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ANTI-INVARIANT RIEMANNIAN SUBMERSIONS FROM ALMOST PRODUCT RIEMANNIAN MANIFOLDS

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Abstract. In this paper, we introduce anti-invariant Riemannian submersions from almost product Riemannian manifolds onto Riemannian manifolds. We give an example, investigate the geometry of foliations which are arisen from the definition of a Riemannian submersion and check the harmonicity of such submersions.

1. Introduction

Given a $C^{\infty}-$submersion π from a Riemannian manifold (M, g) onto a Riemannian manifold (N, g'), there are several kinds of submersions according to the conditions on it: e.g. Riemannian submersion([6], [12]), slant submersion ([7],[13],[14]), anti-invariant Riemannian submersion[15], almost Hermitian submersion [16], quaternionic submersion [8], etc. As we know, Riemannian submersions are related with physics and have their applications in the Yang-Mills theory([3],[17]), Kaluza-Klein theory ([2],[9]), supergravity and superstring theories ([10],[11]), etc. The paper is organized as follows. In Section 2 we recall some notions needed for this paper. In section 3 we introduce the notion of almost product Riemannian submersions. In section 4, we give definition of anti-invariant Riemannian submersions, provide an example and investigate the geometry of leaves of the distributions. Finally we give necessary and sufficient conditions for such submersions to be totally geodesic or harmonic.

2. Preliminaries

In this section, we define almost product Riemannian manifolds, recall the notion of Riemannian submersions between Riemannian manifolds and give a brief review of basic facts of Riemannian submersions.

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Let M be a m- dimensional manifold with a tensor F of type $(1,1)$ such that

$$F^2 = I, (F \neq I).$$

Then, we say that M is an almost product manifold with almost product structure F. We put

$$P = \frac{1}{2}(I + F), \quad Q = \frac{1}{2}(I - F).$$

Then we get

$$P + Q = I, \quad P^2 = P, \quad Q^2 = Q, \quad PQ = QP = 0, \quad F = P - Q.$$

Thus P and Q define two complementary distributions P and Q. We easily see that the eigenvalues of F are $+1$ or -1.
If an almost product manifold M admits a Riemannian metric g such that

$$g(FX, FY) = g(X, Y) \tag{2.1}$$

for any vector fields X and Y on M, then M is called an almost product Riemannian manifold, denoted by (M, g, F).
Denote the Levi-Civita connection on M with respect to g by ∇. Then, M is called a locally product Riemannian manifold[18] if F is parallel with respect to ∇, i.e.,

$$\nabla_X F = 0, X \in \Gamma(TM). \tag{2.2}$$

Let (M, g) and (N, g') be two Riemannian manifolds. A surjective $C^\infty-$map $\pi : M \to N$ is a $C^\infty-$submersion if it has maximal rank at any point of M. Putting $\mathcal{V}_x = ker\pi_{*x}$, for any $x \in M$, we obtain an integrable distribution $\mathcal{V}$, which is called vertical distribution and corresponds to the foliation of M determined by the fibres of π. The complementary distribution $\mathcal{H}$ of $\mathcal{V}$, determined by the Riemannian metric g, is called horizontal distribution. A $C^\infty-$submersion $\pi : M \to N$ between two Riemannian manifolds (M, g) and (N, g') is called a Riemannian submersion if, at each point x of M, π_{*x} preserves the length of the horizontal vectors. A horizontal vector field X on M is said to be basic if X is $\pi-$related to a vector field X' on N. It is clear that every vector field X' on N has a unique horizontal lift X to M and X is basic.

