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REMARKS AND QUESTIONS ON BASE POSITIONAL DIMENSION-LIKE FUNCTIONS OF THE TYPE IND

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Abstract. Let Q be a subset of a space X. A family A of open subsets of X is said to be a p-base for Q in X if the set $\{Q \cap U : U \in A\}$ is a base for the subspace Q. In [12] base positional dimension-like functions of the type ind were introduced. The domain of these functions is the class of all p-bases. These functions were studied only with respect to the property of universality. Here, we study these functions with respect to other standard properties of dimension theory and we present questions concerning these functions.

1. Introduction and preliminaries

All spaces are assumed to be T_0-spaces. We denote by ω the first infinite cardinal. The class of all ordinals is denoted by $\mathcal{O}$. We also consider two extra symbols, "-1" and "∞". It is assumed that $-1 < \alpha < \infty$ for every $\alpha \in \mathcal{O}$, $-1(+)\alpha = \alpha(+)(-1) = \alpha$ for every $\alpha \in \mathcal{O} \cup \{-1, \infty\}$, and $\infty(+)\alpha = \alpha(+)\infty = \infty$ for every $\alpha \in \mathcal{O} \cup \{\infty\}$, where by $(+)$ we denote the natural sum of Hessenberg (see [13]). We recall some properties of natural sum. Let α and β be ordinals. Then,
(1) $\alpha(+)\beta = \beta(+)\alpha$,
(2) if $\alpha_1 < \alpha_2$, then $\alpha_1(+)\beta < \alpha_2(+)\beta$, and
(3) $\alpha(+)n = \alpha + n$ for $n < \omega$.

Let Q be a subset of a space X. We denote by $\mathrm{Cl}_X(Q)$ and $\mathrm{Bd}_X(Q)$ the closure and the boundary of Q in X, respectively. Also, by $|Q|$ we denote the cardinality of the subset Q and by $w(Q)$ the weight of the subspace Q.

By an Alexandroff space (see [1]) we mean a space such that every point has a minimal open neighborhood. For every Alexandroff space X we denote by $\mathbf{U}_x$ the smallest open set of X containing the point x.

The small inductive dimension (see [4] and [14]) of a space X, denoted by $\mathrm{ind}(X)$, is defined as follows:
(i) $\mathrm{ind}(X) = -1$ if and only if $X = \emptyset$.
(ii) $\mathrm{ind}(X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a base B for X such that for every $V \in B$ we have $\mathrm{ind}(\mathrm{Bd}_X(V)) < \alpha$.

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(iii) $\mathrm{ind}(X) = \infty$ if and only if the inequality $\mathrm{ind}(X) \leq \alpha$ does not hold for every $\alpha \in \mathcal{O} \cup \{-1\}$.

Let Q be a subset of a space X. A family A of open subsets of X is said to be a p-base for Q in X if the set $\{Q \cap U : U \in A\}$ is a base for the subspace Q. A p-base A for Q in X is said to be a ps-base if A is a base for the space X.

We denote by $\mathrm{p_1}$-ind (see [5] and [12]) the dimension-like function whose domain is the class of all pairs (Q, X), where Q is a subset of a space X, and whose range is the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:
(i) $\mathrm{p_1}\text{-ind}(Q, X) = -1$ if and only if $Q = \emptyset$.
(ii) $\mathrm{p_1}\text{-ind}(Q, X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a p-base A for Q in X such that for every $U \in A$ we have $\mathrm{p_1}\text{-ind}(Q \cap \mathrm{Bd}_X(U), X) < \alpha$.

If in the above definition instead of the p-base A we consider a ps-base, then the dimension-like function $\mathrm{p_1}$-ind will be denoted by $\mathrm{ps_1}$-ind.

The "relative dimensions" or "positional dimensions" are functions whose domains are classes of pairs (Q, X), where Q is a subset of a space X. Several articles were written on the relative dimensions. About some other relative dimensions see for example [2], [3], [5], [6], [7], [8], [9], [10], [12], [15], [16], and [17]. A survey of the subject can be find in [11].

