^{n-m-1}(s+ia+xD)\left\{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ -y_jP_{k_j}(x)\right]\right\} \tag{3.8}$$

$$= x^{a(m-n)}\frac{V_{n-m}^{(p_1,\dots,p_r,q_1,\dots,q_r,0)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)}{\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ y_jP_{k_j}(x)\right]}.$$

From Equations (3.5) and (3.8), we have the main result.

Proof of the second finite summation formula.

Equation (1.4) can be written as

$$\sum_{n=0}^{\infty}V_n^{(p_1,\dots,p_r,q_1,\dots,q_r,\alpha)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)\,t^n \tag{3.9}$$

$$= x^{-\alpha}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ y_jP_{k_j}(x)\right]\times$$

$$\exp\left(tT_x^{(a,s)}\right)\left\{x^{\alpha}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ -y_jP_{k_j}(x)\right]\right\}.$$

Applying the (1.7) to the Equation (3.9), we have

$$\sum_{n=0}^{\infty}V_n^{(p_1,\dots,p_r,q_1,\dots,q_r,\alpha)}(x;a,k_1,\dots,k_r,y_1,\dots,y_r,s)\,t^n \tag{3.10}$$

$$= x^{-\alpha}\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ y_jP_{k_j}(x)\right]x^{\alpha}\left(1-ax^at\right)^{-\left(\frac{\alpha+s}{a}\right)}\times$$

$$\prod_{j=1}^{r}{}_{p_j}F_{q_j}\left[\begin{array}{l}(a_{p_j});\\(b_{q_j});\end{array}\ -y_jP_{k_j}\left(x(1-ax^at)^{-1/a}\right)\right]$$

$$= (1-ax^a t)^{-\left(\frac{\alpha+s}{a}\right)} \prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]\times$$
$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; -y_j P_{k_j}\left(x(1-ax^a t)^{-1/a}\right)\right].$$

Using the result from Equation (1.11), Equation (3.10) reduces to

$$\sum_{n=0}^{\infty} V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s)\,t^n \tag{3.11}$$
$$= (1-ax^a t)^{-\left(\frac{\beta+s}{a}\right)} \sum_{m=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a t)^m}{m!}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]\times$$
$$\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; -y_j P_{k_j}\left(x(1-ax^a t)^{-1/a}\right)\right]$$

$$= \sum_{m=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a t)^m}{m!}x^{-\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]\times$$
$$\exp\left(tT_x^{a,s}\right)\left\{x^{\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; -y_j P_{k_j}(x)\right]\right\}$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a)^m\, t^{n+m}}{m!n!}x^{-\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]\times$$
$$\left(T_x^{a,s}\right)^n\left\{x^{\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; -y_j P_{k_j}(x)\right]\right\}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a)^m\, t^{n}}{m!\,(n-m)!}x^{-\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]\times$$
$$\left(T_x^{a,s}\right)^{n-m}\left\{x^{\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; -y_j P_{k_j}(x)\right]\right\}.$$

Now equating the coefficient of t^n , we get

$$V_n^{(p_1,\ldots,p_r,q_1,\ldots,q_r,\alpha)}(x;a,k_1,\ldots,k_r,y_1,\ldots,y_r,s) \tag{3.12}$$
$$= \sum_{m=0}^{n}\left(\frac{\alpha-\beta}{a}\right)_m \frac{(ax^a)^m}{m!\,(n-m)!}x^{-\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; y_j P_{k_j}(x)\right]\times$$
$$\left(T_x^{a,s}\right)^{n-m}\left\{x^{\beta}\prod_{j=1}^{r} {}_{p_j}F_{q_j}\left[\begin{matrix}(a_{p_j});\\(b_{q_j});\end{matrix}\; -y_j P_{k_j}(x)\right]\right\}.$$

Using the Equation (1.4) in (3.12), we have the result (3.2).

4. Special Cases

(I) If we take $r = 1$, then the results established in equations (2.1), (2.2), (2.3), (3.1) and (3.2) reduce to the known results in [1].

