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ON HADAMARD TYPE INTEGRAL INEQUALITIES FOR NONCONVEX FUNCTIONS

MEHMET ZEKI SARIKAYA, HAKAN BOZKURT AND NECMETTIN ALP

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Abstract. In this paper, we extend some estimates of the right and left hand side of a Hermite- Hadamard type inequality for nonconvex functions whose derivatives absolute values are φ-convex and quasi-φ-convex was introduced by Noor in [3].

1. Introduction

It is well known that if f is a convex function on the interval $I = [a, b]$ and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

which is known as the Hermite-Hadamard inequality for the convex functions.

Both inequalities hold in the reversed direction if f is concave. We note that Hadamard's inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Hadamard's inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found (see, for example, [1], [2], [8]-[10], [12], [14]-[20]).

In [8] some inequalities of Hermite-Hadamard type for differentiable convex mappings connected with the left part of (1.1) were proved by using the following lemma:

Lemma 1.1. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ (I° is the interior of I) with $a < b$. If $f' \in L([a, b])$, then we have*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\\ &= (b-a)\left[\int_0^{\frac{1}{2}} tf'(ta+(1-t)b)dt + \int_{\frac{1}{2}}^1 (t-1)\,f'(ta+(1-t)b)dt\right].\end{aligned} \tag{1.2}$$

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One more general result related to (1.2) was established in [9]. The main result in [8] is as follows:

Theorem 1.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If the mapping $|f'|$ is convex on $[a, b]$, then*

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)| + |f'(b)|}{2} \right). \tag{1.3}$$

In [1], Dragomir and Agarwal established the following results connected with the right part of (1.1) as well as to apply them for some elementary inequalities for real numbers and numerical integration:

Theorem 1.2. *Let $f : I^\circ \subset \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$, and $f' \in L(a, b)$. If the mapping $|f'|$ is convex on $[a, b]$, then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq (b-a) \left(\frac{|f'(a)| + |f'(b)|}{8} \right). \tag{1.4}$$

In [12], Pearce and Pečarić proved the following theorem:

Theorem 1.3. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$, be a differentiable mapping on I°, $a, b \in I^\circ$ with $a < b$. If the mapping $|f'|^q$ is convex on $[a, b]$ for some $q \geq 1$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}} \tag{1.5}$$

and

$$\left| \frac{1}{b-a} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right) \right| \leq \frac{b-a}{4} \left(\frac{|f'(a)|^q + |f'(b)|^q}{2} \right)^{\frac{1}{q}}. \tag{1.6}$$

We recall that the notion of quasi-convex functions generalizes the notion of convex functions. More precisely, a function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said quasi-convex on $[a, b]$ if

$$f(tx + (1-t)y) \leq \sup \{f(x), f(y)\}$$

for all $x, y \in [a, b]$ and $t \in [0, 1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex (see [11]).

The classical Hermite-Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a, b] \to \mathbb{R}$. Ion in [11] presented some estimates of the right hand side of a Hermite- Hadamard type inequality in which some quasi-convex functions are involved. The main results of [11] are given by the following theorems.

Theorem 1.4. *Assume $a, b \in \mathbb{R}$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ is a differentiable function on (a, b). If $|f'|$ is quasi-convex on $[a, b]$, then the following inequality holds*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)}{4} \sup \{|f'(a)|, |f'(b)|\}.$$

Theorem 1.5. *Assume $a,b\in\mathbb{R}$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b). Assume $p\in\mathbb{R}$ with $p>1$. If $|f'|^{p/p-1}$ is quasi-convex on $[a,b]$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\leq\frac{(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\sup\left\{|f'(a)|^{p/p-1},|f'(b)|^{p/p-1}\right\}\right]^{\frac{p-1}{p}}.$$

Convexity plays a central and fundamental role in mathematical finance, economics, engineering, management sciences and optimizastion theory. In recent years, several extensions and generalizations have been considered for classical convexity. A significant generalization of convex functions is that of φ-convex functions introduced by Noor in [3]. In [3] and [7], the authors have studied the basic properties of the φ-convex functions. It is well-know that the φ-convex functions and φ-sets may not be convex functions and convex sets. This class of nonconvex functions include the classical convex functions and its various classes as special cases. For some recent results related to this nonconvex functions, see the papers [3]-[7]

