PARALLEL LINEAR WEINGARTEN SURFACES IN $E^3$ AND $E^3_1$

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Abstract. In this paper we show that $M$ is a linear Weingarten surface if and only if $M_r$ is a linear Weingarten surface in $E^3$ and $E^3_1$. And also we determine the types of the pair $(M, M_r)$ according to the distance $r$.

1. Introduction

Let $M$ and $M_r$ be two surfaces in Euclidean space. The function
\[ f : M \rightarrow M_r \]
\[ p \rightarrow f(p) = p + rN_p \]
is called the parallelization function between $M$ and $M_r$ and furthermore $M_r$ is called parallel surface to $M$ where $N$ is the unit normal vector field on $M$ and $r$ is a given real number.

The Gaussian curvature and mean curvature of $M_r$ denoted by $K_r$ and $H_r$ are respectively
\begin{equation}
K_r = \frac{K}{1 + rH + r^2K} \quad \text{and} \quad H_r = \frac{H + 2rK}{1 + rH + r^2K},
\end{equation}
where $K$ and $H$ are Gaussian curvature and mean curvature of $M$ [1].

A surface $M$ in 3-dimensional Euclidean space $E^3$ is called a Weingarten surface if there is a relation between its two principal curvatures $k_1$ and $k_2$, that is, if there is a smooth function $W$ of two variables such that $W(k_1, k_2) = 0$ implies a relation $U(K, H) = 0$. In this paper we study Weingarten surfaces that satisfy the simplest case for $U$, that is, that $U$ is of the linear type

\begin{equation}
aH + bK = c,
\end{equation}
where $a, b, c \in \mathbb{R}$. We say that $M$ is a linear Weingarten surface and we abbreviate by LW-surface.
The behaviour of a LW-surface and its qualitative properties strongly depend on the sign of the discriminant $\Delta := a^2 + 4bc$. A surface $M$ is called hyperbolic if $\Delta < 0$, elliptic if $\Delta > 0$ and parabolic if $\Delta = 0$ [2,3].

2. Parallel Linear Weingarten Surfaces in $E^3$

**Theorem 2.1.** $M$ is a linear Weingarten surface if and only if $M_r$ is a linear Weingarten surface in $E^3$.

**Proof.** Let $M$ be a linear Weingarten surface. Then mean curvature $H$ and Gaussian curvature $K$ of $M$ satisfy a relation

$$aH + bK = c$$

where $a, b, c \in \mathbb{R}$. From (1.1) we obtain that

$$K = \frac{K_r}{1 - rH_r + r^2K_r} \quad \text{and} \quad H = \frac{H_r - 2rK_r}{1 - rH_r + r^2K_r}.$$ 

If we use these equations in (2.1) we get

$$a_rH_r + b_rK_r = c_r.$$ 

So that $M_r$ is a linear Weingarten surface.

Conversely we assume that $M_r$ is a linear Weingarten surface. Then the proof can be obtained with similar calculations. □

**Theorem 2.2.** Let $M$ be a LW-surface with $c = 0$ in $E^3$. Then $M$ and $M_r$ are elliptic LW-surface.

**Proof.** Since $\Delta = a^2 > 0$ and from (2.2) $\Delta_r = a^2 > 0$ then $M_r$ is an elliptic LW-surface. □

**Theorem 2.3.** Let $M$ be an elliptic LW-surface with $c > 0$ in $E^3$.

a) If $\frac{1}{c} (-a - \frac{2}{3}\sqrt{3(a^2 + bc)}) < r < \frac{1}{c} (-a + \frac{2}{3}\sqrt{3(a^2 + bc)})$ then $M_r$ is an elliptic LW-surface.

b) If $r < \frac{1}{c} (-a - \frac{2}{3}\sqrt{3(a^2 + bc)})$ or $r > \frac{1}{c} (-a + \frac{2}{3}\sqrt{3(a^2 + bc)})$ then $M_r$ is a hyperbolic LW-surface.

c) If $r = \frac{1}{c} (-a - \frac{2}{3}\sqrt{3(a^2 + bc)})$ or $r = \frac{1}{c} (-a + \frac{2}{3}\sqrt{3(a^2 + bc)})$ then $M_r$ is a parabolic LW-surface.