(II) If we apply the case of Mittage-Leffler function via hypergeometric function, *i.e.*, $E_\alpha = {}_0F_{\alpha-1}\left(;\frac{1}{\alpha},\frac{2}{\alpha},...,\frac{\alpha-1}{\alpha};\frac{z}{\alpha^\alpha},\right)$ and $r = 1, y_1 = 1, p_1 = p, q_1 = q$, all the results established in Equations (2.1), (2.2), (2.3), (3.1) and (3.2) reduce to those identities in [13].

(III) If we apply the Wright function $W(\alpha, \delta; z)$ which is very special case of the hypergeomtric function ${}_pF_q$ and $r = 1, y_1 = 1, p_1 = p, q_1 = q$, all the results established in Equations (2.1), (2.2), (2.3), (3.1) and (3.2) reduce to the results in [12].

5. Matlab implementation

In this section, we choose $p_j = 2; q_j = 1; r = 2$ to establish the program of the sequence of functions given in equation (1.4).

5.1. **Code of new sequence of functions:**

```
function [Vn] = pgnhypergeo(sigma,alpha1,lambda,beta,mu,
delta,alpha,a,k,s,x)
%Graph of Vn(sigma,lambda,mu,alpha,a,k,s,x)V
%=Vn(sigma,lambda,mu,alpha,a,k,s,x)=(1/n!).*x.^(-beta)
%.*hypergeom([sigma,lambda],mu,x.^k).
%*hypergeom([alpha1,beta],delta,x.^k)
%.*Tn.^(a,s)(x.^a.*(s+x.*D)(x.^beta)
%.*hypergeom([sigma,lambda],mu,-x.^k)).
%*hypergeom([alpha1,beta],delta,-x.^k), where n=1,2,3,
syms x
%n=input('please enter n:');
n=4;
W11= hypergeom([sigma,lambda],mu,-x.^k);
W12= hypergeom([alpha1,beta],delta,-x.^k);
y=(x.^alpha).*W11.*W12;
for i=1:n
y=(x.^a).*(s.*y+x.*diff(y));
end
W21=hypergeom([sigma,lambda],mu,x.^k);
W22=hypergeom([alpha1,beta],delta,x.^k);
v=(1./factorial(n)).*(1./(x.^alpha)).*W21.*W22.*y;
Vn=subs(v,x);
end

Plot The Graph:
 hold on
h1= ezplot(pgnhypergeo(1,1,1,1,1,1,1,3,1,3,x),[-.1:.05:.1]);
h2= ezplot(pgnhypergeo(3,3,3,3,3,3,1,3,1,3,x),[-.1:.05:.1]);
```

```
h3= ezplot(pgnhypergeo(5,5,5,5,5,5,1,3,1,3,x),[-.1:.05:.1]);
h4= ezplot(pgnhypergeo(7,7,7,7,7,7,1,3,1,3,x),[-.1:.05:.1]);
title('V_1(a,a,a,a,a,a,1,3,1,3,x);a=1,3,5,7');ylabel('V_1')
xlabel('x-axis')
hold off
set(h1,'color','r')
set(h2,'color','b')
set(h3,'color','g')
set(h4,'color','k')
legend('V_1(1,1,1,1,1,1,1,3,1,3,x)','V_1(3,3,3,3,3,3,1,3,1,3,x)',
'V_1(5,5,5,5,5,5,1,3,1,3,x)','V_1(7,7,7,7,7,7,1,3,1,3,x)')
```

6. Graphs:

Some graphs of new sequence of functions (1.4) are established with the help of using the above matlab program for different values of the parameters and can be easily interpreted, which are listed at the end of the paper.

7. Conclusion

In this paper, we have presented a new sequence of functions involving the a product of the ${}_pF_q$ by using operational techniques. With the help of our main sequence formula, some generating relations and finite summation formula of the sequence are also presented here. Our sequence formula is important due to presence of ${}_pF_q$. On account of the most general nature of the ${}_pF_q$ a large number of sequences and polynomials involving simpler functions can be easily obtained as their special cases but due to lack of space we can not mention here.