2. Preliminaries

Let $f,\varphi:K\to\mathbb{R}^n$, where K is a nonempty closed set in $\mathbb{R}^n$, be continuous functions. First of all, we recall the following well know results and concepts, which are mainly due to Noor and Noor [3] and Noor [7] as follows:

Definition 2.1. Let $u,v\in K$. Then the set K is said to be $\varphi-convex$ at u with respect to φ, if

$$u+te^{i\varphi}(v-u)\in K,\ \forall u,v\in K,\ t\in[0,1].$$

Remark 2.1. We would like to mention that the Definition 2.1 of a $\varphi-convex$ set has a clear geometric interpretation. This definition essentially says that there is a path starting from a point u which is contained in K. We do not require that the point v should be one of the end points of the path. This observation plays an important role in our analysis. Note that, if we demand that v should be an end point of the path for every pair of points, $u,v\in K$, then $e^{i\varphi}(v-u)=v-u$ if and only if, $\varphi=0$, and consequently $\varphi-convexity$ reduces to convexity. Thus, it is true that every convex set is also an $\varphi-convex$ set, but the converse is not necessarily true, see [3],[7] and the references therein.

Definition 2.2. The function f on the $\varphi-convex$ set K is said to be $\varphi-convex$ with respect to φ, if

$$f\left(u+te^{i\varphi}(v-u)\right)\leq(1-t)f(u)+tf(v),\ \forall u,v\in K,\ t\in[0,1].$$

The function f is said to be $\varphi-concave$ if and only if $-f$ is $\varphi-convex$. Note that every convex function is a $\varphi-convex$ function, but the converse is not true. For eample, the function $f:\mathbb{R}\to\mathbb{R}$, $f(x)=-|x|$ is not a convex function, but $f(x)=-|x|$ is a $\varphi-convex$ with respect to φ where

$$\varphi(v,u)=\begin{cases}2k\pi, & u.v\geq 0,\ k\in\mathbb{Z}\\ k\pi, & u.v<0,\ k\in\mathbb{Z}.\end{cases}$$

Definition 2.3. The function f on the $\varphi-convex$ set K is said to be logarithmic $\varphi-convex$ with respect to φ, such that

$$f\left(u+te^{i\varphi}(v-u)\right)\leq(f(u))^{1-t}(f(v))^{t},\ u,v\in K,\ t\in[0,1]$$

where $f(.)>0$.

Now we define a new definition for quasi-$\varphi-$*convex* functions as follows:

Definition 2.4. The function f on the quasi $\varphi-$*convex* set K is said to be quasi $\varphi-$*convex* with respect to φ, if

$$f\left(u+te^{i\varphi}\left(v-u\right)\right)\leq\max\left\{f\left(u\right),f\left(v\right)\right\}.$$

From the above definitions, we have

$$\begin{aligned} f\left(u+te^{i\varphi}\left(v-u\right)\right) &\leq \left(f\left(u\right)\right)^{1-t}\left(f\left(v\right)\right)^{t} \\ &\leq \left(1-t\right)f\left(u\right)+tf\left(v\right) \\ &\leq \max\left\{f\left(u\right),f\left(v\right)\right\}. \end{aligned}$$

In [5], Noor proved the Hermite-Hadamard inequality for the $\varphi-$convex functions as follows:

Theorem 2.1. *Let $f:K=\left[a,a+e^{i\varphi}\left(b-a\right)\right]\rightarrow(0,\infty)$ be a $\varphi-$convex function on the interval of real numbers K^{0} (the interior of K) and $a,b\in K^{0}$ with $a<a+e^{i\varphi}\left(b-a\right)$ and $0\leq\varphi\leq\frac{\pi}{2}$. Then the following inequality holds:*

$$\begin{aligned} f\left(\frac{2a+e^{i\varphi}\left(b-a\right)}{2}\right) &\leq \frac{1}{e^{i\varphi}\left(b-a\right)}\int_{a}^{a+e^{i\varphi}\left(b-a\right)}f\left(x\right)dx \\ &\leq \frac{f\left(a\right)+f\left(a+e^{i\varphi}\left(b-a\right)\right)}{2}\leq\frac{f\left(a\right)+f\left(b\right)}{2}. \end{aligned} \tag{2.1}$$

This inequality can easily show that using the $\varphi-$*convex* function's definition and $f\left(a+e^{i\varphi}\left(b-a\right)\right)<f\left(b\right)$.