In [12] the base positional dimension-like functions $\mathrm{b_0}$-$\mathrm{p_1}$-ind and $\mathrm{b^0}$-$\mathrm{p_1}$-ind were introduced by S.D. Iliadis. The domain of these functions is the class of all p-bases. In section 2 we investigate the relations between of them and we compare these dimensions with the classical small inductive dimension ind. In sections 3 and 4 we present subspace, partition, and sum theorems. Finally, in section 5 we give questions for these functions.

2. On the base positional dimension-like functions $\mathrm{b_0}$-$\mathrm{p_1}$-ind and $\mathrm{b^0}$-$\mathrm{p_1}$-ind

Definition 2.1. (see [12]) We denote by $\mathrm{b_0}$-$\mathrm{p_1}$-ind the dimension-like function whose domain is the class of all triads (Q, A, X), where A is a p-base for a subset Q in a space X and whose range is the class $\mathcal{O} \cup \{-1, \infty\}$ satisfying the following conditions:
(1) $\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q, A, X) = -1$ if and only if $Q = \emptyset$.
(2) $\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q, A, X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a base B for X such that for every $U \in B$ we have $\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q \cap \mathrm{Bd}_X(U), A, X) < \alpha$.

Theorem 2.1. *For every p-base A for a subset Q in a space X we have*
$$\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q, A, X) = \mathrm{ps_1}\text{-ind}(Q, X).$$

Proof. We prove that
$$\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q, A, X) \leq \mathrm{ps_1}\text{-ind}(Q, X). \tag{1}$$
Let $\mathrm{ps_1}\text{-ind}(Q, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality (1) is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality (1) is true for every triad (Q^Y, A^Y, Y) with $\mathrm{ps_1}\text{-ind}(Q^Y, Y) < \alpha$.

Since $\mathrm{ps_1}\text{-ind}(Q, X) = \alpha$, there exists a ps-base A for Q in X such that for every $U \in A$ we have $\mathrm{ps_1}\text{-ind}(Q \cap \mathrm{Bd}_X(U), X) < \alpha$. Then, A is a base for the space X. Also, by inductive assumption, for every $U \in A$ we have
$$\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q \cap \mathrm{Bd}_X(U), A, X) \leq \mathrm{ps_1}\text{-ind}(Q \cap \mathrm{Bd}_X(U), X) < \alpha.$$
Thus, $\mathrm{b_0}\text{-}\mathrm{p_1}\text{-ind}(Q, A, X) \leq \alpha$.

We prove that

$$\mathrm{ps}_1\text{-ind}(Q,X) \leq \mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X). \tag{2}$$

Let $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = \alpha \in \mathcal{O} \cup \{-1,\infty\}$. The inequality (2) is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality (2) is true for every triad (Q^Y, A^Y, Y) with $\mathrm{b}_0\text{-p}_1\text{-ind}(Q^Y, A^Y, Y) < \alpha$.

Since $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = \alpha$, there exists a base B for X such that for every $U \in B$ we have $\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(U), A, X) < \alpha$. By inductive assumption, for every $U \in B$ we have

$$\mathrm{ps}_1\text{-ind}(Q \cap \mathrm{Bd}_X(U), X) \leq \mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(U), A, X) < \alpha.$$

Thus, $\mathrm{ps}_1\text{-ind}(Q,X) \leq \alpha$. □

Theorem 2.2. *For every p-base A for a subset Q in an Alexandroff space X we have* $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) \in \{-1, 0, \infty\}$.

Proof. Let A be a p-base for a subset Q in an Alexandroff space X. If $Q = \emptyset$, then $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = -1$. Suppose that $Q \neq \emptyset$. Then, we have the following two cases:

(1) For every $x \in X$, $Q \cap \mathrm{Bd}_X(\mathbf{U}_x) = \emptyset$. In this case we consider the base $B = \{\mathbf{U}_x : x \in X\}$ of X. Then, for every $x \in X$ we have

$$\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_x), A, X) = \mathrm{b}_0\text{-p}_1\text{-ind}(\emptyset, A, X) = -1.$$

Therefore, $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = 0$.