We recall that the sections of $\mathcal{V}$, respectively $\mathcal{H}$, are called the vertical vector fields, respectively horizontal vector fields. A Riemannian submersion $\pi : M \to N$ determines two $(1,2)$ tensor fields T and A on M, by the formulas:

$$T(E, F) = T_E F = h\nabla_{vE} vF + v\nabla_{vE} hF \tag{2.3}$$

and

$$A(E, F) = A_E F = v\nabla_{hE} hF + h\nabla_{hE} vF \tag{2.4}$$

for any $E, F \in \Gamma(TM)$, where v and h are the vertical and horizontal projections (see [5]). From (2.3) and (2.4), one can obtain

$$\nabla_U W = T_U W + \hat{\nabla}_U W; \tag{2.5}$$

$$\nabla_U X = T_U X + h(\nabla_U X); \tag{2.6}$$

$$\nabla_X U = v(\nabla_X U) + A_X U; \tag{2.7}$$

$$\nabla_X Y = A_X Y + h(\nabla_X Y), \tag{2.8}$$

for any $X, Y \in \Gamma((ker\pi_*)^\perp)$, $U, W \in \Gamma(ker\pi_*)$. Moreover, if X is basic then

$$h(\nabla_U X) = h(\nabla_X U) = A_X U. \tag{2.9}$$

We note that for $U, V \in \Gamma(ker\pi_*)$, $T_U V$ coincides with the second fundamental form of the immersion of the fibre submanifolds and for $X, Y \in \Gamma((ker\pi_*)^\perp)$, $A_X Y = \frac{1}{2}v[X, Y]$ reflecting the complete integrability of the horizontal distribution $\mathcal{H}$. It is known that A is alternating on the horizontal distribution: $A_X Y = -A_Y X$, for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and T is symmetric on the vertical distribution: $T_U V = T_V U$, for $U, V \in \Gamma(ker\pi_*)$.

We now recall the following result which will be useful for later.

Lemma 2.1. *(see* [5],[12]*). If $\pi : M \to N$ is a Riemannian submersion and X, Y basic vector fields on M, $\pi-$related to X' and Y' on N, then we have the following properties*

1. *$h[X, Y]$ is a basic vector field and $\pi_* h[X, Y] = [X', Y'] \circ \pi$;*
2. *$h(\nabla_X Y)$ is a basic vector field $\pi-$related to $(\nabla'_{X'} Y')$, where ∇ and ∇' are the Levi-Civita connection on M and N;*
3. *$[E, U] \in \Gamma(ker\pi_*)$, for any $U \in \Gamma(ker\pi_*)$ and for any basic vector field E.*

Let (M, g) and $(N, g^{'})$ be Riemannian manifolds and $\pi : M \to N$ is a smooth map. Then the second fundamental form of π is given by

$$(\nabla\pi_*)(X, Y) = \nabla_{\pi_* X}\pi_* Y - \pi_*(\nabla_X Y) \tag{2.10}$$

for $X, Y \in \Gamma(TM)$, where we denote conveniently by ∇ the Levi-Civita connections of the metrics g and $g^{'}$. Recall that π is said to be *harmonic* if $trace(\nabla\pi_*) = 0$ and π is called a *totally geodesic* map if $(\nabla\pi_*)(X, Y) = 0$ for $X, Y \in \Gamma(TM)$[1]. It is known that the second fundamental form is symmetric.

3. Almost product Riemannian submersions

In this section, we define the notion of almost product Riemannian submersions. The results given in this section can be find in [7].

Definition 3.1. Let M and N be almost product Riemannian manifolds with almost product structures F and F', respectively. A mapping $\pi : M \to N$ is said to be almost product map if $\pi_* \circ F = F' \circ \pi_*$.

By using the above definition, we are ready to give the following notion.

Definition 3.2. Let (M, F, g) and (N, F', g') be almost product Riemannian manifolds. A Riemannian submersion $\pi : M \to N$ is called an almost product Riemannian submersion if π is an almost product map, i.e. $\pi_* \circ F = F' \circ \pi_*$.

By using the almost product map, we have the following result.

Proposition 3.1. *Let $\pi : (M, F, g) \to (N, F', g')$ be an almost product Riemannian submersion from an almost product manifold M onto an almost product manifold N, and let X be a basic vector field on M, $\pi-$related to X' on N. Then, FX is also a basic vector field $\pi-$related to $F'X'$.*

Next proposition shows that an almost product submersion puts some restrictions on the distributions $ker\pi_*$ and $((ker\pi_*)^\perp)$.

Proposition 3.2. *Let $\pi : (M, F, g) \to (N, F', g')$ be an almost product Riemannian submersion from an almost product manifold M onto an almost product manifold N. Then, the horizontal and vertical distributions are $F-$ invariant.*

In the sequel, we show that base manifold is a locally product manifold if the total manifold is a locally product manifold.