In this article, using functions whose derivatives absolute values are φ-convex and quasi-φ-convex, we obtained new inequalities releted to the right and left side of Hermite-Hadamard inequality. In particular if $\varphi=0$ is taken as, our results obtained reduce to the Hermite-Hadamard type inequality for classical convex functions.

Throughout this study, we always assume that $K=\left[a,a+e^{i\varphi}(b-a)\right]$ and $0\leq\varphi\leq\frac{\pi}{2}$ the interval, unless otherwise specified.

We shall start with the following refinements of the Hermite-Hadamard inequality for $\varphi-$convex functions. Firstly, we give the following results connected with the right part of (2.1):

Theorem 2.2. *Let $f:K\rightarrow(0,\infty)$ be a differentiable mapping on K^{0}. If $|f'|$ is $\varphi-$convex function on the interval of real numbers K^{0} (the interior of K) and $a,b\in K$ with $a<a+e^{i\varphi}(b-a)$. Then, the following inequality holds:*

$$\begin{aligned} &\left|\frac{1}{e^{i\varphi}(b-a)}\int_{a}^{a+e^{i\varphi}(b-a)}f(x)dx-\frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\ &\leq \frac{e^{i\varphi}(b-a)}{8}\left[|f'(a)|+|f'(b)|\right]. \end{aligned} \tag{2.2}$$

Proof. Since K is $\varphi-$convex with respect to φ, for every $t \in [0,1]$, we have $a + te^{i\theta}(b-a) \in K$. Integrating by parts implies that

$$\begin{aligned} & \int_0^1 (1-2t) f'(a+te^{i\varphi}(b-a))dt \\ (2.3) \\ = & \left[\frac{(1-2t)f(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}\right]_0^1 + \frac{2}{e^{i\varphi}(b-a)} \int_0^1 f(a+te^{i\varphi}(b-a))dt \\ = & -\frac{f(a)+f(a+e^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} + \frac{2}{e^{2i\varphi}(b-a)^2} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx \end{aligned}$$

By $\varphi-$convexity of $|f'|$ and (2.3), we have

$$\begin{aligned} & \left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x))dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2} \right| \\ = & \frac{e^{i\varphi}(b-a)}{2} \left| \int_0^1 (1-2t) f'(a+te^{i\varphi}(b-a))dt \right| \\ \leq & \frac{e^{i\varphi}(b-a)}{2} \int_0^1 |1-2t| \left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right] dt \\ = & \frac{e^{i\varphi}(b-a)}{8} \left[\left|f'(a)\right| + \left|f'(b)\right|\right]. \end{aligned}$$

which completes the proof. □

Theorem 2.3. *Let $f : K \to (0,\infty)$ be a differentiable mapping on K^0. Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is $\varphi-$convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. Then, the following inequality holds:*

$$\begin{aligned} & \left| \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2} \right| \\ (2.4) \\ \leq & \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}} \left(\frac{|f'(a)|^{\frac{p}{p-1}} + |f'(b)|^{\frac{p}{p-1}}}{2} \right)^{\frac{p-1}{p}} \end{aligned}$$

Proof. From Hölder's inequality and by using (2.3) in the proof of Theorem 2.2, we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x))dx - \frac{f(a)+f(a+e^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2}\int_0^1 |1-2t|\left|f'(a+te^{i\varphi}(b-a))\right| dt \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1 |1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1 |1-2t|^p\, dt\right)^{\frac{1}{p}}\left(\int_0^1 \left[(1-t)\left|f'(a)\right|^{\frac{p}{p-1}} + t\left|f'(b)\right|^{\frac{p}{p-1}}\right] dt\right)^{\frac{p-1}{p}} \\
=\ & \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}}\left(\frac{|f'(a)|^{\frac{p}{p-1}}+|f'(b)|^{\frac{p}{p-1}}}{2}\right)^{\frac{p-1}{p}}.
\end{aligned}$$

This implies ineqyality (2.4). □

Now, we give the following results connected with the left part of (2.1):

