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FURTHER RESULTS ON THE UNIQUENESS OF MEROMORPHIC FUNCTIONS SHARING THREE SETS

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Abstract. We use the notion of weighted sharing of sets to deal with the problem of uniqueness of meromorphic functions sharing three sets and obtain some results which in turn improve and extend a series of results obtained in [2] and [3]. One example is exhibited to show that one condition in one of our results is the best possible.

1. Introduction, Definitions and Results

Let f and g be two non-constant meromorphic functions defined in the open complex plane $\mathbb{C}$. We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside a possible exceptional set of finite linear measure.
If for some $a \in \mathbb{C} \cup \{\infty\}$, f and g have the same set of a-points with same multiplicities then we say that f and g share the value a CM (counting multiplicities). If we do not take the multiplicities into account, f and g are said to share the value a IM (ignoring multiplicities).

Let S be a set of distinct elements of $\mathbb{C}\cup\{\infty\}$ and $E_f(S) = \bigcup_{a\in S}\{z : f(z)-a = 0\}$, where each zero is counted according to its multiplicity. If we do not count the multiplicity the set $E_f(S) = \bigcup_{a\in S}\{z : f(z) - a = 0\}$ is denoted by $\overline{E}_f(S)$.

If $E_f(S) = E_g(S)$ we say that f and g share the set S CM. On the other hand if $\overline{E}_f(S) = \overline{E}_g(S)$, we say that f and g share the set S IM.

In 1976 F. Gross [5] posed the following question:

Question A *Can one find two finite sets S_j $(j = 1, 2)$ such that any two non-constant entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ for $j = 1, 2$ must be identical ?*

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For meromorphic function it is natural to ask the following question .

Question B[17] *Can one find three finite sets* S_j $(j = 1, 2, 3)$ *such that any two non-constant meromorphic functions* f *and* g *satisfying* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2, 3$ *must be identical ?*

The above question can be considered as the inception of the new era in the study of relationship between two meromorphic functions via their pre-image sets and as a result during the last couple of years or so several authors explored the possible answer to *Question B* under weaker hypothesis. {cf.[1]-[4], [11], [14], [15], [17], [20]}.

In the direction of *Question B* Fang and Xu [4] proved the following result.

Theorem A. [4] *Let* $S_1 = \{z : z^3 - z^2 - 1 = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$. *Suppose that* f *and* g *are two non-constant meromorphic functions satisfying* $\Theta(\infty; f) > \frac{1}{2}$ *and* $\Theta(\infty; g) > \frac{1}{2}$. *If* $E_f(S_j) = E_g(S_j)$ *for* $j = 1, 2, 3$ *then* $f \equiv g$.

In 2002, Qiu and Fang [15] further generalized *Theorem A* as follows.

Theorem B. [15] *Let* $n \geq 3$ *be a positive integer* $S_1 = \{z : z^n - z^{n-1} - 1 = 0\}$, $S_2 = \{0\}$ *and let* f *and* g *be two non-constant meromorphic functions whose poles are of multiplicities at least 2. If* $E_f(\{\infty\}) = E_g(\{\infty\})$ *and* $E_f(S_i) = E_g(S_i)$ *for* $i = 1, 2,$ *then* $f \equiv g$.

In [15] example were provided by the authors to show that the condition that the poles of $f(z)$ and $g(z)$ are of multiplicities at least 2 can not be removed in *Theorem B*.

It should be noted that if two meromorphic functions f and g have no simple pole then clearly $\Theta(\infty, f) \geq \frac{1}{2}$ and $\Theta(\infty, g) \geq \frac{1}{2}$.

Lahiri and Banerjee [11] investigate the situation for $\Theta(\infty, f) \leq \frac{1}{2}$ and $\Theta(\infty, g) \leq \frac{1}{2}$ in *Theorem A* and proved the following result.

Theorem C. [11] *Let* $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$, *where* a, b *are nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no repeated root and* n (≥ 4) *is an integer. If for two non-constant meromorphic functions* f *and* g $E_f(S_i) = E_g(S_i)$ *for* $i = 1, 2, 3$ *and* $\Theta(\infty; f) + \Theta(\infty; g) > 0$, *then* $f \equiv g$.

In 2004 Yi and Lin [20] proved the following theorem.

Theorem D. [20] *Let* $S_1 = \{z : z^n + az^{n-1} + b = 0\}$, $S_2 = \{0\}$ *and* $S_3 = \{\infty\}$, *where a,b are nonzero constants such that* $z^n + az^{n-1} + b = 0$ *has no repeated root and* n (≥ 3) *is an integer. If for two non-constant meromorphic functions* f *and* g, $E_f(S_i) = E_g(S_i)$ *for* $i = 1, 2, 3$ *and* $\Theta(\infty; f) > \frac{1}{2}$, *then* $f \equiv g$.

The introduction of the new notion of the scaling between CM and IM, known as weighted sharing of values and sets by I. Lahiri [8, 9] in 2001 further expedite the investigations in the direction of *Question B*.

Definition 1.1. [8, 9] Let k be a nonnegative integer or infinity. For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k) then f, g share (a, p) for any integer p, $0 \leq p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

Definition 1.2. [8] Let S be a set of distinct elements of $\mathbb{C} \cup \{\infty\}$ and k be a nonnegative integer or ∞. We denote by $E_f(S,k)$ the set $E_f(S,k) = \bigcup_{a \in S} E_k(a;f)$.

Clearly $E_f(S) = E_f(S,\infty)$ and $\overline{E}_f(S) = E_f(S,0)$.

In 2007 the first present author further invigorate the three set sharing problem in the direction of *Question B* by showing that weighted sharing rendered an useful tool to relax the nature of sharing the image sets. Though Banerjee proved a number of results in the direction of *Question B*, we only recall a few of them to make the discussion pertinent in context of the paper.