Theorem 3.1. *Let (M, F, g) be a locally product manifold and (N, F', g') be an almost product manifold. Suppose that $\pi : (M, F, g) \to (N, F', g')$ be an almost product Riemannian submersion. Then (N, F', g') is a locally product Riemannian manifold.*

As the fibers of an almost product submersion is an invariant submanifold of M with respect to F, we have the following.

Corollary 3.1. *Let $\pi : (M, F, g) \to (N, F', g')$ be an almost product submersion from a locally product Riemannian manifold M onto an almost product manifold N. Then, the fibers are locally product manifolds.*

4. Anti-invariant Riemannian Submersions

In this section, we define anti-invariant Riemannian submersions from an almost product Riemannian manifold onto a Riemannian manifold, investigate the integrability of distributions and obtain a necessary and sufficient condition for such submersions to be totally geodesic map.

Definition 4.1. Let (M, g, F) be an almost product Riemannian manifold and (N, g') a Riemannian manifold. Suppose that there exists a Riemannian submersion $\pi : M \to N$ such that $X \in (ker\pi_*)$ is anti-invariant with respect to F, i.e., $F(ker\pi_*) \subseteq (ker\pi_*)^\perp$. Then we say π is an anti-invariant Riemannian submersion.

Let $\pi : (M, g, F) \to (N, g')$ be an anti-invariant Riemannian submersion from an almost product Riemannian manifold (M, g, F) to a Riemannian manifold (N, g'). First of all, from Definition 4.1, we have $F(ker\pi_*)^\perp \cap (ker\pi_*) \neq 0$. We denote the complementary orthogonal distribution to $F(ker\pi_*)$ in $(ker\pi_*)^\perp$ by μ. Then we have

$$(ker\pi_*)^\perp = F(ker\pi_*) \oplus \mu. \tag{4.1}$$

Corollary 4.1. *Let π be an anti-invariant Riemannian submersion from an almost product Riemannian manifold (M, g, F) to a Riemannian manifold $(N, g^{'})$. Then μ is an invariant distribution of $(ker\pi_*)^\perp$, under the endomorphism F.*

Proof. First by using (2.1), we have $g(FX, FU) = 0$ for $U \in \Gamma(ker\pi_*)$, $X \in \Gamma(\mu)$, which shows that FX is orthogonal to $F(ker\pi_*)$. On the other hand, since FU and X are orthogonal we get $g(FU, X) = g(U, FX) = 0$ which shows that FX is orthogonal to $ker\pi_*$. This completes proof. □

For $X \in \Gamma((ker\pi_*)^\perp)$, we have

$$FX = BX + CX, \tag{4.2}$$

where $BX \in \Gamma((ker\pi_*)$ and $CX \in \Gamma(\mu)$. On the other hand, since $\pi_*((ker\pi_*)^\perp) = TN$ and π is a Riemannian submersion, using (4.2) we derive $g'(\pi_* FV, \pi_* CX) = 0$, for every $X \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$, which implies that

$$TN = \pi_*(F(ker\pi_*)) \oplus \pi_*(\mu). \tag{4.3}$$

Note that given an Euclidean space R^4 with coordinates $(x_1, ..., x_4)$, we can canonically choose an almost product structure F on R^4 as follows:

$$F(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3} + a_4\frac{\partial}{\partial x_4}) = -a_2\frac{\partial}{\partial x_1} - a_1\frac{\partial}{\partial x_2} + a_4\frac{\partial}{\partial x_3} + a_3\frac{\partial}{\partial x_4},$$

where $a_1, ..., a_4 \in R$.

Example 4.1. Let $\pi : R^4 \to R^2$ be a map defined $\pi(x_1, x_2, x_3, x_4) = (\frac{x_1+x_3}{\sqrt{2}}, \frac{x_2+x_4}{\sqrt{2}})$. Then, by direct calculations

$$ker\pi_* = Span\{Z_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, Z_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\}$$

and

$$(ker\pi_*)^\perp = Span\{X_1 = \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3}, X_2 = \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_4}\}.$$

Then it is easy to see that π is a Riemannian submersion. Moreover $FZ_1 = X_2$ and $FZ_2 = X_1$ imply that $F(ker\pi_*) = (ker\pi_*)^\perp$. As a result, π is an anti-invariant Riemannian submersion.