(2) There exists $x_0 \in X$ such that $Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}) \neq \emptyset$. We prove that

$$\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = \infty.$$

Suppose that $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = \alpha \in \mathcal{O}$. Then, there exists a base B' for X such that for every $U \in B'$ we have

$$\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(U), A, X) < \alpha.$$

Particularly, since $\mathbf{U}_{x_0} \in B'$, we have

$$\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), A, X) = \beta < \alpha. \tag{3}$$

Hence, there exists a base B'' for X such that for every $U \in B''$ we have

$$\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}) \cap \mathrm{Bd}_X(U), A, X) < \beta.$$

Since $\mathbf{U}_{x_0} \in B''$, we have

$\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}) \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), A, X) =$
$\mathrm{b}_0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), A, X) < \beta$,

which contradicts relation (3). Thus, $\mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X) = \infty$. □

Definition 2.2. (see [12]) We denote by b^0-p_1-ind the dimension-like function whose domain is the class of all triads (Q,A,X), where A is a p-base for a subset Q in a space X and whose range is the class $\mathcal{O} \cup \{-1,\infty\}$ satisfying the following conditions:
(1) $\mathrm{b}^0\text{-p}_1\text{-ind}(Q,A,X) = -1$ if and only if $Q = \emptyset$.
(2) $\mathrm{b}^0\text{-p}_1\text{-ind}(Q,A,X) \leq \alpha$, where $\alpha \in \mathcal{O}$, if and only if there exists a base B for X such that for every $U \in A$ we have $\mathrm{b}^0\text{-p}_1\text{-ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha$.

Theorem 2.3. *For every p-base A for a subset Q in a space X we have*
(1) $\mathrm{ind}(Q) \leq \mathrm{p}_1\text{-ind}(Q,X) \leq \mathrm{b}_0\text{-p}_1\text{-ind}(Q,A,X)$.
(2) $\mathrm{ind}(Q) \leq \mathrm{p}_1\text{-ind}(Q,X) \leq \mathrm{b}^0\text{-p}_1\text{-ind}(Q,A,X)$.

Proof. (1) By Proposition 2.1 of [5] we have $\mathrm{ind}(Q) \leq \mathrm{p}_1\text{-}\mathrm{ind}(Q, X)$. By Proposition 2.2 of [5] we have $\mathrm{p}_1\text{-}\mathrm{ind}(Q, X) \leq \mathrm{ps}_1\text{-}\mathrm{ind}(Q, X)$. Also, by Theorem 2.1 we have $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \mathrm{ps}_1\text{-}\mathrm{ind}(Q, X)$. Therefore, $\mathrm{p}_1\text{-}\mathrm{ind}(Q, X) \leq \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X)$.

(2) We prove that

$$\mathrm{p}_1\text{-}\mathrm{ind}(Q, X) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X). \tag{4}$$

Let $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality (4) is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality (4) is true for every triad (Q^Y, A^Y, Y) with $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q^Y, A^Y, Y) < \alpha$.

Since $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha$, there exists a base B for X such that for every $U \in A$ we have $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha$. By inductive assumption, for every $U \in A$ we have

$$\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), X) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha.$$

Thus, $\mathrm{p}_1\text{-}\mathrm{ind}(Q, X) \leq \alpha$. □

Example 2.1. (1) Let $X = \{a, b, c\}$ with the topology

$$\tau = \{\emptyset, \{c\}, \{a, c\}, \{b, c\}, X\}$$

and $Q = \{a, b\}$. We consider the p-base $A = \{\{a, c\}, \{b, c\}\}$ for Q in X. Then, we have $\mathrm{ind}(Q) = 0$, $\mathrm{ind}(X) = 1$, $\mathrm{p}_1\text{-}\mathrm{ind}(Q, X) = 1$, $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \infty$, and $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \infty$.

(2) Let $\mathbb{R}$ be the space of the real numbers with the natural topology and $\mathbb{Q}$ the set of the rational numbers. It is known that $\mathrm{ind}(\mathbb{Q}) = 0$ and $\mathrm{ind}(\mathbb{R}) = 1$. We consider the bases

$$A_1 = \{(a, b) : a, b \in \mathbb{Q}\} \quad \text{and} \quad A_2 = \{(a, b) : a, b \in \mathbb{R} \setminus \mathbb{Q}\}$$

of the space $\mathbb{R}$. Then, we have $\mathrm{p}_1\text{-}\mathrm{ind}(\mathbb{Q}, \mathbb{R}) = 0$,

$$\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(\mathbb{Q}, A_1, \mathbb{R}) = \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(\mathbb{Q}, A_2, \mathbb{R}) = \mathrm{ps}_1\text{-}\mathrm{ind}(\mathbb{Q}, \mathbb{R}) = 0,$$

$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(\mathbb{Q}, A_1, \mathbb{R}) = 1$, and $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(\mathbb{Q}, A_2, \mathbb{R}) = 0$.