In [2] the first present author proved the following result:

Theorem E. [2] *Let S_1, S_2 and S_3 be defined as in* Theorem C. *If for two non-constant meromorphic functions f and g $E_f(S_1,3) = E_g(S_1,3)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,\infty) = E_g(S_3,\infty)$ and $\Theta(\infty;f) + \Theta(\infty;g) > \max\{0, \frac{20-4n}{7n-11}\}$, then $f \equiv g$.*

Theorem F. [2] *Let S_1, S_2 and S_3 be defined as in* Theorem C. *If for two non-constant meromorphic functions f and g $E_f(S_1,2) = E_g(S_1,2)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,\infty) = E_g(S_3,\infty)$ and $\Theta(\infty;f) + \Theta(\infty;g) > \max\{0, \frac{32-4n}{5n-4}\}$, then $f \equiv g$.*

Further in 2009 the first present author proved a series of results as follows.

Theorem G. [3] *Let S_1, S_2 and S_3 be defined as in* Theorem C. *If for two non-constant meromorphic functions f and g $E_f(S_1,3) = E_g(S_1,3)$, $E_f(S_2,\infty) = E_g(S_2,\infty)$, $E_f(S_3,\infty) = E_g(S_3,\infty)$ and $\Theta(\infty;f) + \Theta(\infty;g) > 0$, then $f \equiv g$.*

Theorem H. [3] *Let S_1, S_2 and S_3 be defined as in* Theorem C. *If for two non-constant meromorphic functions f and g $E_f(S_1,2) = E_g(S_1,2)$, $E_f(S_2,\infty) = E_g(S_2,\infty)$, $E_f(S_3,\infty) = E_g(S_3,\infty)$ and $\Theta(\infty;f) + \Theta(\infty;g) > \frac{4}{3n-5}$, then $f \equiv g$.*

Theorem I. [3] *Let S_1, S_2 and S_3 be defined as in* Theorem D. *If for two non-constant meromorphic functions f and g $E_f(S_1,5) = E_g(S_1,5)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,\infty) = E_g(S_3,\infty)$ and $\Theta(\infty;f) + \Theta(\infty;g) > 1$, then $f \equiv g$.*

Theorem J. [3] *Let S_1, S_2 and S_3 be defined as in* Theorem D. *If for two non-constant meromorphic functions f and g $E_f(S_1,4) = E_g(S_1,4)$, $E_f(S_2,\infty) = E_g(S_2,\infty)$, $E_f(S_3,\infty) = E_g(S_3,\infty)$ and $\Theta(\infty;f) + \Theta(\infty;g) > 1$, then $f \equiv g$.*

From *Theorems E*, *F* and *H* it is to be noted that to further relax the nature of sharing the set S_1 or S_2 from a standard weight in [2]-[3], lower bound of the deficiency has to be increased accordingly. Naturally the following questions are inevitable.

Question 1: Is there any significant contribution of the deficiencies of other values not deviating the lower bound of the same, to improve the above *Theorems E, F, G and H*?

Question 2: Can one further relax the nature of sharing of the set S_1 in the *Theorem I* and S_2 in the *Theorems H* and *J* by taking the possible answer to *Question B* in background ?

The above questions are the motivation of the paper. In the paper we will drastically improve and extend all the results stated so far. Following three theorems are the main results of the paper.

Next we suppose

$$\Theta_f(n) = \begin{cases} \Theta(\infty; f) + \Theta(-a\frac{n-1}{n}; f), & if \;\; n \geq 4 \\ \Theta(\infty; f), & if \;\; n = 3 \end{cases}$$

$\Theta_g(n)$ can be similarly defined.

Theorem 1.1. *Let S_1, S_2 and S_3 be defined as in* Theorem C. *If for two non-constant meromorphic functions f and g $E_f(S_1, 3) = E_g(S_1, 3)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta_f(n) + \Theta_g(n) > 0$, then $f \equiv g$.*

Theorem 1.2. *Let S_1, S_2 and S_3 be defined as in* Theorem C. *If for two non-constant meromorphic functions f and g $E_f(S_1, 2) = E_g(S_1, 2)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta_f(n) + \Theta_g(n) > \frac{4}{6n-15}$, then $f \equiv g$.*

Theorem 1.3. *Let S_1, S_2 and S_3 be defined as in* Theorem D. *If for two non-constant meromorphic functions f and g $E_f(S_1, 4) = E_g(S_1, 4)$, $E_f(S_2, 0) = E_g(S_2, 0)$, $E_f(S_3, \infty) = E_g(S_3, \infty)$ and $\Theta_f(n) + \Theta_g(n) > 1$, then $f \equiv g$.*

Remark 1.1. Clearly *Theorem 1.1* improves *Theorems E* and *G*, *Theorem 1.2* extends and improves *Theorems F* and *H* and that *Theorem 1.3* improves *Theorems I* and *J*.

The following example shows that the condition $\Theta_f(n) + \Theta_g(n) > 0$ is sharp in *Theorem 1.1*

Example 1.1. Let $f = -a\frac{1-e^{(n-1)z}}{1-e^{nz}}$ and $g = -ae^z\frac{1-e^{(n-1)z}}{1-e^{nz}}$, where $n(\geq 3)$ is an integer and $S_i^{'}s$ be as in *Theorem 1.2*. Then $E_f(S_i, \infty) = E_g(S_i, \infty)$ for $i = 1, 2, 3$ because $f^{n-1}(f + a) \equiv g^{n-1}(g + a)$ and $g \equiv e^z f$.