Lemma 4.1. *Let π be an anti-invariant Riemannian submersion from a locally product manifold (M, g, F) to a Riemannian manifold $(N, g^{'})$. Then we have*

$$g(CY, FV) = 0 \tag{4.4}$$

and

$$g(\nabla_X CY, FV) = -g(CY, FA_X V) \tag{4.5}$$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$.

Proof. For $Y \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$, using (2.1) we have

$$g(CY, FV) = g(FY - BY, FV) = g(FY, FV)$$

due to $BY \in \Gamma(ker\pi_*)$ and $FV \in \Gamma((ker\pi_*)^\perp)$. Hence $g(FY, FV) = g(Y, V) = 0$ which is (4.4). Since M is a locally product manifold, using (4.4) we get

$$g(\nabla_X CY, FV) = -g(CY, F\nabla_X V)$$

for $X, Y \in \Gamma((ker\pi_*)^\perp)$ and $V \in \Gamma(ker\pi_*)$. Then using (2.7) we have

$$g(\nabla_X CY, FV) = -g(CY, FA_X V) - g(CY, Fv\nabla_X V).$$

Since $Fv\nabla_X V \in \Gamma(Fker\pi_*)$, we obtain (4.5). □

We now study the integrability of the distribution $(ker\,\pi_*)^\perp$ and then we investigate the geometry of leaves of $ker\,\pi_*$ and $(ker\pi_*)^\perp$. We note that it is known that the distribution $ker\,\pi_*$ is integrable.

Theorem 4.1. *Let π be an anti-invariant Riemannian submersion from a locally product manifold (M, g, F) to a Riemannian manifold $(N, g^{'})$. Then the following assertions are equivalent to each other;*

(i) *$(ker\pi_*)^\perp$ is integrable.*

(ii) $g^{'}((\nabla\pi_*)(Y,BX),\pi_*FV) = g^{'}((\nabla\pi_*)(X,BY),\pi_*FV) + g(CY,FA_XV) - g(CX,FA_YV)$.

(iii) $g(FV,A_XBY - A_YBX) = g(CY,FA_XV) - g(CX,FA_YV)$,

for $X,Y \in \Gamma((ker\pi_*)^{\perp})$ *and* $V \in \Gamma(ker\pi_*)$.

Proof. For $Y \in \Gamma((ker\pi_*)^{\perp})$ and $V \in \Gamma(ker\pi_*)$, we see from Definition 4.1, $FV \in \Gamma((ker\pi_*)^{\perp})$ and $FY \in \Gamma(ker\pi_* \oplus \mu)$. Thus using (2.1) and (2.2) we obtain

$$g([X,Y],V) = g(\nabla_XFY,FV) - g(\nabla_YFX,FV).$$

Then from (4.2) we get

$$\begin{aligned} g([X,Y],V) &= g(\nabla_XBY,FV) + g(\nabla_XCY,FV) \\ &\quad -g(\nabla_YBX,FV) - g(\nabla_YCX,FV). \end{aligned}$$

Since π is a Riemannian submersion, we have

$$\begin{aligned} g([X,Y],V) &= g^{'}(\pi_*\nabla_XBY,\pi_*FV) + g(\nabla_XCY,FV) \\ &\quad -g^{'}(\pi_*\nabla_YBX,\pi_*FV) - g(\nabla_YCX,FV). \end{aligned}$$