**Proof.** Let $M$ be an elliptic LW-surface with $c > 0$ in $E^3$. From (2.2)

$$\Delta_r = -3c^2r^2 - 6acr + \Delta.$$ 

Then the roots of $\Delta_r = 0$ are $r_1 = \frac{1}{c} (-a - \frac{2}{3}\sqrt{3(a^2 + bc)})$ and $r_2 = \frac{1}{c} (-a + \frac{2}{3}\sqrt{3(a^2 + bc)})$. So the proof is obvious. □
Theorem 2.4. Let $M$ be an elliptic LW-surface with $c < 0$ in $E^3$.

a) If $\frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right) < r < \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is an elliptic LW-surface.

b) If $r < \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ or $r > \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is a hyperbolic LW-surface.

c) If $r = \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ or $r = \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is a parabolic LW-surface.

Theorem 2.5. Let $M$ be a hyperbolic LW-surface with $c \neq 0$ in $E^3$.

a) If $a^2 < -bc$ then $M_r$ is a hyperbolic LW-surface.

b) Let $a^2 = -bc$.

b.i) If $r \neq -\frac{a}{c}$ then $M_r$ is a hyperbolic LW-surface.

b.ii) If $r = -\frac{a}{c}$ then $M_r$ is a parabolic LW-surface.

c) Let $-bc < a^2 < -4bc$ and $c > 0$.

c.i) If $\frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right) < r < \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is an elliptic LW-surface.

c.ii) If $r < \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ or $r > \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is a hyperbolic LW-surface.

c.iii) If $r = \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ or $r = \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is a parabolic LW-surface.

d) Let $-bc < a^2 < -4bc$ and $c < 0$.

d.i) If $\frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right) < r < \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is an elliptic LW-surface.

d.ii) If $r < \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ or $r > \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is a hyperbolic LW-surface.

d.iii) If $r = \frac{1}{c} \left(-a + \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ or $r = \frac{1}{c} \left(-a - \frac{2}{3} \sqrt{3(a^2 + bc)}\right)$ then $M_r$ is a parabolic LW-surface.

Theorem 2.6. Let $M$ be a parabolic LW-surface with $c > 0$ and $a > 0$ or $c < 0$ and $a < 0$ in $E^3$.

a) If $r < -\frac{2a}{c}$ or $r > 0$ then $M_r$ is a hyperbolic LW-surface.

b) If $-\frac{2a}{c} < r < 0$ then $M_r$ is an elliptic LW-surface.

c) If $r = 0$ or $r = -\frac{2a}{c}$ then $M_r$ is a parabolic LW-surface.

Theorem 2.7. Let $M$ be a parabolic LW-surface with $c > 0$ and $a < 0$ or $c < 0$ and $a > 0$ in $E^3$.

a) If $r < 0$ or $r > -\frac{2a}{c}$ then $M_r$ is a hyperbolic LW-surface.

b) If $0 < r < -\frac{2a}{c}$ then $M_r$ is an elliptic LW-surface.
Theorem 2.8. Let \( M_r \) be a LW-surface with \( c_r = 0 \) in \( E^3 \). Then \( M \) and \( M_r \) are elliptic LW-surface.

Theorem 2.9. Let \( M_r \) be an elliptic LW-surface with \( c_r > 0 \) in \( E^3 \).

a) If \( \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) < r < \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M_r \) is an elliptic LW-surface.

b) If \( r < \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) or \( r > \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M_r \) is a hyperbolic LW-surface.

c) If \( r = \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M_r \) is a parabolic LW-surface.

Theorem 2.10. Let \( M_r \) be an elliptic LW-surface with \( c_r < 0 \) in \( E^3 \).

a) If \( \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) < r < \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M_r \) is an elliptic LW-surface.

b) If \( r < \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) or \( r > \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M_r \) is a hyperbolic LW-surface.

c) If \( r = \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M_r \) is a parabolic LW-surface.