Then $T(r, f) = (n-1)T(r, e^z) + O(1)$ and $T(r, g) = (n-1)T(r, e^z) + O(1)$. Here

$$\Theta(\infty; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\sum\limits_{k=1}^{n-1} \overline{N}(r, \beta^k; e^z)}{(n-1)T(r, e^z) + O(1)} = 0$$

and

$$\Theta(\infty; g) = 1 - \limsup_{r \longrightarrow \infty} \frac{\sum\limits_{k=1}^{n-1} \overline{N}(r, \beta^k; e^z)}{(n-1)T(r, e^z) + O(1)} = 0,$$

where $\beta = exp\ (\frac{2\pi i}{n})$. We note that the polynomial $(n-1)z^n - nz^{n-1} + 1$ has double zero at the point $z = 1$. Consequently it has $n - 1$ distinct zeros which are denoted as $u_k, k = 1, \ldots, n-1$. So

$$\Theta(-a\frac{n-1}{n}; f) = 1 - \limsup_{r \longrightarrow \infty} \frac{\sum\limits_{k=1}^{n-1} \overline{N}(r, u_k; e^z)}{(n-1)T(r, e^z) + O(1)} = 0$$

and

$$\Theta(-a\frac{n-1}{n}; g) = 1 - \limsup_{r \longrightarrow \infty} \frac{\sum\limits_{j=1}^{n-1} \overline{N}(r, v_j; e^z)}{(n-1)T(r, e^z) + O(1)} = 0,$$

where $v_j = \frac{1}{u_j}, j = 1, \ldots, n-1$. Therefore $\Theta_f(n) + \Theta_g(n) = 0$ but $f \not\equiv g$.

We now explain some notations which are used in the paper.

Definition 1.3. [7] For $a \in \mathbb{C} \cup \{\infty\}$ we denote by $N(r, a; f \mid= 1)$ the counting function of the simple a-points of f. For a positive integer m we denote by $N(r, a; f \mid\leq m)(N(r, a; f \mid\geq m))$ the counting function of those a-points of f whose multiplicities are not greater(less) than m where each a-point is counted according to its multiplicity.

$\overline{N}(r, a; f \mid\leq m)$ $(\overline{N}(r, a; f \mid\geq m))$ are defined similarly, where in counting the a-points of f we ignore the multiplicities.

Also $N(r, a; f \mid< m)$, $N(r, a; f \mid> m)$, $\overline{N}(r, a; f \mid< m)$ *and* $\overline{N}(r, a; f \mid> m)$ are defined analogously.

Definition 1.4. [1] We denote by $\overline{N}(r, a; f| = k)$ the reduced counting function of those a-points of f whose multiplicities are exactly k, where $k \geq 2$ is an integer.

Definition 1.5. [1] Let f and g be two non-constant meromorphic functions such that f and g share (a, k) where $a \in \mathbb{C} \cup \{\infty\}$. Let z_0 be an a-point of f with multiplicity p, an a-point of g with multiplicity q. We denote by $\overline{N}_L(r, a; f)$ the counting function of those a-points of f and g where $p > q$,by $\overline{N}_E^{(k+1}(r, a; f)$ the counting function of those a-points of f and g where $p = q \geq k + 1$; each point in this counting function is counted only once. In the same way we can define $\overline{N}_L(r, a; g)$ and $\overline{N}_E^{(k+1}(r, a; g)$.

Definition 1.6. [8, 9] Let f, g share a value $a \in \mathbb{C}$ IM. We denote by $\overline{N}_*(r, a; f, g)$ the reduced counting function of those a-points of f whose multiplicities differ from the multiplicities of the corresponding a-points of g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ and $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f)+\overline{N}_L(r, a; g)$.

Definition 1.7. [12] Let $a, b_1, b_2, \ldots, b_q \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f \mid g \neq b_1, b_2, \ldots, b_q)$ the counting function of those a-points of f, counted according to multiplicity, which are not b_i-points of g for $i = 1, 2, \ldots, q$.

2. Lemmas

In this section we present some lemmas which will be needed in the sequel. Let F and G be two non-constant meromorphic functions defined as follows.

$$F = \frac{f^{n-1}(f + a)}{-b}, \quad G = \frac{g^{n-1}(g + a)}{-b}, \tag{2.1}$$

where a, b two nonzero constants defined as in *Theorem C*.

Henceforth we shall denote by H, Φ and V the following three functions

$$H = (\frac{F^{''}}{F^{'}} - \frac{2F^{'}}{F - 1}) - (\frac{G^{''}}{G^{'}} - \frac{2G^{'}}{G - 1}),$$

$$\Phi = \frac{F^{'}}{F - 1} - \frac{G^{'}}{G - 1}$$

and

$$V = \frac{f^{'}}{f} - \frac{g^{'}}{g}.$$

Lemma 2.1. *([9], Lemma 1) Let F, G share $(1, 1)$ and $H \not\equiv 0$. Then*

$$N(r, 1; F \mid= 1) = N(r, 1; G \mid= 1) \leq N(r, H) + S(r, F) + S(r, G).$$

Lemma 2.2. *Let S_1, S_2 and S_3 be given as* Theorem 1.1 *and F, G be given as (2.1). If for two non-constant meromorphic functions f and g $E_f(S_1,0) = E_g(S_1,0)$, $E_f(S_2,0) = E_g(S_2,0)$, $E_f(S_3,0) = E_g(S_3,0)$ and $H \not\equiv 0$ then*

$$\begin{aligned} N(r,H) \quad \leq \quad & \overline{N}_*(r,0,f,g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,-a\frac{n-1}{n};g) \\ & + \overline{N}_*(r,\infty;f,g) + \overline{N}_0(r,0;f^{'}) + \overline{N}_0(r,0;g^{'}), \end{aligned}$$

where $\overline{N}_0(r,0;f^{'})$ is the reduced counting function of those zeros of $f^{'}$ which are not the zeros of $f(F-1)$ and $\overline{N}_0(r,0;g^{'})$ is similarly defined.

Proof. Since $E_f(S_1,0) = E_g(S_1,0)$ it follows that F and G share $(1,0)$. We have from (2.1) that

$$F^{'} = \frac{[nf + (n-1)a]f^{n-2}f^{'}}{(-b)}$$

and

$$G^{'} = \frac{[ng + (n-1)a]g^{n-2}g^{'}}{(-b)}.$$

We can easily verify that possible poles of H occur at (i) those zeros of f and g whose multiplicities are different related to f and g, (ii)zeros of $nf + a(n-1)$ and $ng + a(n-1)$, (iii) those poles of f and g whose multiplicities are different related to f and g, (iv) those 1-points of F and G with different multiplicities, (v) zeros of $f^{'}$ which are not the zeros of $f(F-1)$, (vi) zeros of $g^{'}$ which are not zeros of $g(G-1)$.