Theorem 2.4. *Under the assumptions of Theorem 2.2. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\right| \\
&\leq \frac{e^{i\varphi}(b-a)}{8}\left[|f'(a)|+|f'(b)|\right].
\end{aligned} \tag{2.5}$$

Proof. Since K is $\varphi-$convex with respect to φ, for every $t\in[0,1]$, we have $a+te^{i\theta}(b-a)\in K$. Integrating by parts implies that

$$\begin{aligned}
&\int_0^{\frac{1}{2}} tf'(a+te^{i\varphi}(b-a))dt + \int_{\frac{1}{2}}^1 (t-1)f'(a+te^{i\varphi}(b-a))dt \\
=\ & \left[\frac{tf(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}\right]_0^{\frac{1}{2}} + \left[\frac{(t-1)f(a+te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)}\right]_{\frac{1}{2}}^1 \\
&-\frac{1}{e^{i\varphi}(b-a)}\int_0^1 f(a+te^{i\varphi}(b-a))dt \\
=\ & \frac{1}{e^{i\varphi}(b-a)}f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right) - \frac{1}{e^{2i\varphi}(b-a)^2}\int_a^{a+e^{i\varphi}(b-a)} f(x)dt.
\end{aligned} \tag{2.6}$$

By $\varphi-$convexity of $|f'|$, we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\right| \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+te^{i\varphi}(b-a))\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(a+te^{i\varphi}(b-a))\right|dt\right] \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^{\frac{1}{2}} t\left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right]dt + \int_{\frac{1}{2}}^1 (1-t)\left[(1-t)\left|f'(a)\right| + t\left|f'(b)\right|\right]dt\right] \\
\leq\ & e^{i\varphi}(b-a)\left[\frac{|f'(a)|+|f'(b)|}{8}\right].
\end{aligned}$$

The proof is completed. □

Theorem 2.5. *Under the assumptions of Theorem 2.3. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\right| \\
&\qquad (2.7) \\
\leq\ & \frac{e^{i\varphi}(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\left|f'(a)\right|^{\frac{p}{p-1}} + 3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right].
\end{aligned}$$

Proof. From Hölder's inequality and by using (2.6), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(\frac{2a+e^{i\varphi}(b-a)}{2})\right| \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+te^{i\varphi}(b-a))\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(a+te^{i\varphi}(b-a))\right|dt\right] \\
\leq\ & e^{i\varphi}(b-a)\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} \left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
&+e^{i\varphi}(b-a)\left(\int_{\frac{1}{2}}^1 (1-t)^p\, dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 \left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}} dt\right)^{\frac{p-1}{p}} \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\int_0^{\frac{1}{2}}\left[(1-t)\left|f'(a)\right|^{\frac{p}{p-1}} + t\left|f'(b)\right|^{\frac{p}{p-1}}\right]dt\right]^{\frac{p-1}{p}} \\
&+\frac{e^{i\varphi}(b-a)}{2^{1+\frac{1}{p}}(p+1)^{\frac{1}{p}}}\left[\int_{\frac{1}{2}}^1\left[(1-t)\left|f'(a)\right|^{\frac{p}{p-1}} + t\left|f'(b)\right|^{\frac{p}{p-1}}\right]dt\right]^{\frac{p-1}{p}} \\
=\ & \frac{e^{i\varphi}(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}} + \left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}} + \left(\left|f'(a)\right|^{\frac{p}{p-1}} + 3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right]
\end{aligned}$$

which completes the proof. □

Theorem 2.6. *Under the assumptions of Theorem 2.2. Then, the following inequality holds:*

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(\frac{2a+e^{i\varphi}(b-a)}{2})\right| \tag{2.8}$$

$$\leq \frac{e^{i\varphi}(b-a)}{4}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[|f'(a)|+|f'(b)|\right].$$

Proof. We consider the inequality (2.7) i.e

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(\frac{2a+e^{i\varphi}(b-a)}{2})\right| \tag{2.9}$$

$$\leq \frac{e^{i\varphi}(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}+\left(\left|f'(a)\right|^{\frac{p}{p-1}}+3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right].$$