Theorem 2.11. Let \( M_r \) be a hyperbolic LW-surface with \( c_r \neq 0 \) in \( E^3 \).

a) If \( a_r^2 < -b_r c_r \) then \( M \) is a hyperbolic LW-surface.

b) Let \( a_r^2 = -b_r c_r \)

b.i) If \( r \neq \frac{a_r}{c_r} \) then \( M \) is a hyperbolic LW-surface.

b.ii) If \( r = \frac{a_r}{c_r} \) then \( M \) is a parabolic LW-surface.

c) Let \( -b_r c_r, a_r^2 < -4b_r c_r \) and \( c_r > 0 \).

c.i) If \( \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) < r < \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M \) is an elliptic LW-surface.

c.ii) If \( r < \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) or \( r > \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M \) is a hyperbolic LW-surface.

c.iii) If \( r = \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) or \( r = \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M \) is a parabolic LW-surface.

d) Let \( -b_r c_r, a_r^2 < -4b_r c_r \) and \( c_r < 0 \).

d.i) If \( \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) < r < \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M \) is an elliptic LW-surface.

d.ii) If \( r < \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) or \( r > \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3(a_r^2 + b_r c_r)} \right) \) then \( M \) is a hyperbolic LW-surface.
d.iii) If \( r = \frac{1}{c_r} \left( a_r + \frac{2}{3} \sqrt{3}(a_r^2 + b_rc_r) \right) \) or \( r = \frac{1}{c_r} \left( a_r - \frac{2}{3} \sqrt{3}(a_r^2 + b_rc_r) \right) \) then \( M \) is a parabolic LW-surface.

**Theorem 2.12.** Let \( M_r \) be a parabolic LW-surface with \( c_r > 0 \) and \( a_r > 0 \) or \( c_r < 0 \) and \( a_r < 0 \) in \( E^3 \).

a) If \( r < 0 \) or \( r > \frac{2a_r}{c_r} \) then \( M \) is a hyperbolic LW-surface.

b) If \( 0 < r < \frac{2a_r}{c_r} \) then \( M \) is an elliptic LW-surface.

c) If \( r = 0 \) or \( r = \frac{2a_r}{c_r} \) then \( M \) is a parabolic LW-surface.

**Theorem 2.13.** Let \( M_r \) be a parabolic LW-surface with \( c_r > 0 \) and \( a_r < 0 \) or \( c_r < 0 \) and \( a_r > 0 \) in \( E^3 \).

a) If \( r < \frac{-3-2\sqrt{3}}{9} \) or \( r > \frac{-3+2\sqrt{3}}{9} \) then \( M \) is a hyperbolic LW-surface.

b) If \( \frac{-3-2\sqrt{3}}{9} < r < \frac{-3+2\sqrt{3}}{9} \) then \( M \) is an elliptic LW-surface.

c) If \( r = \frac{-3-2\sqrt{3}}{9} \) or \( r = \frac{-3+2\sqrt{3}}{9} \) then \( M \) is a parabolic LW-surface.

**Example 2.1.** Let \( M \) be a sphere surface in \( E^3 \) given with the equation \( y_1^2 + y_2^2 + y_3^2 = 1 \). The Gaussian curvature and the mean curvature of \( M \) are respectively \( K = 1 \) and \( H = \frac{2}{r} \). If we take \( a = 1 \) and \( b = 1 \) then we obtain from the relation (2.1)

\[
\Delta r = \frac{-27}{r^2} - \frac{18}{r} + 13
\]

the roots of which are

\[
r_1 = \frac{-6-8\sqrt{3}}{18} \quad \text{and} \quad r_2 = \frac{-6+8\sqrt{3}}{18}.
\]

Therefore

a) If \( \frac{-3-4\sqrt{3}}{9} < r < \frac{-3+4\sqrt{3}}{9} \) then \( M \) is elliptic.

b) If \( r < \frac{-3-4\sqrt{3}}{9} \) or \( r > \frac{-3+4\sqrt{3}}{9} \) then \( M \) is hyperbolic.

c) If \( r = \frac{-3-2\sqrt{3}}{9} \) or \( r = \frac{-3+2\sqrt{3}}{9} \) then \( M \) is parabolic.

### 3. Parallel Surfaces in \( E^3_1 \)

**Definition 3.1.** Let \( M \) be a pseudo-Euclidean surface in \( E^3_1 \) and \( D \) be the Levi-Civita connection on \( E^3_1 \). Then,

\[
S : \chi(M) \to \chi(M), \quad X \to S(X) = DX\mathbf{N}
\]

is called the shape operator (Weingarten map), where \( \mathbf{N} \) is the unit normal vector on \( M \) [4].

**Definition 3.2.** Let \( M \) be a pseudo-Euclidean surface in \( E^3_1 \) and \( S \) be shape operator on \( M \), for \( p \in M \), \( K \) denotes Gauss curvature of \( M \) and defined as

\[
K : M \to R, \quad p \to K(p) = \varepsilon \det S_p
\]

where \( \varepsilon = \langle N, N \rangle = \pm 1 \) and \( N \) is the unit normal vector field on \( M \) [5].