Since H has only simple poles, the lemma follows from above. This proves the lemma. □

Lemma 2.3. [16] *Let f be a non-constant meromorphic function and $P(f) = a_0 + a_1 f + a_2 f^2 + \ldots + a_n f^n$, where $a_0, a_1, a_2 \ldots, a_n$ are constants and $a_n \neq 0$. Then $T(r,P(f)) = nT(r,f) + O(1)$.*

Lemma 2.4. [3] *Let F and G be given as (2.1). If f, g share $(0,0)$ and 0 is not an Picard exceptional value of f and g. Then $\Phi \equiv 0$ implies $F \equiv G$.*

Lemma 2.5. [3] *Let F and G be given as (2.1), $n \geq 3$ an integer and $\Phi \not\equiv 0$. If F, G share $(1,m)$, f, g share $(0,p)$, (∞,k),where $0 \leq p < \infty$ then*

$$\begin{aligned} & [(n-1)p + (n-2)]\, \overline{N}(r,0;f| \geq p+1) \leq \overline{N}_*(r,1;F,G) + \overline{N}_*(r,\infty;F,G) \\ & + S(r,f) + S(r,g). \end{aligned}$$

Lemma 2.6. *Let f, g be two nonconstant meromorphic functions. Also let F, G be given as (2.1), $n \geq 3$ an integer and $V \not\equiv 0$. If F, G share $(1,m)$ and f, g share $(0,p)$, (∞,k), where $1 \leq m \leq \infty$ then*

$$m\overline{N}_*(r,1;F,G) \leq \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + S(r,f) + S(r,g).$$

Proof. Note that

$$\begin{aligned} & m\overline{N}_*(r,1;F,G) \\ \leq \quad & N(r,0;V) \\ \leq \quad & N(r,V) + S(r,f) + S(r,g) \\ \leq \quad & \overline{N}_*(r,0;f,g) + \overline{N}_*(r,\infty;f,g) + S(r,f) + S(r,g). \end{aligned}$$

□

Lemma 2.7. *Let f, g be two nonconstant meromorphic functions. Also let F, G be given as (2.1), $n \geq 3$ an integer and $V \not\equiv 0$, $\Phi \not\equiv 0$. If F, G share $(1, m)$, where $m \geq 2$, if $n \geq 3$; $m \geq 1$, if $n \geq 4$ and f, g share $(0, 0)$, (∞, k), $0 \leq k \leq \infty$ then*

$$\overline{N}(r, 0; f) \leq \frac{m+1}{(n-2)m-1} \overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g),$$

Similar result holds for g also.

Proof. Using *Lemma 2.5* and *Lemma 2.6* for $p = 0$ we see that

$$\begin{aligned}(n-2)\overline{N}(r, 0; f) &\leq \overline{N}_*(r, 1; F, G) + \overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g)\\ &\leq \frac{1}{m}\overline{N}(r, 0; f) + \frac{m+1}{m}\overline{N}_*(r, \infty; f, g) + S(r, f) + S(r, g),\end{aligned}$$

from which the lemma follows. □

Lemma 2.8. *([10], Lemma 5) If two nonconstant meromorphic functions f, g share $(\infty, 0)$ then for $n \geq 2$*

$$f^{n-1}(f + a)g^{n-1}(g + a) \not\equiv b^2,$$

where a, b are finite nonzero constants.

Lemma 2.9. [13] *If $N(r, 0; f^{(k)} \mid f \neq 0)$ denotes the counting function of those zeros of $f^{(k)}$ which are not the zeros of f, where a zero of $f^{(k)}$ is counted according to its multiplicity then*

$$N(r, 0; f^{(k)} \mid f \neq 0) \leq k\overline{N}(r, \infty; f) + N(r, 0; f \mid < k) + k\overline{N}(r, 0; f \mid \geq k) + S(r, f).$$

Lemma 2.10. *Let F, G be given as (2.1), F, G share $(1, m)$, $2 \leq m < \infty$ and $\Phi \not\equiv 0$ and $n \geq 3$.Also f,g share $(0, 0)$ and (∞, ∞). Then*

$$\overline{N}(r, 0; f) \leq \frac{1}{m(n-2)-1}\ \overline{N}(r, \infty; f) + S(r, f).$$

Proof. Using *Lemma 2.3* and *Lemma 2.9* we see that

$$\begin{aligned}\overline{N}_*(r, 1; F, G) &\leq \overline{N}(r, 1; F \mid \geq m+1)\\ &\leq \frac{1}{m}\ \big(N(r, 1 : F) - \overline{N}(r, 1; F)\big)\\ &\leq \frac{1}{m}\ [\sum_{j=1}^{n}\big(N(r, \omega_j; f) - \overline{N}(r, \omega_j; f)\big)]\\ &\leq \frac{1}{m}\ \Big(N(r, 0; f' \mid f \neq 0)\Big)\\ &\leq \frac{1}{m}\ \big[\overline{N}(r, 0; f) + \overline{N}(r, \infty; f)\big] + S(r, f),\end{aligned}$$

where $\omega_1, \omega_2 \ldots \omega_n$ are the distinct roots of the equation $z^n + az^{n-1} + b = 0$. Rest of the proof follows from the *Lemma 2.5* for $p = 0$.This proves the lemma. □

Lemma 2.11. *Let F, G be given as (2.1), F, G share $(1, m)$, $2 \leq m < \infty$ and $\Phi \not\equiv 0$ and $n \geq 3$.Also f, g share $(0, 0)$ and (∞, ∞). Then*

$$\overline{N}_L(r, 1; F) \leq \frac{m(n-2)}{(m+1)[m(n-2)-1]}\ \overline{N}(r, \infty; f) + S(r, f).$$

