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A CLASS OF ALMOST CONTACT METRIC MANIFOLDS AND DOUBLE-TWISTED PRODUCTS

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Abstract. We determine the Chinea-Gonzales class of almost contact metric manifolds locally realized as double-twisted product manifolds $I \times_{(\lambda_1,\lambda_2)} F$, I being an open interval, F an almost Hermitian manifold and λ_1, λ_2 smooth positive functions. Several subclasses are studied. We also give an explicit expression for the cosymplectic defect of any manifold in the considered class and derive several consequences in dimensions $2n + 1 \geq 5$. Explicit formulas for two algebraic curvature tensor fields are obtained. In particular cases, this allows to state interesting curvature relations.

1. Introduction

Twisted products play an interesting role in clarifying the interrelation between almost Hermitian (a.H.) and almost contact metric (a.c.m.) manifolds. In fact, as stated in [6], any a.c.m. manifold in the Chinea-Gonzales class $\mathcal{C}_{1-5} = \bigoplus_{1\leq i\leq 5} \mathcal{C}_i$ is, locally, a twisted product $]-\varepsilon, \varepsilon[\ \times_\lambda F$, $\varepsilon > 0$, F being an a.H. manifold and $\lambda : I \times F \to \mathbf{R}$ a smooth positive function.

On the other hand, in [12] Ponge and Reckziegel generalized the concept of twisted product introducing the notion of double-twisted product of two pseudo-Riemannian manifolds $(M_1, g_1), (M_2, g_2)$ by means of two positive functions $\lambda_1, \lambda_2 : M_1 \times M_2 \to \mathbf{R}$.
This is the pseudo-Riemannian manifold $M_1 \times_{(\lambda_1,\lambda_2)} M_2 = (M_1 \times M_2, \lambda_1^2 \pi_1^* g_1 + \lambda_2^2 \pi_2^* g_2)$, $\pi_i : M_1 \times M_2 \to M_i$, $i \in \{1, 2\}$, denoting the canonical projections. The same authors proved that any pseudo-Riemannian manifold that admits two complementary foliations L, K whose leaves are totally umbilic and intersect perpendicularly is, locally, isometric to a double-twisted product and L, K correspond to the canonical foliations of the product.

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In this article, given an open interval $I \subset \mathbf{R}$, an a.H. manifold $(F, \widehat{J}, \widehat{g})$ and two smooth positive functions $\lambda_1, \lambda_2 : I \times F \to \mathbf{R}$, on $I \times F$ one considers the double-twisted product metric g of the Euclidean metric on I and $\widehat{g}$ by λ_1, λ_2 and the a.c.m. structure (φ, ξ, η, g) naturally induced by $(\widehat{J}, \widehat{g})$ as in (2.1). The double-twisted product of I and F by (λ_1, λ_2) is the a.c.m. manifold $I \times_{(\lambda_1,\lambda_2)} F = (I \times F, \varphi, \xi, \eta, g)$. In particular, if $\lambda_1 = 1$, $I \times_{(1,\lambda_2)} F$ belongs to the class $\mathcal{C}_{1-5}$ since this manifold is the twisted product of I and F by λ_2. More generally, we prove that $I \times_{(\lambda_1,\lambda_2)} F$ falls in the Chinea-Gonzales class $\bigoplus_{1 \le i \le 5} \mathcal{C}_i \oplus \mathcal{C}_{12}$, briefly denoted by $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$. Combining an algebraic characterization of this class with the Ponge-Reckziegel theorem, one proves that any $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold is, locally, almost contact isometric with a double-twisted product $]-\varepsilon, \varepsilon[\times_{(\lambda_1,\lambda_2)} F$, $\varepsilon > 0$, where F is an a.H. manifold and λ_1, λ_2 are smooth positive functions.

Moreover, given a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold $(M, \varphi, \xi, \eta, g)$, we denote by $\mathcal{D}$ the umbilic foliation associated with $\ker \eta$. Obviously, any leaf N of $\mathcal{D}$ inherits from M the a.H. structure $(J' = \varphi_{|TN}, g' = g_{|TN \times TN})$. One proves that, for any $i \in \{1, 2, 3, 4\}$, M is in the class $\mathcal{C}_i \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$ if and only if each leaf of $\mathcal{D}$ is in the Gray-Hervella class $\mathcal{W}_i$.

Furthermore, one considers the minimal connection D and the Levi-Civita connection ∇ on a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold M ([9]). Since D preserves the a.c.m. structure, all the curvature operators $R^D(X, Y), X, Y \in \mathcal{X}(M)$, commute with φ. This allows to express the cosymplectic defect Λ, acting as $\Lambda(X, Y, Z, W) = R(X, Y, Z, W) - R(X, Y, \varphi Z, \varphi W)$, R being the Riemannian curvature, as a combination of $D\tau_h, \tau_h \otimes \tau_k, h, k \in \{1, 2, 3, 4, 5, 12\}$, where, for any h, τ_h denotes the $\mathcal{C}_h$-component of $\nabla\Phi$.

Several consequences of this result are obtained. For instance, one proves that, in dimensions $2n + 1 \ge 5$, any $\mathcal{C}_i \oplus \mathcal{C}_5$-manifold, $i \in \{1, 2, 3\}$, is locally realized as a warped product $I \times_\lambda F$, $\lambda : I \to \mathbf{R}$ being a smooth positive function and F a $\mathcal{W}_i$-manifold. This improves a result stated in [6].

Then, we study the behaviour of two algebraic curvature tensor fields naturally associated with a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold, that can be expressed in terms of the cosymplectic defect. This allows to derive suitable curvature properties for the manifolds in a particular subclass of $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$. For instance, one gets that the curvature of a $\mathcal{C}_1 \oplus \mathcal{C}_5$-manifold fulfills the k-nullity condition, k being a smooth function depending on the $\mathcal{C}_5$-component, and another identity that generalizes the $(G2)$-condition recently introduced in [11].

In this paper all manifolds are assumed to be connected.

2. Double-twisted product manifolds

Given an a.H. manifold $(F, \widehat{J}, \widehat{g})$, an open interval $I \subset \boldsymbol{R}$ and two smooth functions $\lambda_1, \lambda_2 : I \times F \to \boldsymbol{R}, \lambda_1, \lambda_2 > 0$, on $I \times F$ one considers the a.c.m. structure (φ, ξ, η, g) such that

$$\begin{gathered}\varphi(a\frac{\partial}{\partial t}, U) = (0, \widehat{J}U), \quad \eta(a\frac{\partial}{\partial t}, U) = a\lambda_1, \quad \xi = \frac{1}{\lambda_1}(\frac{\partial}{\partial t}, 0), \\ g = \lambda_1^2 \pi^*(dt \otimes dt) + \lambda_2^2 \sigma^*(\widehat{g}),\end{gathered} \tag{2.1}$$

for any $a \in \mathcal{F}(I \times F), U \in \mathcal{X}(F), \pi : I \times F \to I, \sigma : I \times F \to F$ denoting the canonical projections. Note that g is the double-twisted product metric of the

Euclidean metric g_0 and $\widehat{g}$. The a.c.m. manifold $I \times_{(\lambda_1,\lambda_2)} F = (I \times F, \varphi, \xi, \eta, g)$ is called the double-twisted product manifold of (I, g_0) and $(F, \widehat{J}, \widehat{g})$ by (λ_1, λ_2). If λ_1 is independent of the real coordinate t and λ_2 only depends on t, then $I \times_{(\lambda_1,\lambda_2)} F$ is named the double-warped product of (I, g_0) and $(F, \widehat{J}, \widehat{g})$ by (λ_1, λ_2). If $\lambda_1 = 1$, then $I \times_{\lambda_2} F = I \times_{(1,\lambda_2)} F$ is the twisted product manifold of (I, g_0) and $(F, \widehat{J}, \widehat{g})$ by λ_2. Finally, if λ_2 only depends on the coordinate t, $I \times_{\lambda_2} F$ is the warped product manifold of (I, g_0) and $(F, \widehat{J}, \widehat{g})$ by λ_2 ([6]).

Now, we recall some basic formulas on double-twisted product manifolds, a.c.m. and a.H. manifolds.

Through the paper, we'll identify any vector field U on F with $(0, U) \in \mathcal{X}(I \times F)$. The Levi-Civita connections ∇ of $I \times_{(\lambda_1,\lambda_2)} F$ and $\widehat{\nabla}$ of F are related by

$$(2.2)\quad \nabla_U V = \widehat{\nabla}_U V - g(U,V)\operatorname{grad}\log\lambda_2 + g(U, \operatorname{grad}\log\lambda_2)V + g(V, \operatorname{grad}\log\lambda_2)U,$$

for any $U, V \in \mathcal{X}(F)$, where grad is evaluated with respect to g ([12]).

The following relations are known, also:

$$(2.3)\qquad \begin{aligned} &\nabla_\xi \xi = \xi(\log\lambda_1)\xi - \operatorname{grad}\log\lambda_1, \quad \nabla_\xi U = U(\log\lambda_1)\xi + \xi(\log\lambda_2)U, \\ &\nabla_U \xi = \xi(\log\lambda_2)U, \end{aligned}$$

for any $U \in \mathcal{X}(F)$.

Given an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ with $\dim M = 2n+1$, fundamental form Φ, $\Phi(X,Y) = g(X, \varphi Y)$, and Levi-Civita connection ∇, for any $h \in \{1, ..., 12\}$ we denote by τ_h the projection of $\nabla\Phi$ on the vector bundle $\mathcal{C}_h(M)$ whose fibre at any $x \in M$ is the linear space $\mathcal{C}_h(T_x M)$ considered in [4]. Putting $\mathcal{C}(M) = \bigoplus_{1\le h\le 12} \mathcal{C}_h(M)$, with any section α of $\mathcal{C}(M)$ are associated the 1-forms $c(\alpha), \overline{c}(\alpha)$ expressed, in a local orthonormal frame, by:

$$c(\alpha)(X) = \sum_{1\le i\le 2n+1} \alpha(e_i, e_i, X), \quad \overline{c}(\alpha)(X) = \sum_{1\le i\le 2n+1} \alpha(e_i, \varphi e_i, X).$$

In particular, one has $\overline{c}(\tau_5)(\xi) = \delta\eta$. The 1-form $\nabla_\xi\eta$ only depends on the projection τ_{12}, since one has $(\nabla_\xi\eta)X = \tau_{12}(\xi, \xi, \varphi X)$. The Lee form ω, defined by $\omega = -\frac{1}{2(n-1)}(\delta\Phi\circ\varphi + \nabla_\xi\eta) + \frac{\delta\eta}{2n}\eta$, if $n \ge 2$, $\omega = \nabla_\xi\eta + \frac{\delta\eta}{2}\eta$, if $n = 1$, depends on the projections $\tau_4, \tau_5, \tau_{12}$ according to the relations

$$\begin{aligned} \omega(X) &= \frac{1}{2(n-1)} c(\tau_4)(\varphi X) + \frac{\overline{c}(\tau_5)(\xi)}{2n}\eta(X), n \ge 2, \\ \omega(X) &= \tau_{12}(\xi, \xi, \varphi X) + \frac{\overline{c}(\tau_5)(\xi)}{2}\eta(X), n = 1. \end{aligned}$$

Let (N, J', g') be an a.H. manifold with Levi-Civita connection ∇' and fundamental form Ω', $\Omega'(X,Y) = g'(X, J'Y)$. For any $h \in \{1,2,3,4\}$ let τ'_h be the component of $\nabla'\Omega'$ on the vector bundle $\mathcal{W}_h(N)$ whose fibre at any point $p \in N$ is the linear space $\mathcal{W}_h(T_pN)$ introduced in [10]. If $\dim N = 2m \ge 4$, the Lee form of N is the 1-form $\omega' = -\frac{1}{2(m-1)}\delta'\Omega' \circ J'$ and is expressed, in a local orthonormal frame, by $\omega'(X) = \frac{1}{2(m-1)} \sum_{1\le i\le 2m} \tau'_4(E_i, E_i, J'X)$.