Let $a_1 = 3\left|f'(a)\right|^{\frac{p}{p-1}}$, $b_1 = \left|f'(b)\right|^{\frac{p}{p-1}}$, $a_2 = \left|f'(a)\right|^{\frac{p}{p-1}}$, $b_2 = 3\left|f'(b)\right|^{\frac{p}{p-1}}$. Here $0 < (p-1)/p < 1$, for $p > 1$. Using the fact that,

$$\sum_{k=1}^{n}(a_k+b_k)^s \leq \sum_{k=1}^{n}a_k^s + \sum_{k=1}^{n}b_k^s,$$

for $(0 \leq s < 1)$, $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\frac{e^{i\varphi}(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left[\left(3\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}+\left(\left|f'(a)\right|^{\frac{p}{p-1}}+3\left|f'(b)\right|^{\frac{p}{p-1}}\right)^{\frac{p-1}{p}}\right]$$

$$\leq \frac{e^{i\varphi}(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}\left(3^{\frac{p-1}{p}}+1\right)\left[|f'(a)|+|f'(b)|\right]$$

$$\leq \frac{e^{i\varphi}(b-a)}{16}\left(\frac{4}{p+1}\right)^{\frac{1}{p}}4\left[|f'(a)|+|f'(b)|\right]$$

which completed proof. □

Theorem 2.7. *Let $f :\to (0,\infty)$ be a differentiable mapping on K^0. Assume $q \in \mathbb{R}$ with $q \geq 1$. If $|f'|^q$ is $\varphi-$convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. Then the following inequality holds:*

$$\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\right|$$

$$\leq \frac{e^{i\varphi}(b-a)}{8}\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right].$$

Proof. From Hölder's inequality and by using (2.6), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(\frac{2a+e^{i\varphi}(b-a)}{2})\right| \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+te^{i\varphi}(b-a))\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(a+te^{i\varphi}(b-a))\right|dt\right] \\
\leq\ & e^{i\varphi}(b-a)\left(\int_0^{\frac{1}{2}} tdt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}} t\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}} \\
&+e^{i\varphi}(b-a)\left(\int_{\frac{1}{2}}^1 (1-t)\,dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1 (1-t)\left|f'(a+te^{i\varphi}(b-a))\right|^q dt\right)^{\frac{1}{q}} \\
\leq\ & \frac{e^{i\varphi}(b-a)}{8^{\frac{1}{p}}}\left[\int_0^{\frac{1}{2}} t\left[(1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q\right]dt\right]^{\frac{1}{q}} \\
&+\frac{e^{i\varphi}(b-a)}{8^{\frac{1}{p}}}\left[\int_{\frac{1}{2}}^1 (1-t)\left[(1-t)\left|f'(a)\right|^q + t\left|f'(b)\right|^q\right]dt\right]^{\frac{1}{q}} \\
=\ & \frac{e^{i\varphi}(b-a)}{8}\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

The proof is completed. □

Theorem 2.8. *Under the assumptions of Theorem 2.7. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{8}(\frac{2^{\frac{1}{q}}+1}{3^{\frac{1}{q}}})\left[\left|f'(a)\right|+\left|f'(b)\right|\right].
\end{aligned}$$

Proof. We consider the inequality (2.7) i.e

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f(\frac{2a+e^{i\varphi}(b-a)}{2})\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{8}\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}} + \left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Let $a_1 = 2\left|f'(a)\right|^q/3$, $b_1 = \left|f'(b)\right|^q/3$, $a_2 = \left|f'(a)\right|^q/3$, $b_2 = 2\left|f'(b)\right|^q/3$. Here $0 < 1/q < 1$, for $q \geq 1$. Using the fact that

$$\sum_{k=1}^n (a_k+b_k)^s \leq \sum_{k=1}^n a_k^s + \sum_{k=1}^n b_k^s,$$

for $(0 \leq s < 1)$, $a_1, a_2, ..., a_n \geq 0$, $b_1, b_2, ..., b_n \geq 0$, we obtain

$$\begin{aligned} & \frac{e^{i\varphi}(b-a)}{8}\left[\left(\frac{2\left|f'(a)\right|^q+\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}+\left(\frac{\left|f'(a)\right|^q+2\left|f'(b)\right|^q}{3}\right)^{\frac{1}{q}}\right] \\ \leq & \frac{e^{i\varphi}(b-a)}{8}(\frac{2^{\frac{1}{q}}+1}{3^{\frac{1}{q}}})\left[\left|f'(a)\right|+\left|f'(b)\right|\right]. \end{aligned}$$