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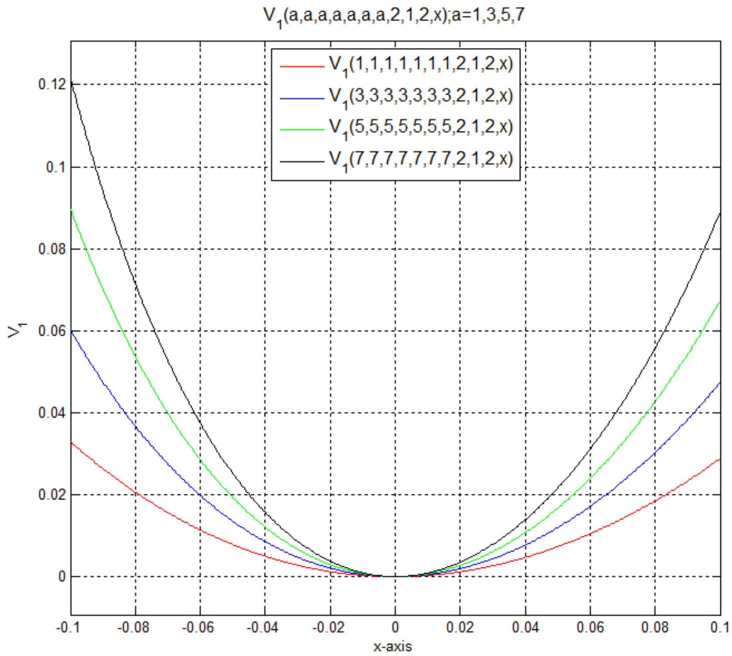