The next results are useful in determining the Chinea-Gonzales class of $I \times_{(\lambda_1,\lambda_2)} F$, $(F, \widehat{J}, \widehat{g})$ being an a.H. manifold, and in relating the covariant derivatives, with respect to the Levi-Civita connections, $\widehat{\nabla}\widehat{\Omega}$, $\nabla\Phi$, where $\widehat{\Omega}, \Phi$ denote the fundamental forms of F, $I \times_{(\lambda_1,\lambda_2)} F$.

Lemma 2.1. *Let $(F, \widehat{J}, \widehat{g})$ be a $2n$-dimensional a.H. manifold, $I \subset \boldsymbol{R}$ an open interval and $\lambda_1, \lambda_2 : I \times F \to \boldsymbol{R}$ smooth positive functions. For the manifold $I \times_{(\lambda_1,\lambda_2)} F$ the following relations hold:*

- **i):** $\nabla_X \xi = -\xi(\log \lambda_2)\varphi^2 X + \eta(X)\nabla_\xi \xi, \quad X \in \mathcal{X}(I \times F)$,
- **ii):** $(\nabla_\xi \varphi)X = \varphi X(\log \lambda_1)\xi - \eta(X)\varphi(\nabla_\xi \xi), \quad X \in \mathcal{X}(I \times F)$,
- **iii):** $\delta\eta = -2n\xi(\log \lambda_2)$,
- **iv):** $\omega = \sigma^*(\widehat{\omega}) - d(\log \lambda_2)$, *if* $n \geq 2$, $\omega = -d(\log \lambda_1) + \xi(\log \frac{\lambda_1}{\lambda_2})\eta$, *if* $n = 1$, $\widehat{\omega}, \omega$ *denoting the Lee forms of* $F, I \times_{(\lambda_1,\lambda_2)} F$.

Proof. Formula (2.3) implies **i), ii), iii)**. If $n = 1$, (2.3) implies **iv)**, also. Moreover, by (2.2), for any vector fields U,V on F, one has:

$$(\nabla_U \varphi)V = (\widehat{\nabla}_U \widehat{J})V + \varphi V(\log \lambda_2)U - V(\log \lambda_2)\varphi U - g(U, \varphi V)\mathrm{grad} \log \lambda_2 + g(U, V)\varphi(\mathrm{grad} \log \lambda_2). \tag{2.4}$$

Let $\{U_i\}_{1 \leq i \leq 2n}$ be a local $\widehat{g}$-orthonormal frame on F, put $e_i = \frac{1}{\lambda_2}U_i$, $i \in \{1, ..., 2n\}$, and consider the g-adapted orthonormal frame $\{e_1, ..., e_{2n}, \xi\}$ on $I \times_{(\lambda_1,\lambda_2)} F$. Then, one gets

$$\begin{aligned}\delta\Phi(U) &= \frac{1}{\lambda_2^2} \sum_{1 \leq i \leq 2n} g((\nabla_{U_i}\varphi)U_i, U) + g(\nabla_\xi \xi, \varphi U) \\ &= \widehat{\delta\Omega}(U) - 2(n-1)\varphi U(\log \lambda_2) - \varphi U(\log \lambda_1).\end{aligned}$$

So, if $n \geq 2$, one has $\omega(U) = \widehat{\omega}(U) - U(\log \lambda_2)$. Since $\omega(\xi) = -\xi(\log \lambda_2)$, **iv)** follows. □

Proposition 2.1. *In the same hypothesis of Lemma 2.1, for any $i \in \{1, 2, 3\}$, the $\mathcal{C}_i$-component of $\nabla\Phi$ vanishes if and only if the $\mathcal{W}_i$-component of $\widehat{\nabla}\widehat{\Omega}$ vanishes. If $n \geq 2$, the $\mathcal{C}_4$-component of $\nabla\Phi$ vanishes if and only if $\sigma^*(\widehat{\omega}) = d(\log \lambda_2) - \xi(\log \lambda_2)\eta$.*

Proof. If $\dim F = 2$, for any $i \in \{1, 2, 3, 4\}$ the $\mathcal{C}_i$-componentb of $\nabla\Phi$, as well as the $\mathcal{W}_i$-component of $\widehat{\nabla}\widehat{\Omega}$ vanish. So, we assume $\dim F = 2n \geq 4$ and consider $U, V, W \in \mathcal{X}(F)$. Applying the theory developed in [4, 10] and Lemma 2.1, one has

$$\tau_4(U, V, W) = \lambda_2^2 \widehat{\tau}_4(U, V, W) + \varphi W(\log \lambda_2)g(U, V) - \varphi V(\log \lambda_2)g(U, W) + W(\log \lambda_2)g(U, \varphi V) - V(\log \lambda_2)g(U, \varphi W), \tag{2.5}$$

$$\tau_i(U, V, W) = 0, \quad i = 5, ...12. \tag{2.6}$$

By (2.4) one obtains

$$\begin{aligned}(\nabla_U \Phi)(V, W) &= \lambda_2^2(\widehat{\nabla}_U \widehat{\Omega})(V, W) - \varphi V(\log \lambda_2)g(U, W) - V(\log \lambda_2)g(U, \varphi W) \\ &\quad + W(\log \lambda_2)g(U, \varphi V) + \varphi W(\log \lambda_2)g(U, V).\end{aligned}$$

It follows that $\sum_{1\le i\le 3}\tau_i(U,V,W) = \lambda_2^2\sum_{1\le i\le 3}\widehat{\tau}_i(U,V,W)$, and then $\tau_i(U,V,W) = \lambda_2^2\widehat{\tau}_i(U,V,W), i\in\{1,2,3\}$. On the other hand, for any $i\in\{1,2,3,4\}$ and X,Y tangent to $I\times F$, one has $\tau_i(\xi,X,Y)=\tau_i(X,Y,\xi)=0$. So, if $i\in\{1,2,3\}$, we have $\tau_i=0$ if and only if $\widehat{\tau}_i=0$. By (2.5) one gets $\tau_4=0$ if and only if $\widehat{\omega}(U)=U(\log\lambda_2)$, $U\in\mathcal{X}(F)$, if and only if $\sigma^*(\widehat{\omega})=d(\log\lambda_2)-\xi(\log\lambda_2)\eta$. □

The next results provide an algebraic characterization of the class $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$ and have a useful application involving double-twisted product manifolds.

Proposition 2.2. *Given an a.c.m. manifold (M,φ,ξ,η,g) with $\dim M=2n+1$, the following conditions are equivalent*

i): *M is a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold,*

ii): $\nabla\eta=-\frac{\delta\eta}{2n}(g-\eta\otimes\eta)+\eta\otimes\nabla_\xi\eta, \nabla_\xi\varphi=-\eta\otimes\varphi(\nabla_\xi\xi)-(\nabla_\xi\eta)\circ\varphi\otimes\xi.$

Proof. In the hypothesis **i)** one puts $\nabla\Phi=\sum_{1\le i\le 5}\tau_i+\tau_{12}$ and applies the theory developed in [4] to evaluate the contribution of each component τ_i in the calculus of $\nabla\eta$, $\nabla_\xi\varphi$. For any X,Y tangent to M, one has:

$$\begin{aligned}\tau_i(\xi,X,,Y) &= 0, i\in\{1,...,5\}\,,\ \tau_i(X,\xi,Y)=0, i\in\{1,2,3,4\}\,,\\ \tau_{12}(\xi,X,Y) &= \eta(X)\tau_{12}(\xi,\xi,Y)-\eta(Y)\tau_{12}(\xi,\xi,X),\\ \tau_5(X,\xi,Y) &= \frac{\overline{c}(\tau_5)(\xi)}{2n}g(X,\varphi Y), \tau_{12}(X,\xi,Y)=\eta(X)\tau_{12}(\xi,\xi,Y).\end{aligned}$$

Then, one obtains

$$g((\nabla_\xi\varphi)X,Y)=-\tau_{12}(\xi,X,Y)=-\eta(X)g(\varphi(\nabla_\xi\xi),Y)-(\nabla_\xi\eta)\varphi X\eta(Y),$$

$$(\nabla_X\eta)Y=(\tau_5+\tau_{12})(X,\xi,\varphi Y)=-\frac{\delta\eta}{2n}(g(X,Y)-\eta(X)\eta(Y))+\eta(X)(\nabla_\xi\eta)Y.$$

Then, **ii)** holds.

Vice versa, we assume **ii)** and write $\nabla\Phi=\sum_{1\le i\le 12}\tau_i$. Then, with respect to a local orthonormal frame $\{e_1,...,e_{2n},\xi\}$ we have

$$c(\tau_6)(\xi)=\sum_{1\le h\le 2n}(\nabla_{e_h}\Phi)(e_h,\xi)=-\sum_{1\le h\le 2n}(\nabla_{e_h}\eta)\varphi e_h=0.$$

Therefore, τ_6 vanishes. Considering X,Y tangent to M, since $\tau_i(\xi,\varphi X,Y)=0$, $i\in\{1,...,10\}$, one has

$$\begin{aligned}(\tau_{11}+\tau_{12})(\xi,\varphi X,Y) &= (\nabla_\xi\Phi)(\varphi X,Y)=-g((\nabla_\xi\varphi)\varphi X,Y)\\ &= -\eta(Y)\tau_{12}(\xi,\xi,\varphi X)=\tau_{12}(\xi,\varphi X,Y).\end{aligned}$$

It follows that $\tau_{11}=0$. Finally, the condition on $\nabla\eta$ entails $\sum_{7\le i\le 10}\tau_i(X,\xi,\varphi Y)=0$. Then, it is easy to verify that all the components $\tau_i, i\in\{7,8,9,10\}$ vanish. It follows that $\nabla\Phi=\sum_{1\le i\le 5}\tau_i+\tau_{12}$ and **i)** holds. □

Corollary 2.1. *For a $2n+1$-dimensional a.c.m. manifold (M,φ,ξ,η,g) in the class $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$ the following equations hold:*

$$d\eta=\eta\wedge\nabla_\xi\eta,\quad d(\nabla_\xi\eta)=(\frac{\delta\eta}{2n}\nabla_\xi\eta-\nabla_\xi(\nabla_\xi\eta))\wedge\eta.$$

Proof. Applying Proposition 2.2, we see that the skew-symmetric part of $\nabla\eta$ is $\eta\wedge\nabla_\xi\eta$, so we get $d\eta=\eta\wedge\nabla_\xi\eta$. Differentiating, one obtains $\eta\wedge d(\nabla_\xi\eta)=0$. Considering $X,Y\in\mathcal{X}(M)$, one has

$$\begin{aligned}2d(\nabla_\xi\eta)(X,Y) &= -\eta(X)(\nabla_Y(\nabla_\xi\eta)(\xi)-\nabla_\xi(\nabla_\xi\eta)(Y))\\ &\quad +\eta(Y)(\nabla_X(\nabla_\xi\eta)(\xi)-\nabla_\xi(\nabla_\xi\eta)(X)).\end{aligned}$$

Moreover, also applying Proposition 2.2, one has

$$\nabla_X(\nabla_\xi\eta)(\xi)=-g(\nabla_\xi\xi,\nabla_X\xi)=\frac{\delta\eta}{2n}(\nabla_\xi\eta)X-\eta(X)g(\nabla_\xi\xi,\nabla_\xi\xi).$$

Then, substituting in the previous formula, one gets the second equation in the statement. □

We remark that, if M is a 5-dimensional a.c.m. manifold, the vector bundles $\mathcal{C}_1(M)$ and $\mathcal{C}_3(M)$ are trivial. So, in dimension 5, by Proposition 2.2 one characterizes the class $\mathcal{C}_2\oplus\mathcal{C}_4\oplus\mathcal{C}_5\oplus\mathcal{C}_{12}$. In dimension 3, the total class is $\mathcal{C}_5\oplus\mathcal{C}_6\oplus\mathcal{C}_9\oplus\mathcal{C}_{12}$ and the class $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$ reduces to $\mathcal{C}_5\oplus\mathcal{C}_{12}$. In this dimension, using the same technique as in Proposition 2.2, one easily obtains the next result.