This concludes the proof. □

3. Hermite-Hadamard type inequalities for quasi $\varphi-$convex functions

In this section, we prove some new inequalities of Hermite-Hadamard for quasi $\varphi-$convex function as follows:

Theorem 3.1. *Let $f : K \to (0, \infty)$ be a differentiable mapping on K^0. If $|f'|$ is quasi $\varphi-$convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. Then the following inequality holds:*

$$\begin{aligned} & \left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\ \leq & \frac{e^{i\varphi}(b-a)}{4}\max\{\left|f'(a)\right|,\left|f'(b)\right|\}. \end{aligned}$$

Proof. By quasi $\varphi-$convexity of $|f'|$ and by using (2.3), we have

$$\begin{aligned} & \left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\ \leq & \frac{e^{i\varphi}(b-a)}{2}\int_0^1 \left|(1-2t)\right|\left|f'(a+te^{i\varphi}(b-a))\right|dt \\ \leq & \frac{e^{i\varphi}(b-a)}{2}\max\{\left|f'(a)\right|,\left|f'(b)\right|\}\int_0^1 \left|(1-2t)\right|dt \\ \leq & \frac{e^{i\varphi}(b-a)}{4}\max\{\left|f'(a)\right|,\left|f'(b)\right|\} \end{aligned}$$

which completes the proof. □

Theorem 3.2. *Let $f : K \to (0, \infty)$ be a differentiable mapping on K^0.Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is quasi $\varphi-$convex function on the interval of real numbers K^0 (the interior of K) and $a, b \in K$ with $a < e^{i\varphi}(b-a)$. Then the following inequality holds:*

$$\begin{aligned} & \left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx-\frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\ \leq & \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\max\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}. \end{aligned}$$

Proof. By quasi $\varphi-$convexity of $|f'|^{p/p-1}$ and by using (2.3), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a)+f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2}\int_0^1 |(1-2t)|\left|f'(a+te^{i\varphi}(b-a))\right|dt \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1 |(1-2t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p-1}{p}}dt\right)^{\frac{p}{p-1}} \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2}\left(\int_0^1 |(1-2t)|^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1 \max\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\}dt\right)^{\frac{p}{p-1}} \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\max\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}
\end{aligned}$$

which completes the proof. □

Theorem 3.3. *Let $f : K \to (0,\infty)$ be a differentiable mapping on K^0 . If $|f'|$ is quasi $\varphi-$convex function on the interval of real numbers K^0 (the interior of K) and $a,b \in K$ with $a < a + e^{i\varphi}(b-a)$. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{4}\max\{|f'(a)|,|f'(b)|\}.
\end{aligned}$$

Proof. By quasi $\varphi-$convexity of $|f'|$ and by using (2.6), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - f\left(\frac{2a+e^{i\varphi}(b-a)}{2}\right)\right| \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+te^{i\varphi}(b-a))\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(a+te^{i\varphi}(b-a))\right|dt\right] \\
\leq\ & e^{i\varphi}(b-a)\max\{|f'(a)|,|f'(b)|\}\left[\int_0^{\frac{1}{2}} tdt + \int_{\frac{1}{2}}^1 (1-t)dt\right] \\
\leq\ & \frac{e^{i\varphi}(b-a)}{4}\max\{|f'(a)|,|f'(b)|\}.
\end{aligned}$$

This concludes the proof. □

Theorem 3.4. *Let $f : K \to (0,\infty)$ be a differentiable mapping on K^0.Assume $p \in \mathbb{R}$ with $p > 1$. If $|f'|^{p/p-1}$ is quasi $\varphi-$convex function on the interval of real numbers K^0 (the interior of K) and $a,b \in K$ with $a < e^{i\varphi}(b-a)$. Then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\max\{|f'(a)|^{\frac{p}{p-1}},|f'(b)|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}.
\end{aligned}$$