(3) Let $\mathbb{R}$ be the set of real numbers with the topology

$$\tau = \{[a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset\} \cup \{\mathbb{R}\}$$

and $Q = \{0, 1, 2\}$. Then, $\mathrm{ind}(Q) = 2$ and $\mathrm{ind}(\mathbb{R}) = \infty$. We consider the p-base

$$A = \{[0, +\infty), [1, +\infty), [2, +\infty)\}$$

of Q in $\mathbb{R}$. Then, we have $\mathrm{p}_1\text{-}\mathrm{ind}(Q, \mathbb{R}) = 2$,

$$\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, \mathbb{R}) = \mathrm{ps}_1\text{-}\mathrm{ind}(Q, \mathbb{R}) = \infty,$$

and $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, \mathbb{R}) = \infty$.

Remark 2.1. The relations between the dimension-like functions of the type ind are summarized in the following diagram, where "$\rightarrow$" means "$\leq$" and "$\nrightarrow$" means that "in general $\nleq$".

$$\begin{tikzcd}
\mathrm{ps}_1\text{-}\mathrm{ind}(Q, X) = \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) \arrow[r, shift right, "/" marking] & \mathrm{p}_1\text{-}\mathrm{ind}(Q, X) \arrow[l, shift right] \arrow[r, shift left] \arrow[d, shift left, "/" marking] & \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) \arrow[l, shift left, "/" marking] \\
 & \mathrm{ind}(Q) \arrow[u, shift left] &
\end{tikzcd}$$

Definition 2.3. Let A_1 be a p-base for a subset Q_1 in a space X_1 and A_2 a p-base for a subset Q_2 in a space X_2. The triads (Q_1, A_1, X_1) and (Q_2, A_2, X_2) are *homeomorphic* if there exists a homeomorphism $h : X_1 \to X_2$ such that $h(Q_1) = Q_2$ and $A_2 = \{h(U) : U \in A_1\}$.

Theorem 2.4. *If the triads (Q_1, A_1, X_1) and (Q_2, A_2, X_2) are homeomorphic, then* $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) = \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2, A_2, X_2)$.

Proof. It suffices to prove that if the triads (Q_1, A_1, X_1) and (Q_2, A_2, X_2) are homeomorphic, then

$$\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) \leq \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2, A_2, X_2).$$

Let $h : X_1 \to X_2$ be a homeomorphism such that $h(Q_1) = Q_2$ and $A_2 = \{h(U) : U \in A_1\}$ and let $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2, A_2, X_2) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality is true for every homeomorphic triads (Q^X, A^X, X) and (Q^Y, A^Y, Y) with $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q^Y, A^Y, Y) < \alpha$.

Since $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2, A_2, X_2) = \alpha$, there exists a base B_2 for X_2 such that for every $U \in B_2$ we have $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2 \cap \mathrm{Bd}_{X_2}(U), A_2, X_2) < \alpha$. We consider the base $B_1 = \{h^{-1}(U) : U \in B_2\}$ for X_1. We prove that for every $U \in B_2$ we have $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1 \cap \mathrm{Bd}_{X_1}(h^{-1}(U)), A_1, X_1) < \alpha$.

Let $U \in B_2$. Then, $h(\mathrm{Bd}_{X_1}(h^{-1}(U))) = h(h^{-1}(\mathrm{Bd}_{X_2}(U))) = \mathrm{Bd}_{X_2}(U)$. Thus, the triads

$$(Q_2 \cap \mathrm{Bd}_{X_2}(U), A_2, X_2)$$

and

$$(Q_1 \cap \mathrm{Bd}_{X_1}(h^{-1}(U)), A_1, X_1)$$

are homeomorphic. By inductive assumption we have
$\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1 \cap \mathrm{Bd}_{X_1}(h^{-1}(U)), A_1, X_1) \leq$
$\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2 \cap \mathrm{Bd}_{X_2}(U), A_2, X_2) < \alpha$. □

The following theorem is proved similarly to Theorem 2.4.