**Definition 3.3.** Let \( M \) be a pseudo-Euclidean surface in \( E^3_1 \) and \( H \) denotes mean curvature of \( M \) and defined as \( H = \varepsilon \| S_p \| \) where \( \varepsilon = \langle N, N \rangle = \pm 1 \) and \( N \) is the unit normal vector field on \( M \) [5].
Note that the principal curvatures of the Weingarten map on $M$ can be obtained easily

$$2k_1 = H + \sqrt{H^2 - 4\varepsilon K}$$

and

$$2k_2 = H - \sqrt{H^2 - 4\varepsilon K}.$$ 

Let $M$ be a pseudo-Euclidean surface with $N = (a_1, a_2, a_3)$ where each $a_i$ is a real valued $C^\infty$ function on $M$ and $-a_1^2 + a_2^2 + a_3^2 = \pm 1$. For any constant $r$ in $R$, let $M_r = \{P + rN_p : P \in M\}$. Thus if $P = (p_1, p_2, p_3)$ is on $M$, then $f(P) = P + rN_p = (p_1 + ra_1(p), p_2 + ra_2(p), p_3 + ra_3(p))$ defines a new surface $M_r$. The map $f$ is called the natural map on $M$ into $M_r$, and if $f$ is univalent, then $M_r$ is a parallel surface of $M$ with unit normal $N$, i.e., $N_{f(p)} = N_p$ for all $P$ in $M$.

**Theorem 3.1.** Let $M$ and $M_r$ be two parallel pseudo-Euclidean surface in $E_1^3$ and $S_r$ be the Weingarten map on $M_r$. Let

$$f : M \to M_r$$

be a parallelization function. Then for $X \in X(M)$,

1. $f_*(X) = X + rS(X)$
2. $S_r(f_*(X)) = S(X)$
3. $f$ preserves principal directions of curvature, that is

$$S_r(f_*(X)) = \frac{k}{1 + rk}f_*(X)$$

where $k$ is a principal curvature of $M$ at $p$ in direction $X$ [4].

**Theorem 3.2.** Let $M$ and $M_r$ be two parallel pseudo-Euclidean surface in $E_1^3$. Then we have

$$K_r = \frac{K}{1 + \varepsilon rH + \varepsilon r^2 K}$$

and

$$H_r = \frac{H + 2rK}{1 + \varepsilon rH + \varepsilon r^2 K}$$

where $\langle N_r, N_r \rangle = \varepsilon$ and Gaussian curvature and mean curvature of $M$ (and $M_r$) be denoted by $K$ (and $K_r$) and $H$ (and $H_r$) [6].

**Theorem 3.3.** Let $M$ is a regular surface with no umbilic points and such that its Gaussian curvature does not vanish.

If $M$ has constant mean curvature $H > 0$, then there exist two surfaces parallel to $M$ such that one has constant positive Gaussian curvature $K_r = \varepsilon H^2$ and the other one has constant mean curvature equal to $-H$.

If $\varepsilon K$ is positive constant, then there exist two surfaces parallel to $M$ at the distance $r = \pm \sqrt{\varepsilon K}$ whose mean curvatures are constant and equal to $H = \pm \varepsilon \sqrt{\varepsilon K}$.

**Proof.** Suppose $M$ has constant mean curvature $H > 0$. Substituting $r = -\frac{\varepsilon}{H}$ into (3.1) and (3.2) we get,
By assumption, we have $K \neq 0$. So the parallel surface at distance $-\frac{\varepsilon}{H}$ has constant Gaussian curvature $\varepsilon H^2$.

Substituting $r = -\frac{2\varepsilon}{H}$ into (3.1) and (3.2) we get,

$$K_r = \frac{K}{1 - \varepsilon \frac{2}{H} H + \varepsilon \frac{1}{H^2} H^2} = \varepsilon H^2$$

$$H_r = \frac{H - 2 \varepsilon \frac{K}{H}}{1 - \varepsilon \frac{2}{H} H + \varepsilon \frac{1}{H^2} H^2} = \varepsilon H^3 - 2 \varepsilon H^2$$

We have

$$-H^2 + 4 \varepsilon K = 0 \iff -(k_1 + k_2)^2 + 4 \varepsilon (k_1 k_2) = (k_1 - k_2)^2 = 0 \iff k_1 = k_2$$