Proof. By quasi $\varphi-$convexity of $|f'|^{p/p-1}$ and by using (2.3), we have

$$\begin{aligned}
&\left|\frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx - \frac{f(a+te^{i\varphi}(b-a))}{2}\right| \\
\leq\ & e^{i\varphi}(b-a)\left[\int_0^{\frac{1}{2}} t\left|f'(a+te^{i\varphi}(b-a))\right|dt + \int_{\frac{1}{2}}^1 (1-t)\left|f'(a+te^{i\varphi}(b-a))\right|dt\right] \\
\leq\ & e^{i\varphi}(b-a)\left(\int_0^{\frac{1}{2}} t^p dt\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
& + e^{i\varphi}(b-a)\left(\int_{\frac{1}{2}}^1 (1-t)^p\, dt\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(a+te^{i\varphi}(b-a))\right|^{\frac{p}{p-1}}dt\right)^{\frac{p-1}{p}} \\
\leq\ & \frac{e^{i\varphi}(b-a)}{2(p+1)^{\frac{1}{p}}}\left[\max\{|f'(a)|^{\frac{p}{p-1}}, |f'(b)|^{\frac{p}{p-1}}\}\right]^{\frac{p-1}{p}}
\end{aligned}$$

which completes the proof. □

References

[1] S.S. Dragomir and R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., 11(5) (1998), 91–95.

[2] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000.

[3] M. Aslam Noor, *Some new classes of nonconvex functions*, Nonl.Funct.Anal.Appl.,11(2006),165-171

[4] M. Aslam Noor, *On Hadamard integral inequalities involving two log-preinvex functions*, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-6, Article 75.

[5] M. Aslam Noor, *Hermite-Hadamard integral inequalities for log-$\varphi-$convex functions*, Nonl. Anal. Forum, (2009).

[6] M. Aslam Noor, *On a class of general variotional inequalities,* J. Adv. Math. Studies, 1(2008), 31-42.

[7] K. Inayat Noor and M. Aslam Noor, *Relaxed strongly nonconvex functions*, Appl. Math. E-Notes, 6(2006), 259-267.

[8] U.S. Kırmacı, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., 147 (2004), 137-146.

[9] U.S. Kırmacı and M.E. Özdemir, *On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comp., 153 (2004), 361-368.

[10] U.S. Kırmacı, *Improvement and further generalization of inequalities for differentiable mappings and applications*, Computers and Math. with Appl., 55 (2008), 485-493.

[11] D.A. Ion, *Some estimates on the Hermite-Hadamard inequality through quasi-convex functions*, Annals of University of Craiova Math. Comp. Sci. Ser., 34 (2007) 82-87.

[12] C.E.M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett., 13(2) (2000), 51–55.

[13] J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, partial ordering and statistical applications*, Academic Press, New York, 1991.

[14] M. Z. Sarikaya, A. Saglam and H. Yıldırım, *New inequalities of Hermite-Hadamard type for functions whose second derivatives absolute values are convex and quasi-convex*, International Journal of Open Problems in Computer Science and Mathematics (IJOPCM), 5(3), 2012.

[15] M. Z. Sarikaya, A. Saglam and H. Yıldırım, *On some Hadamard-type inequalities for h-convex functions*, Journal of Mathematical Inequalities, Volume 2, Number 3 (2008), 335-341.

[16] M. Z. Sarikaya, M. Avci and H. Kavurmaci, *On some inequalities of Hermite-Hadamard type for convex functions*, ICMS Iternational Conference on Mathematical Science. AIP Conference Proceedings 1309, 852 (2010).

[17] M. Z. Sarikaya and N. Aktan, *On the generalization some integral inequalities and their applications* Mathematical and Computer Modelling, Volume 54, Issues 9-10, November 2011, Pages 2175-2182.

[18] M. Z. Sarikaya, E. Set and M. E. Ozdemir, *On some new inequalities of Hadamard type involving h-convex functions*, Acta Mathematica Universitatis Comenianae, Vol. LXXIX, 2(2010), pp. 265-272.

[19] A. Saglam, M. Z. Sarikaya and H. Yildirim, *Some new inequalities of Hermite-Hadamard's type*, Kyungpook Mathematical Journal, 50(2010), 399-410.

[20] Ç. Yıldız, M. Gürbüz and A. O. Akdemir, *The Hadamard type inequalities for m-convex functions*, Konuralp Journal of Math. Volume 1 No. 1 pp. 40–47 (2013).

Department of Mathematics, Faculty of Science and Arts, Düzce University, Düzce-TURKEY

E-mail address: sarikayamz@gmail.com, insedi@yahoo.com and placenn@gmail.com