Proof. Using *Lemma 2.10* and then proceeding as in the proof of *Lemma 2.11* we have

$$\begin{aligned}\overline{N}_L(r,1;F) &\leq \overline{N}(r,1;F\mid\geq m+2)\\ &\leq \frac{1}{m+1}\left[\overline{N}(r,0;f)+\overline{N}(r,\infty;f)\right]+S(r,f).\end{aligned}$$

□

Lemma 2.12. [1] *Let f and g be two nonconstant meromorphic functions sharing $(1,m)$, where $2\leq m<\infty$. Then*

$$\begin{aligned}&\overline{N}(r,1;f\mid=2)+2\,\overline{N}(r,1;f\mid=3)+\ldots+(m-1)\,\overline{N}(r,1;f\mid=m)+m\,\overline{N}_L(r,1;f)\\ &\quad+(m+1)\,\overline{N}_L(r,1;g)+m\,\overline{N}_E^{(m+1}(r,1;g)\leq N(r,1;g)-\overline{N}(r,1;g).\end{aligned}$$

Lemma 2.13. *Let F, G be given by (2.1) and they share $(1,m)$. If f, g share $(0,p)$, (∞,k), where $0\leq p\leq\infty$, $0\leq k\leq\infty$ and $2\leq m<\infty$ and $H\not\equiv 0$.*

$$\begin{aligned}nT(r,f) \leq\; &\overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\ &+\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)\\ &-(m-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G)+S(r,f)+S(r,g).\end{aligned}$$

Similar result holds for g.

Proof. Using *Lemma 2.9* and *Lemma 2.12* we see that

$$\begin{aligned}(2.2)\qquad &\overline{N}_0(r,0;g^{'})+\overline{N}(r,1;F\mid\geq 2)+\overline{N}_*(r,1;F,G)\\ \leq\; &\overline{N}_0(r,0;g^{'})+\overline{N}(r,1;F\mid=2)+\overline{N}(r,1;F\mid=3)+\ldots+\overline{N}(r,1;F\mid=m)\\ &+\overline{N}_E^{(m+1}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_*(r,1;F,G)\\ \leq\; &\overline{N}_0(r,0;g^{'})-\overline{N}(r,1;F\mid=3)-\ldots-(m-2)\overline{N}(r,1;F\mid=m)\\ &-(m-1)\overline{N}_L(r,1;F)-m\overline{N}_L(r,1;G)-(m-1)\overline{N}_E^{(m+1}(r,1;F)\\ &+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_*(r,1;F,G)\\ \leq\; &\overline{N}_0(r,0;g^{'})+N(r,1;G)-\overline{N}(r,1;G)-(m-2)\overline{N}_L(r,1;F)\\ &-(m-1)\overline{N}_L(r,1;G)\\ \leq\; &N(r,0;g^{'}\mid g\neq 0)-(m-2)\overline{N}_L(r,1;F)-(m-1)\overline{N}_L(r,1;G)\\ \leq\; &\overline{N}(r,0;g)+\overline{N}(r,\infty;g)-(m-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G),\end{aligned}$$

where $\overline{N}_0(r,0;g^{'})$ has the same meaning as in the *Lemma 2.2*. Hence using (2.2), *Lemmas 2.1*, *2.2*, *Lemma 2.3* get from second fundamental theorem that

$$
\begin{aligned}
(2.3)\qquad & n\,T(r,f)\\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+N(r,1;F\mid=1)+\overline{N}(r,1;F\mid\geq 2)\\
& -N_0(r,0;f^{'})+S(r,f)\\
\leq\ & \overline{N}(r,0;f)+\overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +\overline{N}_*(r,0;f,g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,1;F,G)+\overline{N}(r,1;F|\geq 2)\\
& +\overline{N}_0(r,0;g^{'})+S(r,f)+S(r,g)\\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +\overline{N}(r,0;f)+\overline{N}(r,0;g)+\overline{N}_*(r,\infty;f,g)+\overline{N}_*(r,0;f,g)\\
& -(m-2)\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G)+S(r,f)+S(r,g).
\end{aligned}
$$

This proves the Lemma. □

Lemma 2.14. *Let f, g be two non-constant meromorphic functions sharing $(0,\infty)$, (∞,∞) and $\Theta_f(n)+\Theta_g(n)>0$. Then $f^{n-1}(f+a)\equiv g^{n-1}(g+a)$ implies $f\equiv g$, where n (≥ 3) is an integer and a is a nonzero finite constant.*

Proof. Let

$$f^{n-1}(f+a)\equiv g^{n-1}(g+a) \tag{2.4}$$

and suppose $f\not\equiv g$. We consider two cases:
Case I Let $y=\frac{g}{f}$ be a constant. Then from (2.4) it follows that $y\neq 1$, $y^{n-1}\neq 1$, $y^n\neq 1$ and $f\equiv -a\frac{1-y^{n-1}}{1-y^n}$, a constant, which is impossible.
Case II Let $y=\frac{g}{f}$ be non-constant. Then

$$f\equiv -a\frac{1-y^{n-1}}{1-y^n}\equiv a\,\left(\frac{y^{n-1}}{1+y+y^2+\ldots+y^{n-1}}-1\right). \tag{2.5}$$

and

$$f+a\frac{(n-1)}{n}\equiv -a\frac{(n-1)y^n-ny^{n-1}+1}{n(1-y^n)}. \tag{2.6}$$

If we assume

$$p(z)=(n-1)z^n-nz^{n-1}+1,$$

then $p(0)\neq 0$ and $p(1)=p^{'}(1)=0$. From (2.5) we see in view of *Lemma 2.3* that

$$T(r,f)=(n-1)T(r,y)+S(r,y).$$

Since f, g share $(0,\infty)$ and (∞,∞) we see that y has no zeros and pole. So from (2.5) and (2.6) we see that

$$\sum_{j=1}^{n-1}\overline{N}(r,u_j;y)\leq\overline{N}(r,-a\frac{n-1}{n};f),\quad \sum_{k=1}^{n-1}\overline{N}(r,\alpha_k;y)\leq\overline{N}(r,\infty;f)$$

where u_j, $j=1,2,\ldots,n-1$ has the same meaning as used in *Example 1.1* and $\alpha_k=exp\,(\frac{2k\pi i}{n})$ for $k=1,2,\ldots,n-1$.