Theorem 2.5. *If the triads (Q_1, A_1, X_1) and (Q_2, A_2, X_2) are homeomorphic, then* $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) = \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_2, A_2, X_2)$.

3. Subspace theorems

Theorem 3.1. (The first subspace theorem) *Let A_1 and A_2 be two p-bases for a subset Q in a space X with $A_1 \subseteq A_2$. Then,*

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A_1, X) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A_2, X).$$

Proof. Let $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A_2, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality is true for every triad (Q^Y, A^Y, Y) with dimension $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q^Y, A^Y, Y) < \alpha$.

Since $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A_2, X) = \alpha$, there exists a base B for X such that for every $U \in A_2$ we have $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha$. Therefore, for every $U \in A_1$ we have $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha$ which means that $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A_1, X) \leq \alpha$. □

Theorem 3.2. *Let X be an Alexandroff space, $Q \subseteq X$, and $A = \{\mathbf{U}_x : x \in Q\}$. Then,* $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) \in \{-1, 0, \infty\}$.

Proof. If $Q = \emptyset$, then $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = -1$. Suppose that $Q \neq \emptyset$. Then, we have the following two cases:

(1) For every $x \in X$, $Q \cap \mathrm{Bd}_X(\mathbf{U}_x) = \emptyset$. In this case we consider the base $B = \{\mathbf{U}_x : x \in X\}$ of X. Then, for every $x \in X$ we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_x), B, X) = \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(\emptyset, B, X) = -1.$$

Therefore, $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = 0$.

(2) There exists $x_0 \in X$ such that $Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}) \neq \emptyset$. We prove that

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \infty.$$

Suppose that $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha \in \mathcal{O}$. Then, there exists a base B' for X such that for every $U \in A$ we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B', X) < \alpha.$$

Particularly, since $\mathbf{U}_{x_0} \in A$, we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), B', X) < \alpha.$$

Since $A \subseteq B'$, by Theorem 3.1, we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), A, X) = \beta < \alpha. \tag{5}$$

Hence, there exists a base B'' for X such that for every $U \in A$ we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}) \cap \mathrm{Bd}_X(U), B'', X) < \beta.$$

Since $\mathbf{U}_{x_0} \in A$, we have
$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}) \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), B'', X) =$
$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), B'', X) < \beta$.

Since $A \subseteq B''$, by Theorem 3.1, we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(\mathbf{U}_{x_0}), A, X) < \beta,$$

which contradicts relation (5). Thus, $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \infty$. □

Theorem 3.3. (The second subspace theorem) *Let A be a p-base for a subset Q in a space X and $Q_1 \subseteq Q$. Then,*
(1) $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A, X) \leq \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X)$.
(2) $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A, X) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X)$.

Proof. (2) We prove the inequality

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A, X) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X). \tag{6}$$

Let $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality (6) is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality (6) is true for every triad (Q^Y, A^Y, Y) with $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q^Y, A^Y, Y) < \alpha$.

Since $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha$, there exists a base B for X such that for every $U \in A$ we have $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha$. For the proof of the inequality (6) it suffices to show that for every $U \in A$ we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1 \cap \mathrm{Bd}_X(U), B, X) < \alpha.$$

Let $U \in A$. Since $Q_1 \cap \mathrm{Bd}_X(U) \subseteq Q \cap \mathrm{Bd}_X(U)$, by inductive assumption we have

$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1 \cap \mathrm{Bd}_X(U), B, X) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha.$$

Thus, $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A, X) \leq \alpha$.

The proof of (1) is similar. □

Theorem 3.4. (The third subspace theorem) *Let A be a p-base for a subset Q in a space X, X_1 a subspace of X, $Q_1 \subseteq X_1 \cap Q$, and $A_1 = \{X_1 \cap V : V \in A\}$. Then,*
(1) $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) \leq \mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X)$.
(2) $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X)$.

