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PARALLEL LINEAR WEINGARTEN SURFACES IN E^3 AND E_1^3

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Abstract. In this paper we show that M is a linear Weingarten surface if and only if M_r is a linear Weingarten surface in E^3 and E_1^3. And also we determine the types of the pair (M, M_r) according to the distance r.

1. Introduction

Let M and M_r be two surfaces in Euclidean space. The function

$$\begin{array}{lccl} f: & M & \to & M_r \\ & p & \to & f(p) = p + rN_p \end{array}$$

is called the parallelization function between M and M_r and furthermore M_r is called parallel surface to M where N is the unit normal vector field on M and r is a given real number.

The Gaussian curvature and mean curvature of M_r denoted by K_r and H_r are respectively

$$K_r = \frac{K}{1 + rH + r^2K} \quad \text{and} \quad H_r = \frac{H + 2rK}{1 + rH + r^2K} \tag{1.1}$$

where K and H are Gaussian curvature and mean curvature of M [1].

A surface M in 3-dimensional Euclidean space E^3 is called a Weingarten surface if there is a relation between its two principal curvatures k_1 and k_2, that is, if there is a smooth function W of two variables such that $W(k_1, k_2) = 0$ implies a relation $U(K, H) = 0$. In this paper we study Weingarten surfaces that satisfy the simplest case for U, that is, that U is of the linear type

$$aH + bK = c, \tag{1.2}$$

where $a, b, c \in R$. We say that M is a linear Weingarten surface and we abbreviate by LW-surface.

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The behaviour of a LW-surface and its qualitative properties strongly depend on the sign of the distriminant $\Delta := a^2 + 4bc$. A surface M is called hyperbolic if $\Delta < 0$, elliptic if $\Delta > 0$ and parabolic if $\Delta = 0$ [2,3]

2. Parallel Linear Weingarten Surfaces in E^3

Theorem 2.1. *M is a linear Weingarten surface if and only if M_r is a linear Weingarten surface in E^3.*

Proof. Let M be a linear Weingarten surface. Then mean curvature H and Gaussian curvature K of M satisfy a relation

$$aH + bK = c \tag{2.1}$$

where $a, b, c \in R$. From (1.1) we obtain that

$$K = \frac{K_r}{1 - rH_r + r^2 K_r} \quad \text{and} \quad H = \frac{H_r - 2rK_r}{1 - rH_r + r^2 K_r}.$$

If we use these equations in (2.1) we get

$$(a + cr)H_r + (b - 2ar - cr^2)K_r = c. \tag{2.2}$$

In (2.2) if we take $a + cr = a_r$, $b - 2ar - cr^2 = b_r$ and $c = c_r$ then

$$a_r H_r + b_r K_r = c_r.$$

So that M_r is a linear Weingarten surface.

Conversely we assume that M_r is a linear Weingarten surface. Then the proof can be obtained with similar calculations. □

Theorem 2.2. *Let M be a LW-surface with $c = 0$ in E^3. Then M and M_r are elliptic LW-surface.*

Proof. Since $\Delta = a^2 > 0$ and from (2.2) $\Delta_r = a^2 > 0$ then M_r is an elliptic LW-surface. □

Theorem 2.3. *Let M be an elliptic LW-surface with $c > 0$ in E^3.*

a) If $\frac{1}{c}\left(-a - \frac{2}{3}\sqrt{3(a^2 + bc)}\right) < r < \frac{1}{c}\left(-a + \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$ then M_r is an elliptic LW-surface.

b) If $r < \frac{1}{c}\left(-a - \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$ or $r > \frac{1}{c}\left(-a + \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$ then M_r is a hyperbolic LW-surface.

c) If $r = \frac{1}{c}\left(-a - \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$ or $r = \frac{1}{c}\left(-a + \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$ then M_r is a parabolic LW-surface.