By the second fundamental theorem we get

$$\begin{aligned}(2n-2)\ T(r,y) &\leq \sum_{j=1}^{n-1}\overline{N}(r,u_j;y)+\sum_{k=1}^{n-1}\overline{N}(r,\alpha_k;y)+S(r,y)\\ &\leq \overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;f)+S(r,y)\\ &\leq (2-\Theta_f(n)+\varepsilon)\ T(r,f)+S(r,y)\\ &= (n-1)\ (2-\Theta_f(n)+\varepsilon)\ T(r,y)+S(r,y)\end{aligned}$$

i.e.,

$$\frac{2n-2}{n-1}\ T(r,y) \leq (2-\Theta_f(n)+\varepsilon)\ T(r,y)+S(r,y), \tag{2.7}$$

where $0<2\varepsilon<\Theta_f(n)+\Theta_g(n)$.

Again putting $y_1=\frac{1}{y}$ and noting that $T(r,y)=T(r,y_1)+O(1)$ and proceeding as above we get that

$$\frac{2n-2}{n-1}\ T(r,y) \leq (2-\Theta_g(n)+\varepsilon)\ T(r,y)+S(r,y). \tag{2.8}$$

Adding (2.7) and (2.8) we get

$$\left(\frac{4n-4}{n-1}-4+\Theta_f(n)+\Theta_g(n)-2\varepsilon\right)\ T(r,y) \leq S(r,y),$$

which is a contradiction.

Hence $f\equiv g$ and this proves the lemma. □

Lemma 2.15. *([19], Lemma 6) If $H\equiv 0$, then F, G share $(1,\infty)$. If further F, G share $(\infty,0)$ then F, G share (∞,∞).*

Lemma 2.16. *Let F, G be given by (2.1) and F, G share $(1,m)$, where $1\leq m<\infty$ and $n(\geq 3)$ is an integer. Also let f, g share $(0,0)$, (∞,k) and $H\equiv 0$. Then $f\equiv g$ if $\Theta_f(n)+\Theta_g(n)>0$.*

Proof. Since $H\equiv 0$ we get from *Lemma 2.13*, F and G share $(1,\infty)$ and (∞,∞) and so f and g share (∞,∞). Also

$$F\equiv \frac{AG+B}{CG+D}, \tag{2.9}$$

where A, B, C, D are constants and $AD-BC\neq 0$. Again

$$T(r,f)=T(r,g)+O(1). \tag{2.10}$$

We now consider the following cases:

Case 1. Let $AC\neq 0$. Since f, g share (∞,∞), it follows from (2.9) that f, g have no pole. Again since

$$F\equiv \frac{A+\frac{B}{G}}{C+\frac{D}{G}},$$

it follows that $F-\frac{A}{C}$ has no zero. So by the second fundamental theorem we get

$$\begin{aligned}T(r,F) &\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,\frac{A}{C};F)+S(r,F)\\ &= \overline{N}(r,0;f)+\overline{N}(r,-a;f)+S(r,f)\\ &\leq 2T(r,f)+S(r,f),\end{aligned}$$

which in view of by *Lemma 2.3* gives a contradiction for $n \geq 3$.
Case 2. Let $A \neq 0$ and $C = 0$. Then $F \equiv \alpha G + \beta$, where $\alpha = \frac{A}{D} \neq 0$ and $\beta = \frac{B}{D}$.
Subcase 2.1: Let $\beta = 0$. Then we get $F \equiv \alpha G$. From the definitions of F, G and the statement of the lemma it follows that 1 can not be a Picard exceptional value (e.v.P.) of F and G. For if it happens, then f omits atleast 3 distinct values, which is impossible. Since F, G share $(1,\infty)$, it follows that $\alpha = 1$ and so $F \equiv G$. This together with the assumption that f and g share $(0,0)$ implie that f and g share $(0,\infty)$. Hence by *Lemma 2.14* we get $f \equiv g$.
Subcase 2.2: Let $\beta \neq 0$. Clearly $\alpha \neq 1$, as F, G share $(1,\infty)$. Since f, g share $(0,0)$, it follows that f, g have no zero. First we note that β can not be an e.v.P. value of F. For, if it happens then f omits the values 0 together with the distinct roots of the equation $z^n + az^{n-1} + b\beta = 0$. Since there can be at most one repeated root of the equation $z^n + az^{n-1} + b\beta = 0$ at the point $z = -a\frac{n-1}{n}$, it follows that f omits 0 and at least $n-1$ non zero distinct values for $n \geq 3$, which is impossible.

First we suppose that $F - \beta$ has no repeated zero. We note that $\overline{N}(r,\beta;F) \leq \overline{N}(r,-a;g)$. So by the second fundamental theorem we get

$$\begin{aligned} nT(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\beta;F) + S(r,F) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,-a;g) + S(r,F) \\ &\leq 2T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 3$.

Next we suppose that $F - \beta$ has repeated zeros. Clearly $F - \beta$ can have only one repeated zero of multiplicity 2. Now in the view of (2.10) we get by the second fundamental theorem that

$$\begin{aligned} (n-1)\,T(r,f) &\leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + \overline{N}(r,\beta;F) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,-a;g) + S(r,f) \\ &\leq 2\,T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction for $n \geq 4$.

When $n = 3$ we get from above

$$2\,T(r,f) \;\leq\; (2 - \Theta_f(n) + \frac{1}{2}\varepsilon)T(r,f) + S(r,f).$$

Similarly we have

$$2\,T(r,g) \leq (2 - \Theta_g(n) + \frac{1}{2}\varepsilon)T(r,g) + S(r,g).$$

Adding we get in view of (2.10),

$$4\,T(r,f) \leq (4 - \Theta_f(n) - \Theta_g(n) + \varepsilon)\,T(r,f) + S(r,f),$$

which is impossible since $\Theta_f(n) + \Theta_g(n) > 0$.