Proof. (2) We prove the inequality
$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X). \tag{7}$$
Let $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha \in \mathcal{O} \cup \{-1, \infty\}$. The inequality (7) is clear if $\alpha = -1$ or $\alpha = \infty$. We suppose that $\alpha \in \mathcal{O}$ and the inequality (7) is true for every triad (Q^Y, A^Y, Y) with $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q^Y, A^Y, Y) < \alpha$. Since $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) = \alpha$, there exists a base B for X such that for every $U \in A$ we have
$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha.$$
For the proof of the inequality (7) it suffices to show that for every $U_1 \in A_1$ we have
$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1 \cap \mathrm{Bd}_{X_1}(U_1), B_1, X_1) < \alpha,$$
where $B_1 = \{X_1 \cap V : V \in B\}$. Let $U_1 \in A_1$. Then, there exists $U \in A$ such that $U_1 = X_1 \cap U$. Since $Q_1 \cap \mathrm{Bd}_{X_1}(U_1) \subseteq Q \cap \mathrm{Bd}_X(U)$, by inductive assumption we have
$$\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1 \cap \mathrm{Bd}_{X_1}(X_1 \cap U), B_1, X_1) \leq \mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), B, X) < \alpha.$$
Thus, $\mathrm{b}^0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q_1, A_1, X_1) \leq \alpha$.
The proof of (1) is similar. □

4. Partition and sum theorems

Definition 4.1. (see [4]) Let A and B be two disjoint subsets of a space X. We say that *a subset L of X is a partition between A and B* if there exist two open subsets U and W of X such that (1) $A \subseteq U$, $B \subseteq W$, (2) $U \cap W = \emptyset$, and (3) $X \setminus L = U \cup W$.

Theorem 4.1. *Let A be a p-base for a subset Q in a regular space X. Then,* $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) \leq \alpha$, *where $\alpha \in \mathcal{O}$ if and only if for every point $x \in X$ and each closed set $F \subseteq X$ such that $x \in X \setminus F$ there exists a partition L between $\{x\}$ and F such that* $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap L, A, X) < \alpha$.

Proof. We suppose that $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) \leq \alpha$, where $\alpha \in \mathcal{O}$. Then, there exists a base B for X such that for every $U \in B$ we have
$$\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), A, X) < \alpha.$$
Let $x \in X$ and $F \subseteq X$ be a closed set such that $x \in X \setminus F$. Since the space X is regular, there exists an open neighbourhood V of the point x such that
$$x \in V \subseteq \mathrm{Cl}_X(V) \subseteq X \setminus F.$$
Therefore, there exists $U \in B$ such that $x \in U \subseteq V \subseteq \mathrm{Cl}_X(V) \subseteq X \setminus F$ and $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap \mathrm{Bd}_X(U), A, X) < \alpha$. Setting $L = \mathrm{Bd}_X(U)$ we have that L is the claimed partition between $\{x\}$ and F.

Conversely, we suppose that the space X satisfies the condition of the theorem. We prove that $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q, A, X) \leq \alpha$. Let $x \in X$ and V be an open neighbourhood of x. Then, there exists a partition L between $\{x\}$ and $X \setminus V$ such that $\mathrm{b}_0\text{-}\mathrm{p}_1\text{-}\mathrm{ind}(Q \cap L, A, X) < \alpha$. Suppose that U and W be open subsets of X such that $\{x\} \subseteq U$,

$X \setminus V \subseteq W$, $U \cap W = \emptyset$, and $X \setminus L = U \cup W$. Then, $\{x\} \subseteq U \subseteq X \setminus W \subseteq V$ and $\mathrm{Bd}_X(U) \subseteq L$. By Theorem 3.3(1) we have

$$\mathrm{b_0\text{-}p_1\text{-}ind}(Q \cap \mathrm{Bd}_X(U), A, X) \leq \mathrm{b_0\text{-}p_1\text{-}ind}(Q \cap L, A, X) < \alpha.$$

Thus, $\mathrm{b_0\text{-}p_1\text{-}ind}(Q, A, X) \leq \alpha$.

□

Corollary 4.1. *Let Q be a subset of a regular space X. Then,* $\mathrm{ps_1\text{-}ind}(Q, X) \leq \alpha$, *where $\alpha \in \mathcal{O}$ if and only if for every point $x \in X$ and each closed set $F \subseteq X$ such that $x \in X \setminus F$ there exists a partition L between $\{x\}$ and F such that* $\mathrm{ps_1\text{-}ind}(Q \cap L, X) < \alpha$.