Proposition 2.3. *Let (M,φ,ξ,η,g) be an a.c.m. manifold with $\dim M=3$. The following conditions are equivalent:*

- **i):** *M is a $\mathcal{C}_5\oplus\mathcal{C}_{12}$-manifold,*
- **ii):** $(\nabla_X\varphi)Y=\frac{\delta\eta}{2}(\eta(Y)\varphi X+g(X,\varphi Y)\xi)-\eta(X)(\eta(Y)\varphi(\nabla_\xi\xi)+(\nabla_\xi\eta)\varphi Y\xi)$,
- **iii):** $\nabla\eta=-\frac{\delta\eta}{2}(g-\eta\otimes\eta)+\eta\otimes\nabla_\xi\eta$.

Propositions 2.2, 2.3 allow to specify the class of double-twisted product manifolds.

In fact, let $(F,\widehat{J},\widehat{g})$ be an a.H. manifold, $I\subset\mathbf{R}$ an open interval and $\lambda_1,\lambda_2:I\times F\to\mathbf{R}$ smooth positive functions. By Lemma 2.1, (2.3) and Propositions 2.2, 2.3, it follows that $I\times_{(\lambda_1,\lambda_2)}F$ belongs to the class $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$ if $n\geq 3$, to $\mathcal{C}_2\oplus\mathcal{C}_4\oplus\mathcal{C}_5\oplus\mathcal{C}_{12}$ if $n\geq 2$, to $\mathcal{C}_5\oplus\mathcal{C}_{12}$ if $n=1$. Also applying Proposition 2.1, under suitable restrictions on the class of $(F,\widehat{J},\widehat{g})$, and on the functions λ_1,λ_2, one obtains that $I\times_{(\lambda_1,\lambda_2)}F$ belongs to a particular subclass of $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$. For instance, if $(F,\widehat{J},\widehat{g})$ is Kähler and $n\geq 2$, then $I\times_{(\lambda_1,\lambda_2)}F$ belongs to $\mathcal{C}_4\oplus\mathcal{C}_5\oplus\mathcal{C}_{12}$, to $\mathcal{C}_5\oplus\mathcal{C}_{12}$ under the additional hypothesis that λ_2 is constant on F. Analogously, if $\lambda_2=1$ and $(F,\widehat{J},\widehat{g})$ is a $\mathcal{W}_i$-manifold, $i\in\{1,2,3,4\}$, then $I\times_{(\lambda_1,1)}F$ is in the class $\mathcal{C}_i\oplus\mathcal{C}_{12}$. Finally, we assume that λ_1 is constant on F. By (2.3) one has $\nabla_\xi\xi=0$ and $I\times_{(\lambda_1,\lambda_2)}F$ belongs to $\mathcal{C}_{1-5}$. In fact, up to a reparametrization of the real coordinate, one writes $g=\pi^*(ds\otimes ds)+\lambda_2^2\sigma^*(\widehat{g})$ and obtains a twisted product a.c.m. structure on $I\times F$.

3. Local description of $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifolds

We are going to describe, locally, the $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifolds and characterize the ones belonging to the classes $\mathcal{C}_5\oplus\mathcal{C}_{12}$, $\mathcal{C}_i\oplus\mathcal{C}_5\oplus\mathcal{C}_{12}$, $i\in\{1,2,3,4\}$. In the sequel, given an a.c.m. manifold (M,φ,ξ,η,g), we'll denote by $\mathcal{D}$, $\mathcal{D}^\perp$ the mutually orthogonal distributions associated to the subbundles of TM $\ker\eta$ and $L(\xi)$. Note that $\mathcal{D}^\perp$ is a totally umbilic foliation with $\nabla_\xi\xi$ as mean curvature vector field. In partricular, $\mathcal{D}^\perp$ is totally geodesic if and only if $\nabla_\xi\eta=0$.

Proposition 3.1. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold. Then, the distribution $\mathcal{D}$ is a totally umbilic foliation and $\mathcal{D}$ is spherical if and only if*

$$d(\overline{c}(\tau_5)(\xi)) = \xi(\overline{c}(\tau_5)(\xi))\eta.$$

Moreover, $\mathcal{D}^\perp$ is spherical if and only if

$$\nabla_\xi(\nabla_\xi\eta) = -\parallel \nabla_\xi\xi \parallel^2 \eta.$$

Proof. Since $d\eta = \eta\wedge\nabla_\xi\eta$, $\mathcal{D}$ is integrable and for any $X\in\Gamma(\mathcal{D})$, one has $\nabla_X\xi = -\frac{\overline{c}(\tau_5)(\xi)}{2n}X$. it follows that any leaf (N,g') of $\mathcal{D}$, g' being the metric induced by g, is a totally umbilic submanifold of M with mean curvature vector field $H = \frac{\overline{c}(\tau_5)(\xi)}{2n}\xi_{|N}$. Moreover, (N,g') is an extrinsic sphere if and only if $0 = \nabla^\perp_X H = \frac{1}{2n}X(\overline{c}(\tau_5)(\xi))\xi$, for any $X\in\mathcal{X}(N)$. Hence, $\mathcal{D}$ is spherical if and only if

$$d(\overline{c}(\tau_5)(\xi)) = \xi(\overline{c}(\tau_5)(\xi))\eta.$$

Finally, $\mathcal{D}^\perp$ is spherical if and only if for any $X\in\Gamma(\mathcal{D})$ one has $\nabla_\xi(\nabla_\xi\eta)(X) = g(\nabla_\xi(\nabla_\xi\xi),X) = 0$. Equivalently, $\mathcal{D}^\perp$ is spherical if and only if

$$\nabla_\xi(\nabla_\xi\eta) = g(\nabla_\xi(\nabla_\xi\xi),\xi)\eta = -\parallel \nabla_\xi\xi \parallel^2 \eta.$$

□

An isometry $f : (M,\varphi,\xi,\eta,g)\to(M',\varphi',\xi',\eta',g')$ between a.c.m. manifolds is called an almost contact (a.c.) isometry if $f_*\circ\varphi = \varphi'\circ f_*$, $f_*\xi = \xi'$.

Theorem 3.1. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold. Then M is, locally, a.c. isometric to a double-twisted product manifold $]-\varepsilon,\varepsilon[\times_{(\lambda_1,\lambda_2)} F$, $\varepsilon>0$, F being an a.H. manifold and $\lambda_1,\lambda_2 :]-\varepsilon,\varepsilon[\times F\to\mathbf{R}$ smooth positive functions. Moreover, M is, locally,*

i): *a double-warped product if and only if*

$$d(\overline{c}(\tau_5)(\xi)) = \xi(\overline{c}(\tau_5)(\xi))\eta.$$
$$\nabla_\xi(\nabla_\xi\eta) = -\parallel \nabla_\xi\xi \parallel^2 \eta,$$

ii): *a twisted product if and only if $\nabla_\xi\eta = 0$.*

Proof. By Proposition 3.1, $\mathcal{D}$ and $\mathcal{D}^\perp$ are complementary foliations whose leaves are totally umbilic and intersect perpendicularly. So, applying the theory developed in [12], given a point $p\in M$, there exist a connected, open neighborhood U of p, a Riemannian manifold $(F,\widehat{g})$, two smooth positive functions $\lambda_1,\lambda_2 : I\times F\to\mathbf{R}$ and an isometry $f :]-\varepsilon,\varepsilon[\times_{(\lambda_1,\lambda_2)} F\to U$ such that the canonical foliations of the product manifold correspond, via f, to $\mathcal{D},\mathcal{D}^\perp$.
It follows that $f^*(g_{|U}) = \lambda_1^2 dt\otimes dt + \lambda_2^2\widehat{g}$, $f_*(\frac{\partial}{\partial t})$ is an integral manifold of $\mathcal{D}^\perp$ and, for any $t\in]-\varepsilon,\varepsilon[$, $f_t(F)$, where $f_t = f(t,\cdot)$, is an integral manifold of $\mathcal{D}$. Since $g(f_*(\frac{\partial}{\partial t}),f_*(\frac{\partial}{\partial t})) = \lambda_1^2$, we can assume that $f_*(\frac{1}{\lambda_1}\frac{\partial}{\partial t}) = \xi_{|U}$. Then, $f^*(\eta_{|U}) = \lambda_1\pi^*(dt)$, $\pi :]-\varepsilon,\varepsilon[\times F\to]-\varepsilon,\varepsilon[$ being the canonical projection, the triplet $(\widehat{\varphi} = f_*^{-1}\circ\varphi_{|U}\circ f_*, \frac{1}{\lambda_1}(\frac{\partial}{\partial t},0), \lambda_1\pi^*(dt))$ is an a.c. structure and $f_*(g_{|U})$ is a compatible metric.

Moreover $(\widehat{J} = \widehat{\varphi}_{|TF},\widehat{g})$ is an a.H. structure on F and $f :]-\varepsilon,\varepsilon[\times_{(\lambda_1,\lambda_2)} F\to(U,\varphi_{|U},\xi_{|U},\eta_{|U},g_{|U})$ is an a.c. isometry.
So, by Proposition 3 in [12], M is, locally, a double-warped product if and only if both the distributions $\mathcal{D}$, $\mathcal{D}^\perp$ are spherical. Then **i)** follows by Proposition 3.1.
Finally, we assume that the function λ_1 is constant, for each of the just considered

isometries $f :]-\varepsilon, \varepsilon[\times_{(\lambda_1,\lambda_2)} F \to U$. Putting $\delta = \lambda_1\varepsilon$, one considers the map $\overline{f} :]-\delta, \delta[\times F \to U$ such that $\overline{f}(s,x) = f(\frac{s}{\lambda_1}, x)$. Then, one has $\overline{f}^*(g_{|U}) = ds \otimes ds + \lambda_2^2 \widehat{g}, \overline{f}_*(\frac{\partial}{\partial s}) = \xi_{|U}$ and for each $s \in]-\delta, \delta[$ $\overline{f}_s(F)$ is an integral manifold of $\mathcal{D}$. It follows that $\overline{f}$ realizes an a.c. isometry between the twisted product $]-\delta, \delta[\times_{\lambda_2} F$ and $(U, \varphi_{|U}, \xi_{|U}, \eta_{|U}, g_{|U})$. This case occurs if and only if $\mathcal{D}^\perp$ is totally geodesic, namely if and only if $\nabla_\xi \eta = 0$. Hence, we obtain **ii)**. □

Since a $\mathcal{C}_{1-5}$-manifold is an a.c.m. manifold in the class $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$ such that $\nabla_\xi \eta = 0$, Theorem 3.1 implies that any $\mathcal{C}_{1-5}$-manifold is, locally, a.c. isometric to a twisted product manifold $]-\varepsilon, \varepsilon[\times_\lambda F$, F being an a.H. manifold and $\lambda : I \times F \to \boldsymbol{R}$ a smooth positive function. This agrees with Theorem 3.1 in [6].