Case 3: Let $A = 0$ and $C \neq 0$. Then $F \equiv \frac{1}{\gamma G + \delta}$, where $\gamma = \frac{C}{B} \neq 0$ and $\delta = \frac{D}{B}$.
Subcae 3.1: Let $\delta = 0$. Then $F \equiv \frac{1}{\gamma G}$. Since F, G share $(1,\infty)$, it follows that $\gamma = 1$ and then $FG \equiv 1$, which is impossible by *Lemma 2.8*.
Subcase 3.2: Let $\delta \neq 0$. Clearly $\gamma \neq 1$, as F, G share $(1,\infty)$. Since f, g share $(0,0)$, it follows that f, g have no zero. Again since F, G share (∞,∞), it follows

that F, G have no pole. Consequently $G+\frac{\delta}{\gamma}$ has no zero. Then by the second fundamental theorem we get

$$\begin{aligned} T(r,G) &\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;G)+\overline{N}(r,\frac{-\delta}{\gamma};G)+S(r,G) \\ &\leq \overline{N}(r,-a;g)+S(r,g), \end{aligned}$$

which is impossible for $n\geq 3$.
This completes the proof of the Lemma. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let F, G be given by (2.1). Then F and G share $(1,3)$, (∞,∞).
We consider the following cases.
Case 1. Let $H\not\equiv 0$. Clarly $F\not\equiv G$. Suppose 0 is not an e.v.P of f and g then by *Lemma 2.4* we get $\Phi\not\equiv 0$.
Subcase 1.1: Suppose that $V\not\equiv 0$. Since f, g share $(0,0)$ it follows that $\overline{N}_*(r,0;f,g)\leq\overline{N}(r,0;f)$. Now from *Lemma 2.7* with $m=3$ and *Lemma 2.13* we obtain for $\varepsilon>0$,

$$\begin{aligned} &nT(r,f) \\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g) \\ & +3\,\overline{N}(r,0;f)-\overline{N}_*(r,1;F,G)-\overline{N}_L(r,1;G)+S(r,f)+S(r,g) \\ \leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g) \\ & +\frac{12}{3n-7}\ \overline{N}_*(r,\infty;f,g)+S(r,f)+S(r,g) \\ \leq\ & \left(2-\Theta_f(n)+\frac{1}{2}\varepsilon\right)T(r,f)+\left(2-\Theta_g(n)+\frac{1}{2}\varepsilon\right)T(r,g)+S(r,f)+S(r,g) \\ \leq\ & (4-\Theta_f(n)-\Theta_g(n)+\varepsilon)\,T(r)+S(r). \end{aligned} \tag{3.1}$$

In a similar way we can obtain

$$nT(r,g)\leq(4-\Theta_f(n)-\Theta_g(n)+\varepsilon)\,T(r)+S(r). \tag{3.2}$$

Combining (3.1) and (3.2) we get

$$(n-4+\Theta_f(n)+\Theta_g(n)-\varepsilon)\,T(r)\leq S(r). \tag{3.3}$$

Since $\varepsilon>0$, (3.3) leads to a contradiction.
Subcase 1.2: Suppose $V\equiv 0$. Then integrating we get $f\equiv cg$, where c is a non-zero constant. Since f and g share $(0,0)$, it follow that f and g share $(0,\infty)$ and hence $\overline{N}_*(r,0;f,g)=0$. Now from *Lemma 2.5* with $p=0$ and *Lemma 2.13* we

obtain for $\varepsilon > 0$

$$
\begin{aligned}
(3.4)\qquad & nT(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g) \\
& +2\ \overline{N}(r,0;f) - \overline{N}_*(r,1;F,G) - \overline{N}_L(r,1;G) + S(r,f) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g) \\
& +\frac{2}{n-2}\ \overline{N}_*(r,1;F,G) - \overline{N}_*(r,1;F,G) + S(r,f) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g) \\
& +S(r,f) + S(r,g) \\
\leq\ & \left(2 - \Theta_f(n) + \frac{1}{2}\varepsilon\right) T(r,f) + \left(2 - \Theta_g(n) + \frac{1}{2}\varepsilon\right) T(r,g) + S(r,f) + S(r,g) \\
\leq\ & (4 - \Theta_f(n) - \Theta_g(n) + \varepsilon)\, T(r) + S(r).
\end{aligned}
$$

So by the similar argument as done in *Subcase 1.1* we get

$$
(3.5)\qquad (n - 4 + \Theta_f(n) + \Theta_g(n) - \varepsilon)\, T(r) \leq S(r).
$$

Since $\varepsilon > 0$, (3.5) leads to a contradiction.
If 0 is an e.v.P of f and g then (3.1) and (3.4) automatically hold.
Case 2. Let $H \equiv 0$. Then the theorem follows from *Lemma 2.16.* □

Proof of Theorem 1.2. Let F, G be given by (2.1). Then F and G share $(1,2)$, (∞,∞).
We consider the following cases.
Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0 is not an e.v.P. of f and g then by *Lemma 2.4* we get $\Phi \not\equiv 0$.
Subcase 1.1: Suppose that $V \not\equiv 0$. Now from *Lemma 2.7* with $m = 2$ and *Lemma 2.13* we obtain for $\varepsilon > 0$,

$$
\begin{aligned}
(3.6)\qquad & nT(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g) \\
& +3\,\overline{N}(r,0;f) - \overline{N}_L(r,1;G) + S(r,f) + S(r,g) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g) \\
& +\frac{9}{2n-5}\ \overline{N}_*(r,\infty;f,g) + S(r,f) + S(r,g) \\
\leq\ & \left(2 - \Theta_f(n) + \frac{1}{2}\varepsilon\right) T(r,f) + \left(2 - \Theta_g(n) + \frac{1}{2}\varepsilon\right) T(r,g) + S(r,f) + S(r,g) \\
\leq\ & (4 - \Theta_f(n) - \Theta_g(n) + \varepsilon)\, T(r) + S(r).
\end{aligned}
$$