FIGURE 1. $V_1(a, a, a, a, a, a, a, 2, 1, 2, x); a = 1, 3, 5, 7$

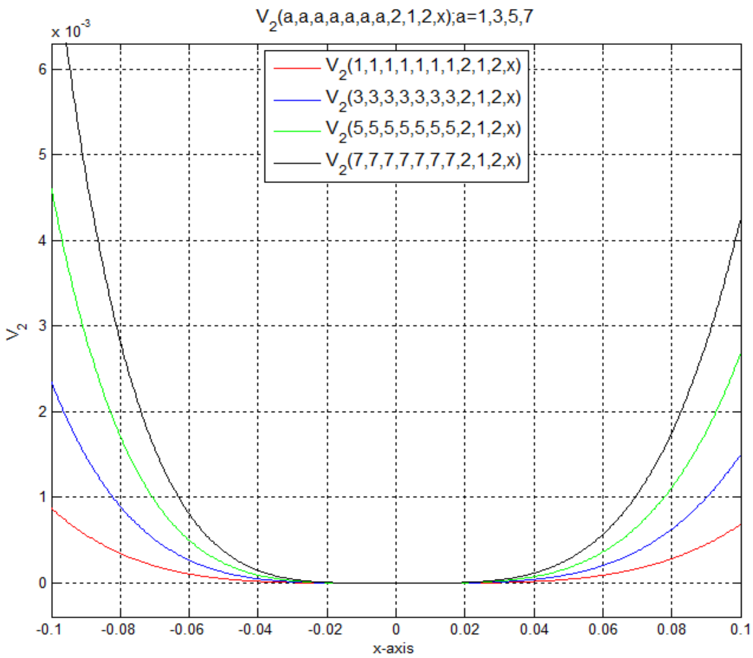

FIGURE 2. $V_2(a, a, a, a, a, a, a, 2, 1, 2, x); a = 1, 3, 5, 7$

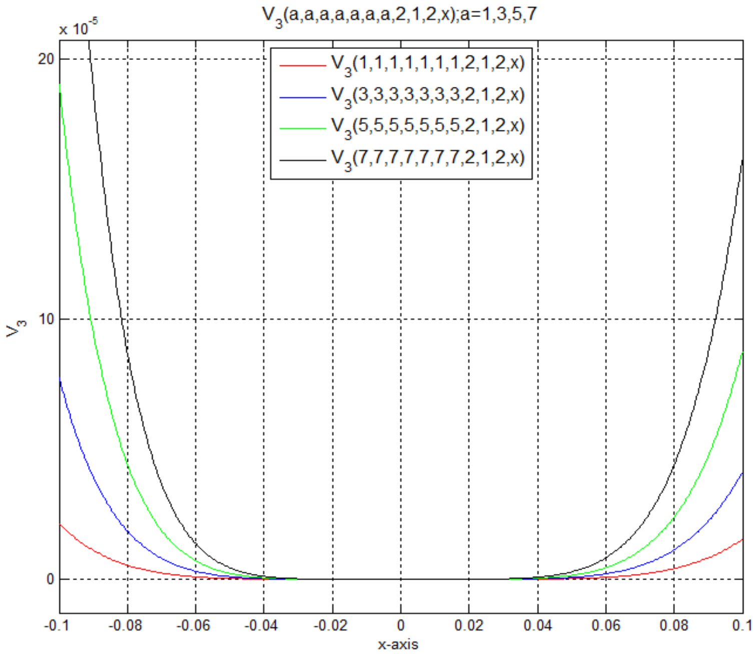

FIGURE 3. $V_3(a, a, a, a, a, a, a, 2, 1, 2, x); a = 1, 3, 5, 7$

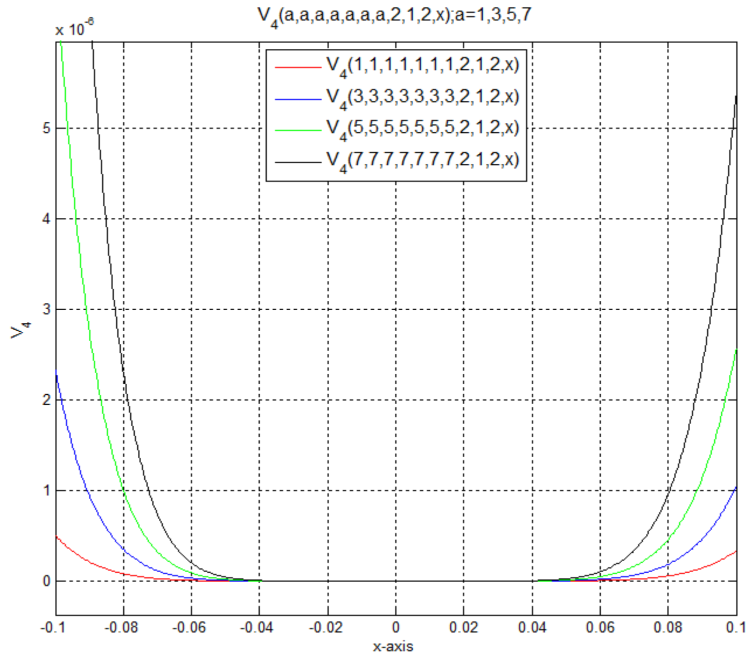

FIGURE 4. $V_4(a, a, a, a, a, a, a, 2, 1, 2, x); a = 1, 3, 5, 7$