As pointed out in Section 2, any 3-dimensional manifold M in $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$ is a $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold. Theorem 3.1 entails that M is locally realized as a double-twisted product manifold $]-\varepsilon, \varepsilon[\times_{(\lambda_1,\lambda_2)} F$, F being a 2-dimensional a.H., hence Kähler, manifold. Analogously, any leaf of $\mathcal{D}$ inherits from M a Kähler structure.

More generally, given $i \in \{1,2,3,4\}$, we say that a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold is foliated by $\mathcal{W}_i$-leaves if any leaf $(N, J' = \varphi_{|TN}, g' = g_{|TN \times TN})$ of $\mathcal{D}$ is in the Gray-Hervella class $\mathcal{W}_i$. We are going to characterize, in dimensions $2n+1 \geq 5$, the $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifolds that are foliated by $\mathcal{W}_i$-leaves. To this aim, for any $i \in \{1,2,3,4\}$, we list the defining condition of the manifolds in $\mathcal{C}_i \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$. These characterizations are obtained combining the theory developed in [4] with the technique used in the proof of Proposition 2.2.

$\mathcal{C}_1 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$:

$$(\nabla_X \varphi)X = \frac{\delta\eta}{2n}\eta(X)\varphi X - \eta(X)((\nabla_\xi \eta)(\varphi X)\xi + \eta(X)\varphi(\nabla_\xi \xi)),$$

$$\nabla \eta = -\frac{\delta\eta}{2n}(g - \eta \otimes \eta) + \eta \otimes \nabla_\xi \eta.$$

$\mathcal{C}_2 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$:

$$d\Phi = -\frac{\delta\eta}{n}\eta \wedge \Phi, \nabla\eta = -\frac{\delta\eta}{2n}(g - \eta \otimes \eta) + \eta \otimes \nabla_\xi \eta.$$

$\mathcal{C}_3 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$:

$$(\nabla_X \varphi)Y = (\nabla_{\varphi X}\varphi)\varphi Y + \frac{\delta\eta}{2n}\eta(Y)\varphi X - \eta(X)((\nabla_\xi \eta)(\varphi Y)\xi + \eta(Y)\varphi(\nabla_\xi \xi)),$$

$$\delta\Phi \circ \varphi = -\nabla_\xi \eta.$$

$\mathcal{C}_4 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$:

$$\begin{aligned}(\nabla_X \varphi)Y &= \omega(Y)\varphi X + \omega(\varphi Y)\varphi^2 X + g(X, \varphi Y)B - g(\varphi X, \varphi Y)\varphi B \\ &\quad -\eta(X)((\nabla_\xi \eta)(\varphi Y)\xi + \eta(Y)\varphi(\nabla_\xi \xi)),\ B = \omega^\sharp.\end{aligned}$$

Theorem 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold with $\dim M = 2n+1 \geq 5$. For any $i \in \{1,2,3,4\}$ the following conditions are equivalent:*

i): *M is foliated by $\mathcal{W}_i$-leaves,*

ii): *M is a $\mathcal{C}_i \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold.*

Proof. Let (N, J', g') be a leaf of $\mathcal{D}$. Since (N, g') is a totally umbilical submanifold of M with mean curvature vector field $\frac{\delta\eta}{2n}\xi_{|N}$, the covariant derivative $\nabla' J'$,

∇' denoting the Levi-Civita connection of N, satisfies

$$(\nabla_X\varphi)Y = (\nabla'_X J')Y + \frac{\delta\eta}{2n} g'(X, J'Y)\xi, \quad X, Y \in TN. \tag{3.1}$$

So, given two vector fields X, Y on M such that $\varphi^2 X, \varphi^2 Y$ are tangent to N, one writes $X = -\varphi^2 X + \eta(X)\xi$, $Y = -\varphi^2 Y + \eta(Y)\xi$, applies polarization, (3.1) and Proposition 2.2, then obtaining

$$\begin{aligned}(\nabla_X\varphi)Y = (\nabla'_{\varphi^2 X} J')\varphi^2 Y + \frac{\delta\eta}{2n}(g(X, \varphi Y)\xi + \eta(Y)\varphi X)\\ - \eta(X)((\nabla_\xi\eta)(\varphi Y)\xi + \eta(Y)\varphi(\nabla_\xi\xi)).\end{aligned} \tag{3.2}$$

Then, in each case, the equivalence **i)$\Longleftrightarrow$ii)** is proved by direct calculus, applying (3.1), (3.2) and the defining condition of $\mathcal{W}_i$-manifold ([10]). □

Corollary 3.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold. Then M is foliated by Kähler leaves if and only if M is in the class* $\mathrm{C}_5 \oplus \mathrm{C}_{12}$.

Now, we examine another consequence of Proposition 2.2 and (3.1).

With any a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ are associated the $(1, 2)$-tensor field τ and the connection D acting as

$$\begin{aligned}\tau(X, Y) &= -\frac{1}{2}\varphi((\nabla_X\varphi)Y) + (\nabla_X\eta)Y\xi - \frac{1}{2}\eta(Y)\nabla_X\xi\\ &= \frac{1}{2}((\nabla_X\varphi)\varphi Y + (\nabla_X\eta)Y\xi) - \eta(Y)\nabla_X\xi,\end{aligned} \tag{3.3}$$

$$D_X Y = \nabla_X Y + \tau(X, Y), \tag{3.4}$$

for any $X, Y \in \mathcal{X}(M)$.

Following [9], D is called the minimal $U(n)$-connection of M. Note that D is metric and preserves both φ and η, so it is a $U(n)$-connection. Obviously, the tensor field τ and then the torsion Σ of D, $\Sigma(X, Y) = \tau(X, Y) - \tau(Y, X)$, can be explicitely expressed by means of the $\mathcal{C}_h(M)$-components of $\nabla\Phi$. Moreover, by direct calculus, Proposition 2.2 and (3.1), one proves the following result.

Proposition 3.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold and (N, J', g') a leaf of D. For any vector fields X, Y on N, one has: $D_X Y = \nabla'_X Y - \frac{1}{2}J'((\nabla'_X J')Y)$.*

Proposition 3.2 means that, starting by a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold, the minimal connection induces a unitary connection on each leaf of D.

In fact, given an a.H. manifold (N, J', g') with Levi-Civita connection ∇', one considers the unitary connection D' acting as $D'_X Y = \nabla'_X Y - \frac{1}{2}J'((\nabla'_X J')Y)$. The connection D' plays a useful role in explaining several results on a.H. manifolds that are strictly related with the Gray-Hervella work and with the study of the curvature formulated by Tricerri and Vanhecke ([8],[13]). In particular, suitable components of the Riemann curvature tensor introduced in [13] have been explicitely expressed by means of the tensor fields $D'\tau'_i$, $\tau'_i \odot \tau'_j$, $i, j \in \{1, 2, 3, 4\}$, $\odot$ denoting the symmetric product ([7]).

This motivates the subject of Sections $4, 5$, where the cosymplectic defect and suitable related tensor fields associated with a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold are expressed as a combination of $D\tau_i$, $\tau_i \otimes \tau_j$, $i, j \in \{1, 2, 3, 4, 5, 12\}$.

4. The cosymplectic defect

Given an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$ with minimal connection D, one considers the $(0,3)$-tensor field τ defined by

$$\tau(X,Y,Z) = g(D_XY - \nabla_XY, Z) = -\frac{1}{2}(\nabla_X\Phi)(\varphi Y, Z) + \frac{1}{2}\eta(Z)(\nabla_X\eta)Y - \eta(Y)(\nabla_X\eta)Z. \tag{4.1}$$

Since both D and ∇ preserve the metric, τ satisfies $\tau(X,Y,Z) = -\tau(X,Z,Y)$.

We denote by R^D, R the curvatures of D, ∇ and use the same notation for the g-associated $(0,4)$-tensor fields, defined according to the convention: $R^D(X,Y,Z,W) = -g(R^D(X,Y,Z),W)$, $R(X,Y,Z,W) = -g(R(X,Y,Z),W)$. Obviously, by (4.1), for any vector fields X, Y, Z, W one has

$$\begin{aligned}(R^D - R)(X,Y,Z,W) = &-(D_X\tau)(Y,Z,W) + (D_Y\tau)(X,Z,W)\\ &- \tau(\Sigma(X,Y),Z,W) - \tau(X,W,\tau(Y,Z))\\ &+ \tau(Y,W,\tau(X,Z)).\end{aligned} \tag{4.2}$$

Since τ depends on the $\mathcal{C}_h(M)$-components of $\nabla\Phi$, it follows that $R^D - R$ can be expressed as a combination of the tensor fields $D\tau_h$ and $\tau_h \otimes \tau_k$, $h, k \in \{1, ..., 12\}$. Since D preserves the a.c.m. structure, it is easy to verify that, for any vector field X, $D_X\tau_h$ is a section of $\mathcal{C}_h(M)$ and R^D satisfies: $R^D(X,Y,Z,W) = R^D(X,Y,\varphi Z,\varphi W)$. Formula (4.2) also allows to express the cosymplectic defect, namely the tensor field Λ defined by $\Lambda(X,Y,Z,W) = R(X,Y,Z,W) - R(X,Y,\varphi Z,\varphi W)$, as follows:

$$\begin{aligned}\Lambda(X,Y,Z,W) = &(D_X\tau)(Y,Z,W) - (D_X\tau)(Y,\varphi Z,\varphi W)\\ &- (D_Y\tau)(X,Z,W) + (D_Y\tau)(X,\varphi Z,\varphi W)\\ &+ \tau(\Sigma(X,Y),Z,W) - \tau(\Sigma(X,Y),\varphi Z,\varphi W)\\ &+ \tau(X,W,\tau(Y,Z)) - \tau(X,\varphi W,\tau(Y,\varphi Z))\\ &- \tau(Y,W,\tau(X,Z)) + \tau(Y,\varphi W,\tau(X,\varphi Z)).\end{aligned} \tag{4.3}$$

Furthermore, we recall that, given a $(0,2)$-tensor field Q, the Kulkarni-Nomizu product $g \curlywedge Q$ of g and Q acts as

$$\begin{aligned}g \curlywedge Q(X,Y,Z,W) \;=\; &g(X,Z)Q(Y,W) + g(Y,W)Q(X,Z) - g(X,W)Q(Y,Z)\\ &-g(Y,Z)Q(X,W).\end{aligned}$$

In particular, to simplify the notation, one puts $\pi_1 = \frac{1}{2} g \curlywedge g$.