By assumption $M$ has no umbilic points, so $-H^2 + 4 \varepsilon K \neq 0$. So the parallel surface at distance $-\frac{2\varepsilon}{H}$ has constant mean curvature $-H$. The rest of the theorem can be proven with similar arguments. \[\square\]

**Theorem 3.4.** Let $M \subset E_1^3$ be a regular surface.

i) If $M$ has non-zero Gaussian curvature and constant mean curvature $H = -\frac{\varepsilon}{r}$, then the parallel surface $M_r$ has constant Gaussian curvature $K_r = \frac{\varepsilon}{r^2}$.

ii) If $M$ has Gaussian curvature $K \neq \frac{\varepsilon}{4r^2}$ and constant mean curvature $H = -\frac{\varepsilon}{r}$, then the parallel surface $M_{2r}$ has constant mean curvature $H_{2r} = \frac{\varepsilon}{r}$.

iii) If $M$ has Gaussian curvature $K = \frac{\varepsilon}{r^2}$ and constant mean curvature $H \neq \pm \frac{2\varepsilon}{r}$, then the parallel surface $M_{\pm r}$ has constant mean curvature $H_{\pm r} = \pm \frac{\varepsilon}{r}$.

**Proof.** If $H = -\frac{\varepsilon}{r}$, then it follows from (3.1), that

$$K_r = \frac{K}{1 - \frac{1}{r} + \varepsilon r^2 K} = \frac{\varepsilon}{r^2}$$

(ii) and (iii) follow from (3.1) and (3.2) in similar fashion. \[\square\]

**Theorem 3.5.** Let $M \subset E^3$ be a regular surface with constant positive curvature $\varepsilon a^{-2}$ where $a > 0$. Let $M_r$ denote the surface parallel to $M$ at a distance $r$. Suppose

$$K_r = \frac{K}{1 - \frac{1}{r} + \varepsilon r^2 K} = \frac{\varepsilon}{r^2}$$

(i) and (ii) follow from (3.1) and (3.2) in similar fashion.
that the umbilic points of $M$ are isolated. If $M_r$ has constant mean curvature, then $r = \pm a$.

**Proof.** Fix $r$, and suppose that $H_r$ is constant on $M_r$. Then (3.2) implies that

$$
\varepsilon k_1 (1 + rk_2) + \varepsilon k_2 (1 + rk_1) = H_r (1 + rk_1)(1 + rk_2)
$$

$$
\varepsilon (k_1 + k_2) + 2ra^{-2} = H_r (1 + r(k_1 + k_2) + r^2a^{-2}).
$$

Hence

(3.3) 
$$(k_1 + k_2)(\varepsilon - rH_r) = H_r + H_r r^2a^{-2} - 2r\varepsilon a^{-2}. $$

By hypothesis, the right hand of (3.3) is constant. But if the left hand of (3.3) constant, it must vanish at the nonumbilic points of $M_r$. Hence

(3.4) 
$$\varepsilon - rH_r = 0 $$

at the nonumbilic points of $M_r$. Then (3.3) and (3.4) imply that

$$
0 = H_r + H_r r^2a^{-2} - 2\varepsilon ra^{-2}
$$

$$
= H_r (1 + r^2a^{-2}) - 2\varepsilon ra^{-2}
$$

$$
= \frac{\varepsilon}{r} \left( 1 + \frac{r^2}{a^2} \right) - \frac{2\varepsilon r}{a^2}
$$

$$
= \frac{\varepsilon}{r} - \frac{\varepsilon r}{a^2}.
$$

Therefore, $r = \pm a$. 

4. **Parallel Linear Weingarten Surfaces in $E^3_1$**

**Theorem 4.1.** $M$ is a linear Weingarten surface if and only if $M_r$ is a linear Weingarten surface in $E^3_1$.

**Proof.** It can be proved easily following the same procedure as in the Theorem 2.1. 

Let $M$ (or $M_r$) be a timelike surface. Since $\varepsilon = 1$ the Gaussian and the mean curvature of $M$ (or $M_r$) are

$$
K = \frac{K_r}{1 - rH_r + r^2K_r} \quad \text{and} \quad H = \frac{H_r - 2rK_r}{1 - rH_r + r^2K_r}
$$

or

$$
K_r = \frac{K}{1 + rH + r^2K} \quad \text{and} \quad H_r = \frac{H + 2rK}{1 + rH + r^2K}.
$$

These formulas are the same for any surface in $E^3$. Therefore Theorem 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 are valid for $M$ and Theorem 2.8, 2.9, 2.10, 2.11, 2.12, 2.13 are valid for $M_r$ in $E^3_1$. 