Proof. Is straightforward verification of Theorem 4.1 and Proposition 4.2 of [5]. □

Theorem 4.2. *Let A be a p-base for a subset Q in a space X. If there exist two subsets Q_1 and Q_2 of Q such that $Q = Q_1 \cup Q_2$, then*

$$\mathrm{b_0\text{-}p_1\text{-}ind}(Q, A, X) \leq \mathrm{b_0\text{-}p_1\text{-}ind}(Q_1, A, X)(+)\mathrm{b_0\text{-}p_1\text{-}ind}(Q_2, A, X) + 1.$$

Proof. Is straightforward verification of Theorem 2.1 and Proposition 4.2 of [5]. □

5. Questions

(1) Let A be a p-base for a subset Q in a Tychonoff space X, where $\mathrm{ind}(Q) < \omega$. By Proposition 2.9 of [5] we have

$$\mathrm{p_1\text{-}ind}(Q, X) \leq \mathrm{ind}(Q) + 1.$$

Are true the following relations:

$$\mathrm{p_1\text{-}ind}(Q, X) \leq \mathrm{b_0\text{-}p_1\text{-}ind}(Q, A, X) \leq \mathrm{ind}(Q) + 1$$

and

$$\mathrm{p_1\text{-}ind}(Q, X) \leq \mathrm{b^0\text{-}p_1\text{-}ind}(Q, A, X) \leq \mathrm{ind}(Q) + 1?$$

(2) Find a space X, a subset Q of X, and a p-base A for Q in X such that $\mathrm{b^0\text{-}p_1\text{-}ind}(Q, A, X) < \mathrm{b_0\text{-}p_1\text{-}ind}(Q, A, X)$.

(3) Are true the Theorems 4.1 and 4.2 for the base positional dimension-like function $\mathrm{b^0\text{-}p_1\text{-}ind}$?

(4) Can be found product theorems for the dimension-like functions $\mathrm{b_0\text{-}p_1\text{-}ind}$ and $\mathrm{b^0\text{-}p_1\text{-}ind}$?

(5) Can be found compactifications theorems for the dimension-like functions $\mathrm{b_0\text{-}p_1\text{-}ind}$ and $\mathrm{b^0\text{-}p_1\text{-}ind}$?

(6) Let *df* be one of the following base positional dimension-like functions $\mathrm{b_0\text{-}p_1\text{-}ind}$ and $\mathrm{b^0\text{-}p_1\text{-}ind}$. For every space X we consider the class of ordinals

$$Sp_{df}(X) = \{df(Q, A, X) : Q \subseteq X \text{ and } A \text{ is a p-base for Q in X}\}.$$

(a) Find the class of all spaces X such that $Sp_{df}(X) = \{0, 1, 2, \ldots, n\}$, where $n \in \omega$.
(b) Find the class of all spaces X such that $Sp_{df}(X) = \{\infty\}$.
(c) Find the class of all spaces X such that $Sp_{df}(X) = \omega$.

(7) For every $\alpha \in \mathcal{O}$ find a space $X(\alpha)$, a subset $Q(\alpha)$ of $X(\alpha)$, and a p-base $A(\alpha)$ for $Q(\alpha)$ in $X(\alpha)$ such that $\mathrm{b_0\text{-}p_1\text{-}ind}(Q(\alpha), A(\alpha), X(\alpha)) = \alpha$.

(8) For every $\alpha \in \mathcal{O}$ find a space $X(\alpha)$, a subset $Q(\alpha)$ of $X(\alpha)$, and a p-base $A(\alpha)$ for $Q(\alpha)$ in $X(\alpha)$ such that $\mathrm{b^0\text{-}p_1\text{-}ind}(Q(\alpha), A(\alpha), X(\alpha)) = \alpha$.

(9) Let $n \in \omega$. Find a space X and a subset Q of X such that for every p-base A for Q in X, $\mathrm{b_0\text{-}p_1\text{-}ind}(Q, A, X) \in \{0, 1, 2, \ldots, n\}$.

(10) Let $n \in \omega$. Find a space X and a subset Q of X such that for every p-base A for Q in X, b^0-p_1-$\mathrm{ind}(Q, A, X) \in \{0, 1, 2, \ldots, n\}$.

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