Theorem 4.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold with $\dim M = 2n+1$. With respect to a local orthonormal frame $\{e_1, ..., e_{2n}, \xi\}$, for any $X, Y, Z, W \in$*

$\mathcal{X}(M)$, *one has:*

$$\begin{aligned}
\Lambda(X,Y,Z,W) \quad = \quad & -\sum_{1\le i\le 4}((D_X\tau_i)(Y,\varphi Z,W)-(D_Y\tau_i)(X,\varphi Z,W)) \\
& +\frac{1}{2n}g\curlywedge(d\overline{c}(\tau_5)(\xi)\otimes\eta)(X,Y,Z,W) \\
& +\eta(Y)((D_X\tau_{12})(\xi,\xi,\varphi Z)\eta(W)-(D_Y\tau_{12})(\xi,\xi,\varphi W)\eta(Z)) \\
& -\eta(X)((D_Y\tau_{12})(\xi,\xi,\varphi Z)\eta(W)-(D_Y\tau_{12})(\xi,\xi,\varphi W)\eta(Z)) \\
& +\frac{1}{2}\sum_{1\le q\le 2n}\sum_{1\le i,h\le 4}(\tau_i(X,Y,\varphi e_q)-\tau_i(Y,X,\varphi e_q))\tau_h(e_q,Z,\varphi W) \\
& -\frac{\overline{c}(\tau_5)(\xi)}{2n}\sum_{1\le i\le 4}(\eta(Y)\tau_i(X,Z,\varphi W)-\eta(X)\tau_i(Y,Z,\varphi W)) \\
& -(\eta(X)(\nabla_\xi\eta)Y-\eta(Y)(\nabla_\xi\eta)X)(\eta(Z)(\nabla_\xi\eta)W-\eta(W)(\nabla_\xi\eta)Z) \\
& -\frac{1}{2}\eta(Z)\sum_{1\le i\le 4}(\eta(X)\tau_i(Y,W,\varphi(\nabla_\xi\xi))-\eta(Y)\tau_i(X,W,\varphi(\nabla_\xi\xi))) \\
& +\frac{1}{2}\eta(W)\sum_{1\le i\le 4}(\eta(X)\tau_i(Y,Z,\varphi(\nabla_\xi\xi))-\eta(Y)\tau_i(X,Z,\varphi(\nabla_\xi\xi))) \\
& -(\frac{\overline{c}(\tau_5(\xi)}{2n})^2(\pi_1(X,Y,Z,W)-\pi_1(X,Y,\varphi Z,\varphi W)) \\
& +\frac{\overline{c}(\tau_5)(\xi)}{2n}g\curlywedge(\eta\otimes\nabla_\xi\eta)(X,Y,Z,W) \\
& -\frac{\overline{c}(\tau_5(\xi)}{2n}g\curlywedge(\eta\otimes\nabla_\xi\eta)(X,Y,\varphi Z,\varphi W).
\end{aligned}$$

Proof. We outline the proof, omitting detailed and long calculation. Firstly, one writes $\nabla\Phi=\sum_{1\le i\le 5}\tau_i+\tau_{12}$ and recalls the relations

$$\tau_5(X,Y,Z)=\frac{\overline{c}(\tau_5)(\xi)}{2n}(g(X,\varphi Z)\eta(Y)-g(X,\varphi Y)\eta(Z)),$$
$$\tau_{12}(X,Y,Z)=\eta(X)(\eta(Y)\tau_{12}(\xi,\xi,Z)-\eta(Z)\tau_{12}(\xi,\xi,Y)).$$

Applying (4.1), for any $X,Y,Z\in\mathcal{X}(M)$, one has

$$\begin{aligned}
\tau(X,Y,Z)=&-\frac{1}{2}\sum_{1\le i\le 4}\tau_i(X,\varphi Y,Z) \\
&+\frac{\overline{c}(\tau_5)(\xi)}{2n}(g(X,Z)\eta(Y)-g(X,Y)\eta(Z)) \\
&+\eta(X)(\eta(Z)(\nabla_\xi\eta)Y-\eta(Y)(\nabla_\xi\eta)Z),
\end{aligned}\tag{4.4}$$

and then

$$\begin{aligned}
\tau(X,Y)=&-\frac{1}{2}\sum_{1\le q\le 2n}\sum_{1\le i\le 4}\tau_i(X,\varphi Y,e_q)e_q \\
&+\frac{\overline{c}(\tau_5)(\xi)}{2n}(\eta(Y)X-g(X,Y)\xi) \\
&+\eta(X)((\nabla_\xi\eta)Y\xi-\eta(Y)\nabla_\xi\xi).
\end{aligned}$$

Hence, by a straightforwad calculus, one obtains

$$\begin{aligned}
&(D_X\tau)(Y,Z,W)-(D_X\tau)(Y,\varphi Z,\varphi W)\\
&\quad=-\sum_{1\le i\le 4}(D_X\tau_i)(Y,\varphi Z,W)\\
&\qquad-\frac{1}{2n}X(\overline{c}(\tau_5)(\xi))(g(Y,Z)\eta(W)-g(Y,W)\eta(Z))\\
&\qquad+\eta(Y)((D_X\tau_{12})(\xi,\xi,\varphi Z)\eta(W)-(D_X\tau_{12})(\xi,\xi,\varphi W)\eta(Z)),
\end{aligned}$$

$$\begin{aligned}
&\tau(\Sigma(X,Y),Z,W)-\tau(\Sigma(X,Y),\varphi Z,\varphi W)\\
&\quad=\frac{1}{2}\sum_{1\le q\le 2n}\sum_{1\le i,h\le 4}(\tau_i(X,Y,\varphi e_q)-\tau_i(Y,X,\varphi e_q))\tau_h(e_q,Z,\varphi W)\\
&\qquad-\frac{\overline{c}(\tau_5)(\xi)}{2n}\sum_{1\le i\le 4}(\eta(Y)\tau_i(X,\varphi Z,W)-\eta(X)\tau_i(Y,\varphi Z,W))\\
&\qquad+\frac{\overline{c}(\tau_5)(\xi)}{4n}\sum_{1\le i\le 4}((\tau_i(X,\varphi Y,Z)-\tau_i(Y,\varphi X,Z))\eta(W)\\
&\qquad-(\tau_i(X,\varphi Y,W)-\tau_i(Y,\varphi X,W))\eta(Z))\\
&\qquad-(\frac{\overline{c}(\tau_5)(\xi)}{2n})^2 g\curlywedge(\eta\otimes\eta)(X,Y,Z,W)\\
&\qquad-(\eta(X)(\nabla_\xi\eta)Y-\eta(Y)(\nabla_\xi\eta)X)(\eta(Z)(\nabla_\xi\eta)W-\eta(W)(\nabla_\xi\eta)Z),
\end{aligned}$$

$$\begin{aligned}
&\tau(X,W,\tau(Y,Z))-\tau(X,\varphi W,\tau(Y,\varphi Z))\\
&\quad=\tau(Y,W,\tau(X,Z))-\tau(Y,\varphi W,\tau(X,\varphi Z))\\
&\qquad-\frac{\overline{c}(\tau_5)(\xi)}{4n}\sum_{1\le i\le 4}(\,(\tau_i(X,\varphi Y,Z)-\tau_i(Y,\varphi X,Z))\eta(W)\\
&\qquad-(\tau_i(X,\varphi Y,W)-\tau_i(Y,\varphi X,W))\eta(Z))\\
&\qquad-\frac{1}{2}\eta(Z)\sum_{1\le i\le 4}(\eta(X)\tau_i(Y,W,\varphi(\nabla_\xi\xi))-\eta(Y)\tau_i(X,W,\varphi(\nabla_\xi\xi)))\\
&\qquad+\frac{1}{2}\eta(W)\sum_{1\le i\le 4}(\eta(X)\tau_i(Y,Z,\varphi(\nabla_\xi\xi))-\eta(Y)\tau_i(X,Z,\varphi(\nabla_\xi\xi)))\\
&\qquad+(\frac{\overline{c}(\tau_5)(\xi)}{2n})^2(g\curlywedge(\eta\otimes\eta)(X,Y,Z,W)-\pi_1(X,Y,Z,W)+\pi_1(X,Y,\varphi Z,\varphi W))\\
&\qquad-\frac{\overline{c}(\tau_5)(\xi)}{2n}(g\curlywedge(\eta\otimes\nabla_\xi\eta)(X,Y,Z,W)-g\curlywedge(\eta\otimes\nabla_\xi\eta)(X,Y,\varphi Z,\varphi W)).
\end{aligned}$$

So, also applying (4.3), one gets the statement. □

Several consequences can be derived by Theorem 4.1. Before stating new results, we point out that, given a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold, the covariant derivatives $D\tau_{12}$, $\nabla(\nabla_\xi\eta)$ are related by

$$(D_X\tau_{12})(\xi,\xi,\varphi Y) = \nabla_X(\nabla_\xi\eta)(Y) + \frac{1}{2}\sum_{1\le i\le 4}\tau_i(X,Y,\varphi(\nabla_\xi\xi)) + \eta(Y)(\eta(X)\parallel\nabla_\xi\xi\parallel^2 - \frac{\overline{c}(\tau_5)(\xi)}{2n}(\nabla_\xi\eta)X). \tag{4.5}$$

In particular, with respect to a local orthonormal frame $\{e_1, ..., e_{2n}, \xi\}$, one has:

$$\sum_{1\le q\le 2n}(D_{e_q}\tau_{12})(\xi,\xi,\varphi e_q) = -\delta(\nabla_\xi\eta) + \parallel\nabla_\xi\xi\parallel^2 + \frac{1}{2}c(\tau_4)(\varphi(\nabla_\xi\xi)). \tag{4.6}$$

The next result easily follows by Theorem 4.1 and (4.6).

Corollary 4.1. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold with* $\dim M = 2n+1$. *For any $X, Y, Z \in \mathcal{X}(M)$ one has*

$$\begin{aligned} R(X,Y,\xi,Z) \quad = \quad & \frac{1}{2n}(X(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi Z) - Y(\overline{c}(\tau_5)(\xi))g(\varphi X,\varphi Z)) \\ & +\eta(X)(D_Y\tau_{12})(\xi,\xi,\varphi Z) - \eta(Y)(D_X\tau_{12})(\xi,\xi,\varphi Z) \\ & -(\eta(X)(\nabla_\xi\eta)Y - \eta(Y)(\nabla_\xi\eta)X)(\nabla_\xi\eta)Z \\ & -\frac{1}{2}\sum_{1\le i\le 4}(\eta(X)\tau_i(Y,Z,\varphi(\nabla_\xi\xi)) - \eta(Y)\tau_i(X,Z,\varphi(\nabla_\xi\xi))) \\ & -(\frac{\overline{c}(\tau_5)(\xi)}{2n})^2(\eta(X)g(Y,Z) - \eta(Y)g(X,Z)). \end{aligned}$$

Moreover, the Ricci tensor satisfies:

$$\rho(\xi,\xi) = \xi(\overline{c}(\tau_5)(\xi)) - \delta(\nabla_\xi\eta) - \frac{\overline{c}(\tau_5)(\xi)^2}{2n},$$

$$\rho(X,\xi) = \frac{2n-1}{2n}(X - \eta(X)\xi)(\overline{c}(\tau_5)(\xi)) + \eta(X)\rho(\xi,\xi),$$

for any $X \in \mathcal{X}(M)$.