Because of that in this section we give the theorems for only spacelike surfaces.

**Theorem 4.2.** Let $M$ be a spacelike LW-surface with $c = 0$ in $E^3_1$. Then $M$ and $M_r$ are elliptic LW-surface.

**Theorem 4.3.** Let $M$ be a spacelike elliptic LW-surface with $c > 0$ in $E^3_1$

a) If $a^2 < bc$ then $M$ is an elliptic LW-surface.

b) Let $a^2 = bc$.

b.i) If $r \neq \frac{a}{c}$ or then $M_r$ is an elliptic LW-surface.

b.ii) If $r = \frac{a}{c}$ then $M_r$ is a spacelike parabolic LW-surface.

c) Let $a^2 > bc$ and $c > 0$.

c.i) If $\frac{1}{c} \left( a - \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right) < r < \frac{1}{c} \left( a + \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ then $M_r$ is an hyperbolic LW-surface.

c.ii) If $r < \frac{1}{c} \left( a - \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ or $r > \frac{1}{c} \left( a + \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ then $M_r$ is an elliptic LW-surface.

***c.iii) If $r = \frac{1}{c} \left( a - \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ or $r = \frac{1}{c} \left( a + \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ then $M_r$ is a parabolic LW-surface.***

**Theorem 4.4.** Let $M$ be a spacelike hyperbolic LW-surface with $c \neq 0$ in $E^3_1$.

a) If $a^2 < bc$ then $M_r$ is an elliptic LW-surface.

b) Let $a^2 = bc$.

b.i) If $r \neq \frac{a}{c}$ or then $M_r$ is an elliptic LW-surface.

b.ii) If $r = \frac{a}{c}$ then $M_r$ is a parabolic LW-surface.

c) Let $bc < a^2 < -4bc$ and $c > 0$.

c.i) If $\frac{1}{c} \left( a - \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right) < r < \frac{1}{c} \left( a + \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ then $M_r$ is an hyperbolic LW-surface.

c.ii) If $r < \frac{1}{c} \left( a - \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ or $r > \frac{1}{c} \left( a + \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ then $M_r$ is an elliptic LW-surface.

***c.iii) If $r = \frac{1}{c} \left( a - \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ or $r = \frac{1}{c} \left( a + \frac{2}{\sqrt{5}} \sqrt{5(a^2 - bc)} \right)$ then $M_r$ is a parabolic LW-surface.***

d) Let $bc < a^2 < -4bc$ and $c < 0$. 


Theorem 4.8. Let \( M \) be a spacelike parabolic LW-surface with \( c > 0 \) and \( a > 0 \) or \( c < 0 \) and \( a < 0 \) in \( E^3 \).

a) If \( r < \frac{2a}{c} \), then \( M \) is an elliptic LW-surface.

b) If \( 0 < r < \frac{2a}{c} \), then \( M \) is a hyperbolic LW-surface.

c) If \( r = 0 \) or \( r > \frac{2a}{c} \), then \( M \) is a parabolic LW-surface.

Theorem 4.7. Let \( M \) be an elliptic LW-surface with \( c > 0 \) and \( a > 0 \). Then \( M \) is a hyperbolic LW-surface.

Theorem 4.6. Let \( M \) be a spacelike parabolic LW-surface with \( c > 0 \) and \( a < 0 \). Then \( M \) is an elliptic LW-surface.

Theorem 4.5. Let \( M \) be a spacelike parabolic LW-surface with \( c > 0 \) and \( a > 0 \) or \( c < 0 \) and \( a < 0 \) in \( E^3 \).

a) If \( r < \frac{2a}{c} \), then \( M \) is an elliptic LW-surface.

b) If \( 0 < r < \frac{2a}{c} \), then \( M \) is a hyperbolic LW-surface.

c) If \( r = 0 \) or \( r > \frac{2a}{c} \), then \( M \) is a parabolic LW-surface.