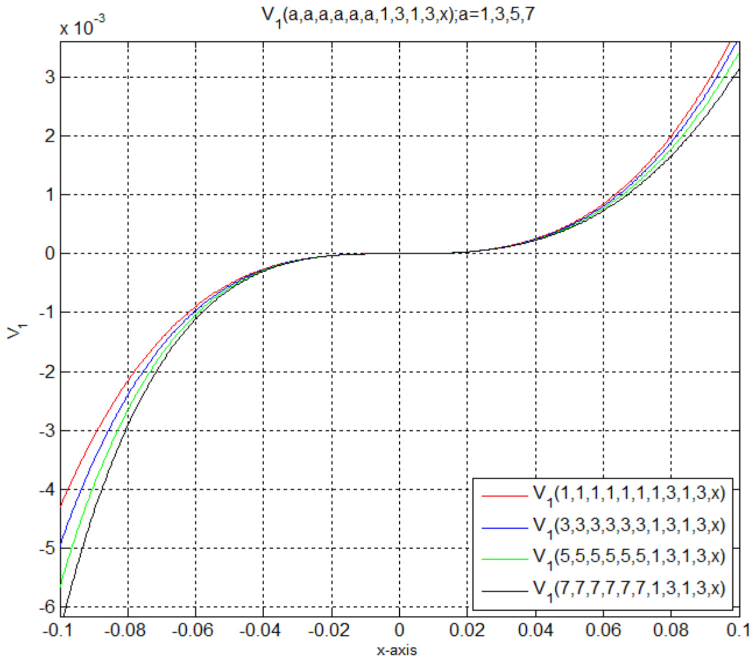

FIGURE 5. $V_1(a, a, a, a, a, a, 1, 3, 1, 3, x); a = 1, 3, 5, 7$

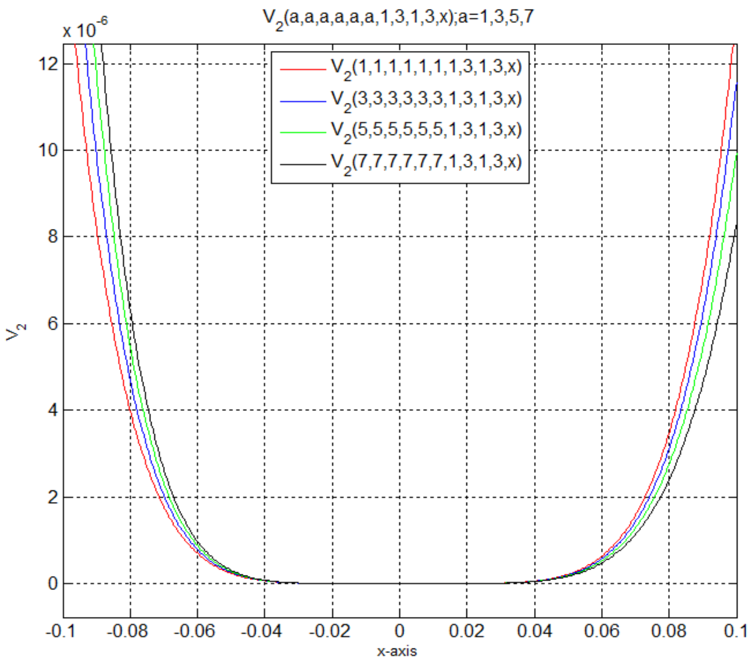

FIGURE 6. $V_2(a, a, a, a, a, a, 1, 3, 1, 3, x); a = 1, 3, 5, 7$

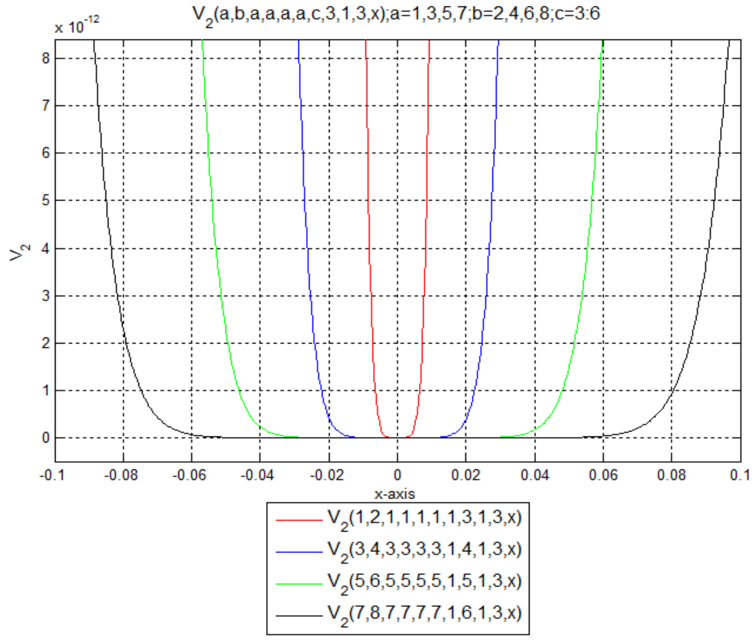

FIGURE 7. $V_2(a, b, a, a, a, a, c, 3, 1, 3, x); a = 1 : 2 : 7; b = 2 : 2 : 8; c = 3 : 6$