Proposition 4.1. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold with* $\dim M = 2n+1$. *For any $Y, Z, W \in \mathcal{X}(M)$ one has*

$$\begin{aligned} 2n\sum_{1\le i\le 4}(D_\xi\tau_i)(Y,Z,\varphi W) \quad = \quad & \overline{c}(\tau_5)(\xi)\sum_{1\le i\le 4}\tau_i(Y,Z,\varphi W) \\ & -Z(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi W) + W(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi Z) \\ & +\varphi Z(\overline{c}(\tau_5)(\xi))g(Y,\varphi W) - \varphi W(\overline{c}(\tau_5)(\xi))g(Y,\varphi Z) \\ & +\xi(\overline{c}(\tau_5)(\xi))(g(Y,W)\eta(Z) - g(Y,Z)\eta(W)) \\ & +\overline{c}(\tau_5)(\xi)((\nabla_\xi\eta)Zg(\varphi Y,\varphi W) - (\nabla_\xi\eta)Wg(\varphi Y,\varphi Z) \\ & -(\nabla_\xi\eta)\varphi Zg(Y,\varphi W) + (\nabla_\xi\eta)\varphi Wg(Y,\varphi Z)). \end{aligned}$$

Proof. Let Y, Z, W be vector fields on M. Since R is an algebraic curvature tensor field, one has

$$\Lambda(\xi,Y,Z,W) - R(Z,W,\xi,Y) + R(\varphi Z,\varphi W,\xi,Y) = 0.$$

Hence, applying Theorem 4.1 and Corollary 4.1, we obtain:

$$\begin{aligned}
0 = &\sum_{1\le i\le 4}(D_\xi\tau_i)(Y,Z,\varphi W) + \frac{1}{2n}(Z(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi W)\\
&- W(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi Z) - \varphi Z(\overline{c}(\tau_5)(\xi))g(Y,\varphi W)\\
&- \varphi W(\overline{c}(\tau_5)(\xi))g(Y,\varphi Z))\\
&+ \frac{1}{2n}\xi(\overline{c}(\tau_5)(\xi))(g(Y,Z)\eta(W) - g(Y,W)\eta(Z))\\
&- ((D_{Y-\eta(Y)\xi}\tau_{12})(\xi,\xi,\varphi W) - (D_W\tau_{12})(\xi,\xi,\varphi Y))\eta(Z)\\
&+ ((D_{Y-\eta(Y)\xi}\tau_{12})(\xi,\xi,\varphi Z) - (D_Z\tau_{12})(\xi,\xi,\varphi Y))\eta(W)\\
&- \frac{\overline{c}(\tau_5)(\xi)}{2n}\sum_{1\le i\le 4}\tau_i(Y,Z,\varphi W)\\
&+ \frac{1}{2}\sum_{1\le i\le 4}(\eta(Z)(\tau_i(Y,W,\varphi(\nabla_\xi\xi)) - \tau_i(W,Y,\varphi(\nabla_\xi\xi)))\\
&- \eta(W)(\tau_i(Y,Z,\varphi(\nabla_\xi\xi)) - \tau_i(Z,Y,\varphi(\nabla_\xi\xi)))\\
&- \frac{\overline{c}(\tau_5)(\xi)}{2n}((\nabla_\xi\eta)Zg(\varphi Y,\varphi W) - (\nabla_\xi\eta)Wg(\varphi Y,\varphi Z)\\
&- (\nabla_\xi\eta)\varphi Zg(Y,\varphi W) + (\nabla_\xi\eta)\varphi Wg(Y,\varphi Z)).
\end{aligned}$$

Then, one proves that the block of terms in the previous formula involving $D\tau_{12}(\xi,\xi,\cdot)\otimes\eta$, $\sum_{1\le i\le 4}\tau_i(\cdot,\cdot,\varphi(\nabla_\xi\xi))\otimes\eta$ vanishes, so obtaining the statement. In fact, (4.5) and Corollary 2.1 entail:

$$\begin{aligned}
&(D_{Y-\eta(Y)\xi}\tau_{12})(\xi,\xi,\varphi Z) - (D_Z\tau_{12})(\xi,\xi,\varphi Y)\\
&\quad - \frac{1}{2}\sum_{1\le i\le 4}(\tau_i(Y,Z,\varphi(\nabla_\xi\xi)) - \tau_i(Z,Y,\varphi(\nabla_\xi\xi)))\\
&\qquad = 2d(\nabla_\xi\eta)(Y,Z) - \eta(Y)(\nabla_\xi(\nabla_\xi\eta)(Z) + \eta(Z)\parallel\nabla_\xi\xi\parallel^2)\\
&\qquad\quad - \frac{\overline{c}(\tau_5)(\xi)}{2n}(\eta(Z)(\nabla_\xi\eta)Y - \eta(Y)(\nabla_\xi\eta)Z)\\
&\qquad = -(\nabla_\xi(\nabla_\xi\eta)(Y) + \eta(Y)\parallel\nabla_\xi\xi\parallel^2)\eta(Z).
\end{aligned}$$

□

In dimension 3, the formula stated in Proposition 4.1 reduces to an identity. In fact, in this case, considering a manifold (M,φ,ξ,η,g) in $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$, all the projections τ_i's,$i\in\{1,2,3,4\}$, vanish. Moreover, we consider the tensor field S acting as

$$\begin{aligned}
S(Y,Z,W) \;=\; & Z(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi W) - W(\overline{c}(\tau_5)(\xi))g(\varphi Y,\varphi Z)\\
&-\varphi Z(\overline{c}(\tau_5)(\xi))g(Y,\varphi W) + \varphi W(\overline{c}(\tau_5)(\xi))g(Y,\varphi Z)\\
&+\xi(\overline{c}(\tau_5)(\xi))(g(Y,Z)\eta(W) - g(Y,W)\eta(Z))\\
&-\overline{c}(\tau_5)(\xi)(g(\varphi Y,\varphi W)(\nabla_\xi\eta)Z - g(\varphi Y,\varphi Z)(\nabla_\xi\eta)W\\
&-g(Y,\varphi W)(\nabla_\xi\eta)\varphi Z + g(Y,\varphi Z)(\nabla_\xi\eta)\varphi W).
\end{aligned}$$

By direct calculus, given a point $p \in M$ and an orthonormal basis $\{X, \varphi X, \xi\}$ of T_pM, for any $Y \in T_pM$ we have

$$S_p(Y, X, \varphi X) = S_p(Y, \varphi X, X) = S_p(Y, X, \xi) = S_p(Y, \varphi X, \xi) = 0.$$

It follows that $S = 0$.
We examine some consequences of Proposition 4.1 in dimensions $2n + 1 \geq 5$.

Proposition 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold with $\dim M = 2n + 1 \geq 5$. Then, one has:*

$$\begin{aligned} D_\xi \tau_i &= \frac{\overline{c}(\tau_5)(\xi)}{2n} \tau_i, \qquad i \in \{1, 2, 3\}, \\ (D_\xi c(\tau_4))\varphi W &= \frac{\overline{c}(\tau_5)(\xi)}{2n} c(\tau_4)(\varphi W) \\ &\quad + \frac{n-1}{n}((W - \eta(W)\xi)(\overline{c}(\tau_5)(\xi)) - \overline{c}(\tau_5)(\xi)(\nabla_\xi \eta)W), \end{aligned}$$

for any $W \in \mathcal{X}(M)$.

Proof. Let Y, Z, W be vector fields on M. By Proposition 4.1, using the properties

$$\begin{aligned} \tau_i(Y, Z, \varphi W) &= -\tau_i(\varphi Y, \varphi Z, \varphi W), \ i \in \{1, 2\}, \\ \tau_i(Y, Z, \varphi W) &= \tau_i(\varphi Y, \varphi Z, \varphi W), \ i \in \{3, 4\}, \\ (D_\xi \tau_i)(Y, Z, \varphi W) &= -(D_\xi \tau_i)(\varphi Y, \varphi Z, \varphi W), \ i \in \{1, 2\}, \\ (D_\xi \tau_i)(Y, Z, \varphi W) &= (D_\xi \tau_i)(\varphi Y, \varphi Z, \varphi W), \ i \in \{3, 4\}, \end{aligned}$$

one has:

$$\sum_{1 \leq i \leq 2} ((D_\xi \tau_i)(Y, Z, \varphi W) - \frac{\overline{c}(\tau_5)(\xi)}{2n} \tau_i(Y, Z, \varphi W)) = 0.$$

Since moreover $(D_\xi \tau_i)(Y, Z, \xi) = \tau_i(Y, Z, \xi) = 0$ and $D_\xi \tau_i - \frac{\overline{c}(\tau_5)(\xi)}{2n} \tau_i$ is a section of $\mathcal{C}_i(M)$, $i \in \{1, 2\}$, one obtains $D_\xi \tau_i = \frac{\overline{c}(\tau_5)(\xi)}{2n} \tau_i$, $i \in \{1, 2\}$. Let $\{e_1, ..., e_{2n}, \xi\}$ be a local orthonormal frame. By Proposition 4.1 we have

$$\begin{aligned} (D_\xi c(\tau_4))\varphi W &= \sum_{1 \leq q \leq 2n} (D_\xi \tau_4)(e_q, e_q, \varphi W) = \frac{\overline{c}(\tau_5)(\xi)}{2n} c(\tau_4)(\varphi W) \\ &\quad + \frac{n-1}{n}((W - \eta(W)\xi)(\overline{c}(\tau_5)(\xi)) - \overline{c}(\tau_5)(\xi)(\nabla_\xi \eta)W). \end{aligned}$$

On the other hand, applying the definition of τ_4, ([4]), one gets:

$$\begin{aligned} 2(n-1)(D_\xi c(\tau_4))(Y, Z, \varphi W) &= g(Y, \varphi Z)(D_\xi c(\tau_4))W - g(Y, \varphi W)(D_\xi c(\tau_4))Z \\ &\quad + g(\varphi Y, \varphi Z)(D_\xi c(\tau_4))\varphi W \\ &\quad - g(\varphi Y, \varphi W)(D_\xi c(\tau_4))\varphi Z. \end{aligned}$$

So, we again apply Proposition 4.1, use the just stated relations and obtain $D_\xi \tau_3 = \frac{\overline{c}(\tau_5)(\xi)}{2n} \tau_3$. □

Theorem 4.2. *Let $(M, \varphi, \xi, \eta, g)$ be an a.c.m. manifold with $\dim M \geq 5$. If M falls in the class $\mathcal{C}_i \oplus \mathcal{C}_5$, $i \in \{1, 2, 3\}$, then M is, locally, a.c. isometric to a warped product manifold $I \times_\lambda F$, where $I \subset \boldsymbol{R}$ is an open interval, $\lambda : I \to \boldsymbol{R}$ a smooth positive function and F an almost Hermitian manifold in the Gray-Hervella class $\mathcal{W}_i$.*

Proof. Fixed $i \in \{1,2,3\}$, since M is a $\mathcal{C}_i \oplus \mathcal{C}_5$-manifold, by Proposition 4.2 we get

$$d\overline{c}(\tau_5)(\xi) = \xi(\overline{c}(\tau_5)(\xi))\eta.$$

By Theorem 3.1 in [6] M is, locally, a.c. isometric to a warped product manifold $]-\varepsilon,\varepsilon[\times_\lambda F$, $\varepsilon > 0$, $(F, \widehat{J}, \widehat{g})$ being an a. H. manifold and $\lambda :]-\varepsilon,\varepsilon[\to \boldsymbol{R}$ a smooth positive function. Obviously, the manifold $]-\varepsilon,\varepsilon[\times_\lambda F$ is in the class $\mathcal{C}_i \oplus \mathcal{C}_5$. Hence Proposition 2.1 entails that $(F, \widehat{J}, \widehat{g})$ is a $\mathcal{W}_i$-manifold. □