Thus, from (2.10) and (4.5) we obtain

$$\begin{aligned} g([X,Y],V) &= g^{'}(-(\nabla\pi_*)(X,BY) + (\nabla\pi_*)(Y,BX),\pi_*FV) \\ &\quad -g(CY,FA_XV) + g(CX,FA_YV) \end{aligned}$$

which proves (i)⇔(ii). On the other hand, using (2.10) we have

$$(\nabla\pi_*)(Y,BX) - (\nabla\pi_*)(X,BY) = -\pi_*(\nabla_YBX - \nabla_XBY).$$

Then (2.7) implies that

$$(\nabla\pi_*)(Y,BX) - (\nabla\pi_*)(X,BY) = -\pi_*(A_YBX - A_XBY).$$

Since $A_XBY - A_YBX \in \Gamma((ker\pi_*)^{\perp})$, this shows that (ii)⇔(iii). □

Theorem 4.2. *Let* π *be an anti-invariant Riemannian submersion from a locally product manifold* (M,g,F) *to a Riemannian manifold* $(N,g^{'})$. *Then the following assertions are equivalent to each other;*

(i) $(ker\pi_*)^{\perp}$ *defines a totally geodesic foliation on* M.

(ii) $g(A_XBY,FV) = g(CY,FA_XV)$.

(iii) $g^{'}((\nabla\pi_*)(X,BY),\pi_*FV) = -g(CY,FA_XV)$,

for $X,Y \in \Gamma((ker\pi_*)^{\perp})$ *and* $V \in \Gamma(ker\pi_*)$.

Proof. From (2.1), (2.2) and (2.7) we obtain

$$g(\nabla_XY,V) = g(A_XBY,FV) + g(\nabla_XCY,FV)$$

for $X,Y \in \Gamma((ker\pi_*)^{\perp})$ and $V \in \Gamma(ker\pi_*)$. Then using (4.5) we have

$$g(\nabla_XY,V) = g(A_XBY,FV) - g(FA_XV,CY)$$

which shows (i)⇔(ii). On the other hand from (2.7) and (2.10) we get

$$g(A_XBY,FV) = g^{'}(-(\nabla\pi_*)(X,BY),\pi_*FV).$$

This shows (ii)⇔(iii). □

Theorem 4.3. *Let* π *be an anti-invariant Riemannian submersion from a locally product manifold* (M,g,F) *to a Riemannian manifold* $(N,g^{'})$. *Then the following assertions are equivalent to each other;*

(i) $(ker\pi_*)$ *defines a totally geodesic foliation on* M.
(ii) $g^{'}((\nabla\pi_*)(V, FX), \pi_*FW) = 0$.
(iii) $T_V BX + A_{CX}V \in \Gamma(\mu)$,

for $X \in \Gamma((ker\pi_*)^{\perp})$ *and* $V, W \in \Gamma(ker\pi_*)$.

Proof. Using (2.1) and (2.2) we have $g(\nabla_V W, X) = g(\nabla_V FW, FX)$. Hence we get $g(\nabla_V W, X) = -g(h\nabla_V FX, FW)$. Then Riemannian submersion π and (2.10) imply that

$$g(\nabla_V W, X) = g^{'}((\nabla\pi_*)(V, FX), \pi_*FW)$$

which is (i)⇔(ii). By direct calculation, we derive

$$-g(\nabla_V FX, FW) = g^{'}((\nabla\pi_*)(V, FX), \pi_*FW).$$

Using (4.2) we obtain

$$-g(\nabla_V BX + \nabla_V CX, FW) = g^{'}((\nabla\pi_*)(V, FX), \pi_*FW).$$

Hence we have

$$-g(\nabla_V BX + [V, CX] + \nabla_{CX}V, FW) = g^{'}((\nabla\pi_*)(V, FX), \pi_*FW).$$

Since $[V, CX] \in \Gamma(ker\pi_*)$, using (2.5) and (2.7), we get

$$-g(T_V BX + A_{CX}V, FW) = g^{'}((\nabla\pi_*)(V, FX), \pi_*FW).$$

This shows (ii)⇔(iii). □

We say that an anti-invariant Riemannian submersion is a Lagrangian Riemannian submersion if $F(ker\pi_*) = (ker\pi_*)^{\perp}$. If $\mu \neq \{0\}$, then π is called a proper anti-invariant Riemannian submersion.
We note that the anti-invariant Riemannian submersion given in Example 4.1 is a Lagrangian Riemannian submersion.

If π is a Lagrangian submersion, then (4.3) implies that $TN = \pi_*(F(ker\pi_*))$. Hence we have the following.