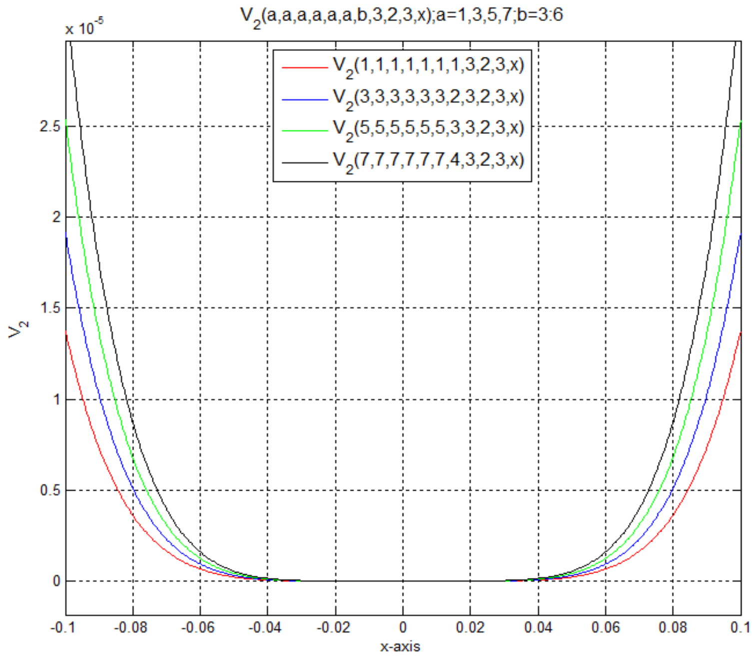

FIGURE 8. $V_2(a, a, a, a, a, a, b, 3, 2, 3, x); a = 1, 3, 5, 7; b = 3 : 6$

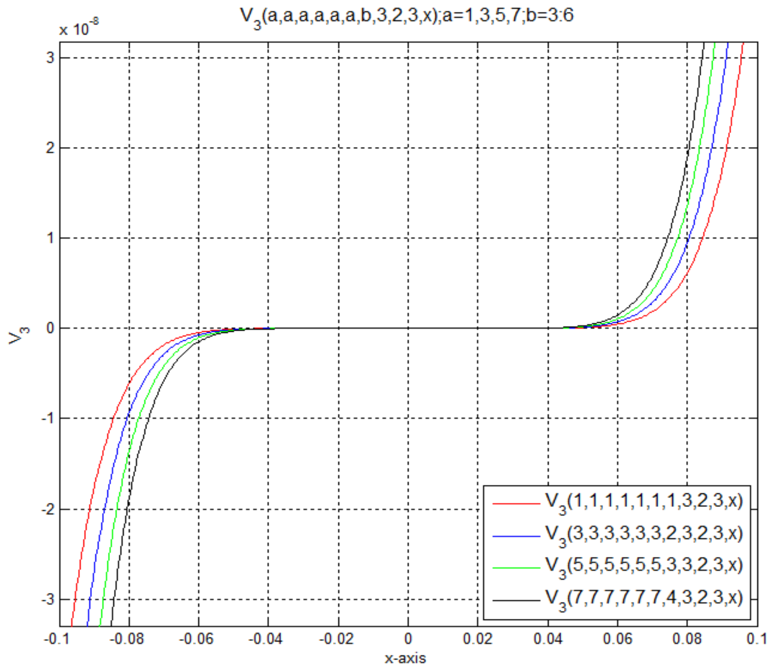

FIGURE 9. $V_3(a, a, a, a, a, a, b, 3, 2, 3, x); a = 1, 3, 5, 7; b = 3 : 6$