Proposition 4.3. *Let (M,φ,ξ,η,g) be an a.c.m. manifold in the class $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_5 \oplus \mathcal{C}_{12}$ with $\dim M = 2n+1 \geq 5$. Then, the Lee form is closed.*

Proof. Since in this case $\tau_4 = 0$, the Lee form is $\omega = \frac{\overline{c}(\tau_5)(\xi)}{2n}\eta$ and, by Proposition 4.2, we have

$$d\overline{c}(\tau_5)(\xi) = \xi(\overline{c}(\tau_5)(\xi))\eta + \overline{c}(\tau_5)(\xi)\nabla_\xi\eta.$$

It follows:

$$d\omega = \frac{\overline{c}(\tau_5)(\xi)}{2n}(\nabla_\xi\eta \wedge \eta + d\eta)$$

and, applying Corollary 2.1, one gets $d\omega = 0$. □

Proposition 4.4. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold with $\dim M = 2n+1 \geq 5$. Then, M is a locally conformal $\mathcal{C}_{12}$-manifold.*

Proof. The hypothesis implies that $\nabla\varphi$ acts as

$$(4.7) \qquad \begin{aligned}(\nabla_X\varphi)Y = {} & \frac{\overline{c}(\tau_5)(\xi)}{2n}(\eta(Y)\varphi X + g(X,\varphi Y)\xi) \\ & - \eta(X)((\nabla_\xi\eta)\varphi Y\xi + \eta(Y)\varphi(\nabla_\xi\xi)),\end{aligned}$$

and the Lee form $\omega = \frac{\overline{c}(\tau_5)(\xi)}{2n}\eta$ is closed. So, we consider an open covering $\{U_i\}_{i\in I}$ of M and, for any i, a function $\sigma_i \in \mathcal{F}(U_i)$ such that $\omega_{|U_i} = d\sigma_i$. Putting $\varphi_i = \varphi_{|U_i}$, $\xi_i = \exp(-\sigma_i)\xi_{|U_i}$, $\eta_i = \exp\sigma_i\eta_{|U_i}$, $g_i = \exp 2\sigma_i g_{|U_i}$, we prove that the a.c.m. manifold $(U_i,\varphi_i,\xi_i,\eta_i,g_i)$ is in the class $\mathcal{C}_{12}$. In fact, the Levi-Civita connections of the local metrics g_i's fit up to the Weyl connection $\widetilde{\nabla}$ of (M,g) acting as

$$(4.8) \qquad \widetilde{\nabla}_X Y = \nabla_X Y + \omega(X)Y + \omega(Y)X - g(X,Y)B, \ B = \omega^\sharp.$$

In particular, fixed $i \in I$, one has $\widetilde{\nabla}_{\xi_i}\xi_i = \exp(-2\sigma_i)\nabla_\xi\xi_{|U_i}$. Considering $X, Y \in \mathcal{X}(M)$, by (4.7), (4.8), in U_i we obtain

$$\begin{aligned}(\widetilde{\nabla}_X\varphi_i)Y &= -\eta(X)((\nabla_\xi\eta)\varphi Y\xi + \eta(Y)\varphi(\nabla_\xi\xi)) \\ &= -\eta_i(X)((\widetilde{\nabla}_{\xi_i}\eta_i)\varphi_i Y\xi_i + \eta_i(Y)\varphi_i(\widetilde{\nabla}_{\xi_i}\xi_i)).\end{aligned}$$

□

Remark 4.1. It is easy to prove that any 3-dimensional a.c.m. manifold is locally conformal cosymplectic if and only if it is a $\mathcal{C}_5 \oplus \mathcal{C}_{12}$-manifold with closed Lee form.

5. Other curvature relations

The results stated in Section 4, in particular Theorem 4.1, allow to describe the behaviour of some algebraic curvature tensor fields naturally associated with a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}-$ manifold.

Firstly, we recall that, if S is an algebraic curvature tensor field on a Riemannian manifold (M, g), putting $S(X, Y) = S(X, Y, X, Y)$, for any $X, Y, Z, W \in \mathcal{X}(M)$, one has:

$$\begin{aligned} 6S(X, Y, Z, W) = & \ S(X, Y + Z) - S(X, Y + W) + S(Y, X + W) \\ & - S(Y, X + Z) + S(Z, X + W) - S(Z, Y + W) \\ & + S(W, Y + Z) - S(W, X + Z) + S(X + Z, Y + W) \\ & - S(X + W, Y + Z) + S(X, W) - S(X, Z) \\ & + S(Y, Z) - S(Y, W). \end{aligned}$$

It follows that S is uniquely determined by the values $S(X, Y)$, for any pair (X, Y) of vector fields.

Given an a.c.m. manifold $(M, \varphi, \xi, \eta, g)$, let T_2, T_3 be the algebraic curvature tensor fields on M acting as:

$$\begin{aligned} T_2(X, Y, Z, W) = & \ R(X, Y, Z, W) + R(\varphi X, \varphi Y, \varphi Z, \varphi W) - R(\varphi X, \varphi Y, Z, W) \\ & - R(X, Y, \varphi Z, \varphi W) - R(\varphi X, Y, \varphi Z, W) - R(X, \varphi Y, Z, \varphi W) \\ & - R(\varphi X, Y, Z, \varphi W) - R(X, \varphi Y, \varphi Z, W), \end{aligned}$$

$$T_3(X, Y, Z, W) = R(X, Y, Z, W) - R(\varphi X, \varphi Y, \varphi Z, \varphi W).$$

We recall that the vanishing of T_3 means that M satisfies the $K_{3\varphi}$-identity ([3]), as well as M fulfills the $(G3)$-identity if and only if $T_3 = g \curlywedge (\eta \otimes \eta)$ ([11]).

Proposition 5.1. *Let $(M, \varphi, \xi, \eta, g)$ be a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold with* $\dim M = 2n + 1 \geq 5$*. With respect to a local orthonormal frame $\{e_1, ..., e_{2n}, \xi\}$, the tensor field T_2 depends on $D\tau_2$, $D\tau_{12}$, $(2\tau_1 - \tau_2) \odot \tau_3$, $\tau_2 \odot \tau_4$, $\tau_2 \odot \tau_5$, $\tau_2 \odot \tau_{12}$, $\tau_{12} \odot \tau_{12}$, according* *to the formula:*

$$\begin{aligned}
T_2(X,Y) &= 2((D_X\tau_2)(Y,Y,\varphi X) + (D_Y\tau_2)(X,X,\varphi Y) + (D_{\varphi X}\tau_2)(Y,Y,X) \\
&\quad +(D_{\varphi Y}\tau_2)(X,X,Y)) + \eta(X)^2((D_Y\tau_{12})(\xi,\xi,\varphi Y) + D_{\varphi Y}\tau_{12})(\xi,\xi,Y)) \\
&\quad +\eta(Y)^2((D_X\tau_{12})(\xi,\xi,\varphi X) + (D_{\varphi X}\tau_{12})(\xi,\xi,X)) \\
&\quad -\eta(X)\eta(Y)((D_X\tau_{12})(\xi,\xi,\varphi Y) + (D_{\varphi X}\tau_{12})(\xi,\xi,Y) \\
&\quad +(D_Y\tau_{12})(\xi,\xi,\varphi X) + (D_{\varphi Y}\tau_{12})(\xi,\xi,X)) \\
&\quad -2\sum_{1\le q\le 2n}(2\tau_1-\tau_2)(e_q,X,Y)\tau_3(e_q,X,Y) \\
&\quad +\frac{1}{n-1}(\tau_2(X,X,Y)c(\tau_4)(Y) - \tau_2(X,X,\varphi Y)c(\tau_4)(\varphi Y) \\
&\quad +\tau_2(Y,Y,X)c(\tau_4)(X) - \tau_2(Y,Y,\varphi X)c(\tau_4)(\varphi X)) \\
&\quad -\frac{\overline{c}(\tau_5)(\xi)}{n}(\eta(X)\tau_2(Y,Y,\varphi X) + \eta(Y)\tau_2(X,X,\varphi Y)) \\
&\quad -\eta(X)^2\tau_2(Y,Y,\varphi(\nabla_\xi\xi)) - \eta(Y)^2\tau_2(X,X,\varphi(\nabla_\xi\xi)) \\
&\quad +\eta(X)\eta(Y)(\tau_2(X,Y,\varphi(\nabla_\xi\xi)) + \tau_2(Y,X,\varphi(\nabla_\xi\xi))) \\
&\quad -(\eta(X)(\nabla_\xi\eta)Y - \eta(Y)(\nabla_\xi\eta)X)^2 \\
&\quad +(\eta(X)(\nabla_\xi\eta)\varphi Y - \eta(Y)(\nabla_\xi\eta)\varphi X)^2.
\end{aligned}$$

Proof. For any $X,Y \in \mathcal{X}(M)$, one has:

$$\begin{aligned}
T_2(X,Y) &= \Lambda(X,Y,X,Y) - \Lambda(\varphi X,\varphi Y,X,Y) - \Lambda(\varphi X,Y,\varphi X,Y) \\
&\quad -\Lambda(X,\varphi Y,\varphi X,Y) - \eta(X)(R(\varphi X,Y,\xi,\varphi Y) + R(X,\varphi Y,\xi,\varphi Y)).
\end{aligned}$$

Applying Theorem 4.1, Corollary 4.1 and using the theory developed in [4], after a long and detailed calculus one gets the statement.We only point out that the block of terms in the final expression of $T_2(X,Y)$ involving $D\tau_i, i \in \{1,3,4\}$ vanishes since for any $U,V,Z,W \in \mathcal{X}(M)$ one has:

$$(D_Z\tau_1)(U,U,V) = 0, (D_Z\tau_i)(\varphi U,\varphi V,W) = (D_Z\tau_i)(U,V,W), i \in \{3,4\}\,.$$

□

As remarked in [6], given an a.H. manifold $(F,\widehat{J},\widehat{g})$ in the class $\mathcal{W}_i$ $i \in \{1,2,3\}$, an open interval $I \subset \mathbf{R}$ and a smooth positive function $\lambda : I \times F \to \boldsymbol{R}$, the twisted product manifold $I \times_\lambda F$ falls in the class $\mathcal{C}_i \oplus \mathcal{C}_4 \oplus \mathcal{C}_5$. Proposition 5.1 entails that, if F is either a nearly-Kähler or a $\mathcal{W}_3$-manifold, then the curvature of $I \times_\lambda F$ satisfies the identity

$$\begin{aligned}
0 &= R(X,Y,Z,W) + R(\varphi X,\varphi Y,\varphi Z,\varphi W) - R(\varphi X,\varphi Y,Z,W) \\
&\quad - R(X,Y,\varphi Z,\varphi W) - R(\varphi X,Y,\varphi Z,W) - R(X,\varphi Y,\varphi Z,W) \\
&\quad - R(\varphi X,Y,Z,\varphi W) - R(X,\varphi Y,Z,\varphi W).
\end{aligned} \tag{5.1}$$

As far as regards the tensor field T_3 associated with a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold, one starts by the relation

$$T_3(X,Y) = \Lambda(X,Y,X,Y) + \Lambda(\varphi X,\varphi Y,X,Y),$$

argues as in the proof of Proposition 5.1 and obtains the next result.