Theorem 4.4. *Let* π *be a Lagrangian Riemannian submersion from a locally product manifold* (M, g, F) *to a Riemannian manifold* $(N, g^{'})$. *Then the following assertions are equivalent to each other;*

(i) $(ker\pi_*)$*defines a totally geodesic foliation on* M.
(ii) $(\nabla\pi_*)(V, FX) = 0$.
(iii) $T_V FW = 0$,

for $X \in \Gamma((ker\pi_*)^{\perp})$ *and* $V, W \in \Gamma(ker\pi_*)$.

Proof. (i)⇔(ii) is clear from Theorem 4.3. We only prove (ii)⇔(iii). From (2.10), we get

$$g^{'}((\nabla\pi_*)(V, FX), \pi_*FW) = -g(\nabla_V FX, FW) = g(\nabla_V FW, FX)$$

for $X \in \Gamma((ker\pi_*)^{\perp})$ and $V, W \in \Gamma(ker\pi_*)$. Then using (2.6) we have

$$g^{'}((\nabla\pi_*)(V, FX), \pi_*FW) = g(T_V FW, FX).$$

Since $T_V FW \in \Gamma(ker\pi_*)$, we get (ii)⇔(iii). □

We note that a differentiable map π between two Riemannian manifolds is called totally geodesic if $\nabla\pi_* = 0$.

Theorem 4.5. *Let π be a Lagrangian Riemannian submersion from a locally product manifold (M, g, F) to a Riemannian manifold $(N, g^{'})$. Then π is a totally geodesic map if and only if*

$$T_W FV = 0, \quad \forall W, V \in \Gamma(ker\pi_*)$$

and

$$A_X FW = 0, \quad \forall X \in \Gamma((ker\pi_*)^{\perp}).$$

Proof. First of all, we recall that the second fundamental form of a Riemannian submersion satisfies

$$(\nabla\pi_*)(X, Y) = 0, \;\; X, Y \in \Gamma((ker\pi_*)^{\perp}). \tag{4.6}$$

For $W, V \in \Gamma(ker\pi_*)$, by using (2.2), (2.6) and (2.10), we get

$$(\nabla\pi_*)(W, V) = -\pi_*(FT_W FV). \tag{4.7}$$

On the other hand, from (2.1), (2.2) and (2.10), we have

$$(\nabla\pi_*)(X, W) = -\pi_*(F\nabla_X FW)$$

for $X \in \Gamma((ker\pi_*)^{\perp})$. Then using (2.8), we get

$$(\nabla\pi_*)(X, W) = -\pi_*(FA_X FW). \tag{4.8}$$

Since F is non-singular, proof comes from (4.6)-(4.8) □

Finally, we give a necessary and sufficient condition for a Lagrangian Riemannian submersion to be harmonic.

Theorem 4.6. *Let π be a Lagrangian Riemannian submersion from a locally product manifold (M, g, F) to a Riemannian manifold $(N, g^{'})$. Then π is harmonic if and only if $TraceFT_V = 0$ for $V \in \Gamma(ker\pi_*)$.*

Proof. From [4] we know that π is harmonic if and only if π has minimal fibres. Thus π is harmonic if and only if $\sum_{i=1}^{r} T_{e_i} e_i = 0$. On the other hand, from (2.2), (2.5) and (2.6), we obtain

$$T_V FW = FT_V W \tag{4.9}$$

for any $V, W \in \Gamma(ker\pi_*)$. Using (4.9), we get

$$\sum_{i=1}^{r} g(T_{e_i} Fe_i, V) = \sum_{i=1}^{r} g(T_{e_i} e_i, FV)$$

for any $V \in \Gamma(ker\pi_*)$. Thus skew-symmetric T with respect to g implies that

$$-\sum_{i=1}^{r} g(T_{e_i} V, Fe_i) = \sum_{i=1}^{r} g(T_{e_i} e_i, FV)$$

Since T is symmetric, we obtain

$$-\sum_{i=1}^{r} g(T_V e_i, Fe_i) = \sum_{i=1}^{r} g(T_{e_i} e_i, FV).$$

□

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