Theorem 4.8. Let \( M_r \) be a spacelike elliptic LW-surface with \( c_r \neq 0 \) in \( E^3 \).

a) If \( a^2 < b_r c_r \), then \( M \) is a parabolic LW-surface.

b) Let \( a_r^2 = b_r c_r \).

c) If \( r \neq - \frac{a_r}{c_r} \) or \( r > - \frac{a_r}{c_r} \), then \( M \) is an elliptic LW-surface.

b.ii.) If \( r = - \frac{a_r}{c_r} \), then \( M \) is a parabolic LW-surface.

Theorem 4.8. Let \( M \) be a hyperbolic LW-surface.

d) If \( a^2 > b_r c_r \) and \( c_r > 0 \).

c) If \( \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a^2_r - b_r c_r)} \right) < r < \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a^2_r - b_r c_r)} \right) \), then \( M \) is a hyperbolic LW-surface.

Theorem 4.8. Let \( M \) be an elliptic LW-surface.

c) If \( r < \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a^2_r - b_r c_r)} \right) \) or \( r > \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a^2_r - b_r c_r)} \right) \), then \( M \) is an elliptic LW-surface.

d) Let \( a^2 > b_r c_r \) and \( c_r < 0 \).

c) If \( \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a^2_r - b_r c_r)} \right) < r < \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a^2_r - b_r c_r)} \right) \), then \( M \) is a hyperbolic LW-surface.
Theorem 4.11. 

c \quad \text{or} \quad c \\
then \ M \text{ is a hyperbolic LW-surface.}

\text{d.ii) If } r = \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ or } r = \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is a parabolic LW-surface.}

Theorem 4.9. Let \ M_r \ be a spacelike hyperbolic LW-surface with \ c_r \neq 0 \ in \ E^1_3.

\ a) \text{ If } a_r^2 < b_r c_r \text{ then } M \text{ is an elliptic LW-surface.}

\ b) \text{ Let } a_r^2 = b_r c_r.

\ b.i) \text{ If } r < -\frac{a_r}{c_r} \text{ or } r > \frac{a_r}{c_r} \text{ then } M \text{ is an elliptic LW-surface.}

\ b.ii) \text{ If } r = -\frac{a_r}{c_r} \text{ then } M \text{ is a parabolic LW-surface.}

\ c) \text{ Let } b_r c_r < a_r^2 < -4 b_r c_r \text{ and } c_r > 0.

\ c.i) \text{ If } \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) < r < \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is a hyperbolic LW-surface.}

\ c.ii) \text{ If } r < \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ or } r > \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is an elliptic LW-surface.}

\ c.iii) \text{ If } r = \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ or } r = \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is a parabolic LW-surface.}

\ d) \text{ Let } a_r^2 > b_r c_r \text{ and } c_r < 0.

\ d.i) \text{ If } \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) < r < \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is a hyperbolic LW-surface.}

\ d.ii) \text{ If } r < \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ or } r > \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is an elliptic LW-surface.}

\ d.iii) \text{ If } r = \frac{1}{c_r} \left( -a_r + \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ or } r = \frac{1}{c_r} \left( -a_r - \frac{2}{5} \sqrt{5(a_r^2 - b_r c_r)} \right) \text{ then } M \text{ is a parabolic LW-surface.}

Theorem 4.10. Let \ M_r \ be a spacelike parabolic LW-surface with \ c_r > 0 \ and \ a_r > 0 \ or \ c_r < 0 \ and \ a_r < 0 \ in \ E^1_3.

\ a) \text{ If } r < -\frac{2a_r}{c_r} \text{ or } r > 0 \text{ then } M \text{ is an elliptic LW-surface.}

\ b) \text{ If } \frac{-2a_r}{c_r} < r < 0 \text{ then } M \text{ is a hyperbolic LW-surface.}

\ c) \text{ If } r = 0 \text{ or } r = \frac{-2a_r}{c_r} \text{ then } M \text{ is a parabolic LW-surface.}

Theorem 4.11. Let \ M_r \ be a spacelike parabolic LW-surface with \ c_r > 0 \ and \ a_r < 0 \ or \ c_r < 0 \ and \ a_r > 0 \ in \ E^1_3.

\ a) \text{ If } r < 0 \text{ or } r > -\frac{2a_r}{c_r} \text{ then } M \text{ is an elliptic LW-surface.}

\ b) \text{ If } 0 < r < -\frac{2a_r}{c_r} \text{ then } M \text{ is a hyperbolic LW-surface.}

\ c) \text{ If } r = 0 \text{ or } r = -\frac{2a_r}{c_r} \text{ then } M \text{ is a parabolic LW-surface.}
References


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