Proposition 5.2. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_{1-5}\oplus\mathcal{C}_{12}$-manifold, with $\dim M = 2n+1 \geq 5$. With respect to a local orthonormal frame $\{e_1,...,e_{2n},\xi\}$ one has:*

$$\begin{aligned}
T_3(X,Y) \;=\; & \sum_{2\leq i\leq 4}((D_X\tau_i)(Y,Y,\varphi X) + (D_Y\tau_i)(X,X,\varphi Y)\\
&+(D_{\varphi X}\tau_i)(\varphi Y,\varphi Y,X) + (D_{\varphi Y}\tau_i)(\varphi X,\varphi X,Y))\\
&+\frac{1}{2n}g\curlywedge(d\overline{c}(\tau_5)(\xi)\otimes\eta)(X,Y,X,Y)\\
&+\frac{1}{2n}g\curlywedge(d\overline{c}(\tau_5)(\xi)\otimes\eta)(\varphi X,\varphi Y,X,Y)\\
&+\eta(Y)((D_X\tau_{12})(\xi,\xi,\varphi X)\eta(Y) - (D_X\tau_{12})(\xi,\xi,\varphi Y)\eta(X))\\
&+\eta(X)((D_Y\tau_{12})(\xi,\xi,\varphi Y)\eta(X) - (D_Y\tau_{12})(\xi,\xi,\varphi X)\eta(Y))\\
&+\sum_{1\leq q\leq 2n}\sum_{1\leq i\leq 4}((\tau_3+\tau_4)(X,Y,\varphi e_q) - (\tau_3+\tau_4)(Y,X,\varphi e_q))\tau_i(e_q,X,\varphi Y)\\
&-\frac{\overline{c}(\tau_5)(\xi)}{2n}\sum_{2\leq i\leq 4}(\eta(X)\tau_i(Y,Y,\varphi X) + \eta(Y)\tau_i(X,X,\varphi Y))\\
&-(\eta(X)(\nabla_\xi\eta)Y - \eta(Y)(\nabla_\xi\eta)X)^2\\
&-\frac{1}{2}\sum_{2\leq i\leq 4}(\eta(X)^2\tau_i(Y,Y,\varphi(\nabla_\xi\xi)) + \eta(Y)^2\tau_i(X,X,\varphi(\nabla_\xi\xi))\\
&-\eta(X)\eta(Y)(\tau_i(X,Y,\varphi(\nabla_\xi\xi)) + \tau_i(Y,X,\varphi(\nabla_\xi\xi))))\\
&-(\frac{\overline{c}(\tau_5)(\xi)}{2n})^2(\eta(X)^2g(Y,Y) - 2\eta(X)\eta(Y)g(X,Y) + \eta(Y)^2g(X,X))\\
&-\frac{\overline{c}(\tau_5)(\xi)}{2n}((\eta(X)g(X,Y) - \eta(Y)g(X,X))(\nabla_\xi\eta)Y\\
&+(\eta(Y)g(X,Y) - \eta(X)g(Y,Y))(\nabla_\xi\eta)X\\
&+g(X,\varphi Y)(\eta(X)(\nabla_\xi\eta)\varphi Y - \eta(Y)(\nabla_\xi\eta)\varphi X)).
\end{aligned}$$

Corollary 5.1. *Let (M,φ,ξ,η,g) be a $\mathcal{C}_1\oplus\mathcal{C}_5$-manifold with $\dim M = 2n+1 \geq 5$. Then, the curvature of M satisfies the k-nullity condition and the identity:*

$$\begin{aligned}
R(X,Y,Z,W) - &R(\varphi X,Y,Z,\varphi W) - R(X,\varphi Y,Z,\varphi W) - R(X,Y,\varphi Z,\varphi W)\\
&= k(g(X,Z)\eta(Y) - g(Y,Z)\eta(X))\eta(W),
\end{aligned}$$

where

$$k = \frac{1}{2n}(\xi(\overline{c}(\tau_5)(\xi)) - \frac{\overline{c}(\tau_5)(\xi)^2}{2n}).$$

Proof. Let k be the smooth function defined in the statement. We apply Propositions 5.1, 4.2 and obtain

$$T_3(X,Y) = kg\curlywedge(\eta\otimes\eta)(X,Y),\ X,Y\in\mathcal{X}(M).$$

Hence R satisfies the identity

$$\begin{aligned}
R(X,Y,Z,W) - R(\varphi X,\varphi Y,\varphi Z,\varphi W)&\\
= k(g(X,Z)\eta(Y)\eta(W) + g(Y,W)\eta(X)\eta(Z)&\\
- g(Y,Z)\eta(X)\eta(W) - g(X,W)\eta(Y)\eta(Z)).&
\end{aligned} \tag{5.2}$$

In particular, (5.2) implies

$$R(X,Y,\xi) = k(g(Y,Z)X - g(X,Z)Y),$$

namely R satisfies the k-nullity condition. Finally, since in this case Proposition 5.1 entails $T_2 = 0$, by repeated applications of (5.2) we get the identity in the statement. □

Remark 5.1. We recall that a nearly Kenmotsu manifold is a $\mathcal{C}_1 \oplus \mathcal{C}_5$- manifold such that $\bar{c}(\tau_5)(\xi) = -2n$. Hence, the curvature of a nearly Kenmotsu manifold satisfies the k-nullity condition and the identity in Corollary 5.1 with $k = -1$.
In [11] the authors give explicit examples of a.c.m. manifolds satisfying the so-called (G2)-identity, namely a.c.m. manifolds whose curvature verifies:

$$\begin{aligned} R(X,Y,Z,W) - &R(\varphi X,Y,Z,\varphi W) - R(X,\varphi Y,Z,\varphi W) - R(X,Y,\varphi Z,\varphi W) \\ &= (g(X,Z)\eta(Y) - g(Y,Z)\eta(X))\eta(W). \end{aligned}$$

Other explicit formulas involving the curvature of a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold follow by Theorem 4.1 and Proposition 5.2. We pay our attention to a $(0,2)$-tensor field defined in terms of the trace of T_3. Considering a local orthonormal frame $\{e_1, ..., e_{2n}, \xi\}$ on a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold, for any vector field X we get:

$$\rho(X,X) - \rho(\varphi X, \varphi X) = \sum_{1 \le q \le 2n} T_3(X, e_q) + T_3(X, \xi).$$

It follows that the tensor field ρ_φ acting as $\rho_\varphi(X,Y) = \rho(X,Y) - \rho(\varphi X, \varphi Y)$ depends on $D\tau_h$, $h \in \{2,4,5,12\}$, $\tau_2 \odot \tau_h$, $h \in \{3,4,5\}$, $\tau_3 \odot \tau_1, \tau_3 \odot \tau_3, \tau_{12} \odot \tau_{12}, \tau_4 \odot \tau_h, h \in \{4,5,12\}$.

Concerning the $*$-Ricci tensor ρ^*, which is locally defined by

$$\rho^*(X,Y) = \sum_{1 \le q \le 2n} R(X, e_q, \varphi Y, \varphi e_q),$$

via Corollary 4.1 one obtains

$$\rho^*(\xi, X) = \frac{1}{2n}(X - \eta(X)\xi)(\bar{c}(\tau_5)(\xi)).$$

By Proposition 4.1 it follows that $\rho^*(\xi, X) = 0$, for any vector field X on a $\mathcal{C}_1 \oplus \mathcal{C}_2 \oplus \mathcal{C}_3 \oplus \mathcal{C}_5$-manifold. Furthermore, by a long calculus, one proves that the skew-symmetric part ρ^*_{alt} of ρ^* depends on $D\tau_h, h \in \{2,3,4,5\}, \tau_h \odot \tau_5, h \in \{1,2\}$ and $\tau_h \odot \tau_4, h \in \{1,2,3\}$.

Finally, we pay our attention to the interrelation between the results stated in this section and the ones dealing with the curvature of a. H. manifolds. Let $(N, J' = \varphi_{|TN}, g' = g_{|Tn \times TN})$ be a leaf of the distribution $\mathcal{D}$ associated with a $\mathcal{C}_{1-5} \oplus \mathcal{C}_{12}$-manifold $(M, \varphi, \xi, \eta, g)$. We use the symbol $'$ (prime) to denote the geometrical objects associated with N. For instance, Ω' stands for the fundamental form of N and for any $i \in \{1,2,3,4\}$ τ'_i denotes the $\mathcal{W}_i$-component of $\nabla'\Omega'$. By (3.1) one gets $\tau'_i(X,Y,Z) = \tau_i(X,Y,Z)$, for any X,Y,Z tangent to N. Moreover, since the minimal connection D on N induces the unitary connection D' acting as $D'_X Y = \nabla'_X Y - \frac{1}{2}J'((\nabla'_X J')Y)$, for any vector fields X,Y,Z,W on N we have $(D'_X \tau'_i)(Y,Z,W) = (D_X \tau_i)(Y,Z,W), i \in \{1,2,3,4\}$. Furthermore, applying the Gauss equation, Theorem 4.1 and the previous relations, one expresses the Kähler defect of N as follows. Considering a local orthonormal frame $\{e_1, ..., e_{2n}\}$ on N, for any $X, Y, Z, W \in \mathcal{X}(N)$ one has:

$$\begin{aligned} R'(X,Y,Z,W) &= R'(X,Y,J'Z,J'W) + \Lambda(X,Y,Z,W) \\ &+ (\frac{\overline{c}(\tau_5)(\xi)}{2n})^2(\pi_1(X,Y,Z,W) - \pi_1(X,Y,\varphi Z,\varphi W)) \\ &= -\sum_{1\leq i\leq 4}((D'_X\tau'_i)(Y,J'Z,W) - (D'_Y\tau'_i)(X,J'Z,W)) \\ &+ \frac{1}{2}\sum_{1\leq q\leq 2n}\sum_{1\leq i,h\leq 4}(\tau'_i(X,Y,J'e_q) - \tau'_i(Y,X,J'e_q))\tau'_h(e_q,Z,J'W). \end{aligned}$$

This is consistent with the expression of the Kähler defect associated with any a. H. manifold given in [7]. Finally, we consider the algebraic curvature tensor fields on N, denoted by $C_5, C_6 + C_7 + C_8$, acting as

$$\begin{aligned} C_5(X,Y,Z,W) &= \frac{1}{8}(R'(X,Y,Z,W) + R'(J'X,J'Y,J'Z,J'W) \\ &-R'(J'X,J'Y,Z,W) - R'(X,Y,J'Z,J'W) \\ &-R'(J'X,Y,J'Z,W) - R'^{(}(X,J'Y,Z,J'W) \\ &-R'(J'X,Y,Z,J'W) - R'(X,J'Y,J'Z,W)), \end{aligned}$$

$$(C_6 + C_7 + C_8)(X,Y,Z,W) = \frac{1}{2}(R'(X,Y,Z,W) - R'(J'X,J'Y,J'Z,J'W)).$$

In this case, for any $X, Y \in \mathcal{X}(N)$, we have:

$$C_5(X,Y) = \frac{1}{8}T_2(X,Y),\ (C_6 + C_7 + C_8)(X,Y) = \frac{1}{2}T_3(X,Y).$$

Therefore, applying Propositions 5.1, 5.2, one gets that C_5 depends on $D'\tau'_2, \tau'_1 \odot \tau'_3, \tau'_2 \odot \tau'_3, \tau'_2 \odot \tau'_4$, as well as $C_6 + C_7 + C_8$ depends on $D'\tau'_i, i \in \{2,3,4\}$,and $(\tau'_3 + \tau'_4) \odot \tau'_i, i \in \{1,2,3,4\}$. This agrees with the analogous results proved in [7].

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