So by the similar argument as done in *Subcase 1.1* of *Theorem 1.1* we obtain

$$
(3.7)\qquad (n - 4 + \Theta_f(n) + \Theta_g(n) - \varepsilon)\, T(r) \leq S(r).
$$

Since $\varepsilon > 0$, (3.7) leads to a contradiction.
Subcase 1.2: Suppose $V \equiv 0$. Then integrating we get $f \equiv cg$, where c is a

non-zero constant. Now from *Lemma 2.5* with $p = 0$, *Lemma 2.11* and *Lemma 2.13* we obtain for $\varepsilon > 0$

$$
\begin{aligned}
(3.8)\qquad & nT(r,f)\\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g)\\
& +2\,\overline{N}(r,0;f) - \overline{N}_L(r,1;G) + S(r,f) + S(r,g)\\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g)\\
& +\frac{2}{n-2}\ \overline{N}_*(r,\infty;f,g) - \overline{N}_*(r,1;F,G) + \overline{N}_L(r,1;F)\\
& +S(r,f) + S(r,g)\\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,-a\frac{n-1}{n};f) + \overline{N}(r,\infty;g) + \overline{N}(r,-a\frac{n-1}{n};g)\\
& +\frac{2n-4}{6n-15}\ \overline{N}(r,\infty;f) + S(r,f) + S(r,g)\\
\leq\ & \left(2 - \Theta_f(n) + \frac{1}{2}\varepsilon\right) T(r,f) + \left(2 - \Theta_g(n) + \frac{1}{2}\varepsilon\right) T(r,g) + \frac{2n-4}{6n-15}\ T(r,f)\\
& +S(r,f) + S(r,g)\\
\leq\ & \left(4 + \frac{2n-4}{6n-15} - \Theta_f(n) - \Theta_g(n) + \varepsilon\right) T(r) + S(r).
\end{aligned}
$$

In a similar way we can obtain

$$
(3.9)\qquad nT(r,g) \leq \left(4 + \frac{2n-4}{6n-15} - \Theta_f(n) - \Theta_g(n) + \varepsilon\right) T(r) + S(r).
$$

Combining (3.8) and (3.9) we get

$$
(3.10)\qquad \left(n - 4 + \frac{8-2n}{6n-15} - \frac{4}{6n-15} + \Theta_f(n) + \Theta_g(n) - \varepsilon\right) T(r) \leq S(r).
$$

Since $\varepsilon > 0$, (3.10) leads to a contradiction.
If 0 is an e.v.P of f and g then (3.6) and (3.8) automatically hold.
Case 2. Let $H \equiv 0$. Then the theorem follows from *Lemma 2.16.* □

Proof of Theorem 1.3. Let F, G be given by (2.1). Then F and G share $(1,4)$, (∞,∞). We consider the following cases.
Case 1. Let $H \not\equiv 0$. Then $F \not\equiv G$. Suppose 0 is not an e.v.P. of f and g then by *Lemma 2.4* we get $\Phi \not\equiv 0$.
Subcase 1.1: Suppose that $V \not\equiv 0$. Now from *Lemma 2.7* with $m = 4$ and *Lemma*

2.13 we obtain for $\varepsilon > 0$,

$$
\begin{aligned}
(3.11)\quad & nT(r,f)\\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +3\,\overline{N}(r,0;f)-2\ \overline{N}_{*}(r,1;F,G)-\overline{N}_{L}(r,1;G)+S(r,f)+S(r,g)\\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +\frac{15}{4n-9}\ \overline{N}_{*}(r,\infty;f,g)+S(r,f)+S(r,g)\\
\leq\ & \left(2-\Theta_f(n)+\frac{1}{2}\varepsilon\right)T(r,f)+\left(2-\Theta_g(n)+\frac{1}{2}\varepsilon\right)T(r,g)+S(r,f)+S(r,g)\\
\leq\ & (4-\Theta_f(n)-\Theta_g(n)+\varepsilon)\,T(r)+S(r).
\end{aligned}
$$

By the same argument as done in the proof of *Theorem 1.1* we arrive at a contradiction.

Subcase 1.2: Suppose $V\equiv 0$. Then integrating we get $f\equiv cg$, where c is a non-zero constant. Now from *Lemma 2.5* with $p=0$ and *Lemma 2.13* we obtain for $\varepsilon>0$

$$
\begin{aligned}
(3.12)\quad & nT(r,f)\\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +2\,\overline{N}(r,0;f)-2\,\overline{N}_{*}(r,1;F,G)-\overline{N}_{L}(r,1;G)+S(r,f)+S(r,g)\\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +\frac{2}{n-2}\ \overline{N}_{*}(r,1;F,G)-2\ \overline{N}_{*}(r,1;F,G)+S(r,f)+S(r,g)\\
\leq\ & \overline{N}(r,\infty;f)+\overline{N}(r,-a\frac{n-1}{n};f)+\overline{N}(r,\infty;g)+\overline{N}(r,-a\frac{n-1}{n};g)\\
& +S(r,f)+S(r,g)\\
\leq\ & \left(2-\Theta_f(n)+\frac{1}{2}\varepsilon\right)T(r,f)+\left(2-\Theta_g(n)+\frac{1}{2}\varepsilon\right)T(r,g)+S(r,f)+S(r,g)\\
\leq\ & (4-\Theta_f(n)-\Theta_g(n)+\varepsilon)\,T(r)+S(r).
\end{aligned}
$$

In this case also we arrive at a contradiction.

If 0 is an e.v.P of f and g then (3.11) and (3.12) automatically hold.

Case 2. Let $H\equiv 0$. Then the theorem follows from *Lemma 2.16*. □

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