Proof. Let M be an elliptic LW-surface with $c > 0$ in E^3. From (2.2)

$$\Delta_r = -3c^2 r^2 - 6acr + \Delta.$$

Then the roots of $\Delta_r = 0$ are $r_1 = \frac{1}{c}\left(-a - \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$ and $r_2 = \frac{1}{c}\left(-a + \frac{2}{3}\sqrt{3(a^2 + bc)}\right)$.
So the proof is obvious. □

Theorem 2.4. *Let M be an elliptic LW-surface with $c<0$ in E^3.*

a) If $\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right) < r < \frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is an elliptic LW-surface.*

b) If $r<\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *or* $r>\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is a hyperbolic LW-surface.*

c) If $r=\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *or* $r=\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is a parabolic LW-surface.*

Theorem 2.5. *Let M be a hyperbolic LW-surface with $c\neq 0$ in E^3.*

a) If $a^2<-bc$ then M_r is a hyperbolic LW-surface.

b) Let $a^2=-bc$.

b.i) If $r\neq -\frac{a}{c}$ then M_r is a hyperbolic LW-surface.

b.ii) If $r=-\frac{a}{c}$ then M_r is a parabolic LW-surface.

c) Let $-bc<a^2<-4bc$ and $c>0$.

c.i) If $\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right) < r < \frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is an elliptic LW-surface.*

c.ii) If $r<\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *or* $r>\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is a hyperbolic LW-surface.*

c.iii) If $r=\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *or* $r=\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is a parabolic LW-surface.*

d) Let $-bc<a^2<-4bc$ and $c<0$

d.i) If $\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right) < r < \frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is an elliptic LW-surface.*

d.ii) If $r<\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *or* $r>\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is a hyperbolic LW-surface.*

d.iii) If $r=\frac{1}{c}\left(-a+\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *or* $r=\frac{1}{c}\left(-a-\frac{2}{3}\sqrt{3(a^2+bc)}\right)$ *then M_r is a parabolic LW-surface.*

Theorem 2.6. *Let M be a parabolic LW-surface with $c>0$ and $a>0$ or $c<0$ and $a<0$ in E^3.*

a)If $r<-\frac{2a}{c}$ or $r>0$ then M_r is a hyperbolic LW-surface.

b) If $-\frac{2a}{c}<r<0$ then M_r is an elliptic LW-surface.

c) If $r=0$ or $r=-\frac{2a}{c}$ then M_r is a parabolic LW-surface.

Theorem 2.7. *Let M be a parabolic LW-surface with $c>0$ and $a<0$ or $c<0$ and $a>0$ in E^3.*

a) If $r<0$ or $r>-\frac{2a}{c}$ then M_r is a hyperbolic LW-surface.

b) If $0<r<-\frac{2a}{c}$ then M_r is an elliptic LW-surface.

c) If $r = 0$ *or* $r = -\dfrac{2a}{c}$ *then* M_r *is a parabolic LW-surface.*

Theorem 2.8. *Let* M_r *be a LW-surface with* $c_r = 0$ *in* E^3*. Then* M *and* M_r *are elliptic LW-surface.*

Theorem 2.9. *Let* M_r *be an elliptic LW-surface with* $c_r > 0$ *in* E^3*.*

a) If $\dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right) < r < \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M_r *is an elliptic LW-surface.*

b) If $r < \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r > \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M_r *is a hyperbolic LW-surface.*

c) If $r = \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r = \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M_r *is a parabolic LW-surface.*

Theorem 2.10. *Let* M_r *be an elliptic LW-surface with* $c_r < 0$ *in* E^3*.*

a) If $\dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right) < r < \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M_r *is an elliptic LW-surface.*

b) If $r < \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r > \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M_r *is a hyperbolic LW-surface.*

c) If $r = \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r = \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M_r *is a parabolic LW-surface.*

Theorem 2.11. *Let* M_r *be a hyperbolic LW-surface with* $c_r \neq 0$ *in* E^3*.*

a) If $a_r^2 < -b_r c_r$ *then* M *is a hyperbolic LW-surface.*

b) Let $a_r^2 = -b_r cr$

b.i) If $r \neq \dfrac{a_r}{c_r}$ *then* M *is a hyperbolic LW-surface.*

b.ii) If $r = \dfrac{a_r}{c_r}$ *then* M *is a parabolic LW-surface.*

c) Let $-b_r c_r < a_r^2 < -4 b_r c_r$ *and* $c_r > 0$*.*

c.i) If $\dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right) < r < \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M *is an elliptic LW-surface.*

c.ii) If $r < \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r > \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M *is a hyperbolic LW-surface.*

c.iii) If $r = \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r = \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M *is a parabolic LW-surface.*

d) Let $-b_r c_r < a_r^2 < -4 b_r c_r$ *and* $c_r < 0$*.*

d.i) If $\dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right) < r < \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M *is an elliptic LW-surface.*

d.ii) If $r < \dfrac{1}{c_r}\left(a_r + \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r > \dfrac{1}{c_r}\left(a_r - \dfrac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M *is a hyperbolic LW-surface.*

d.iii) If $r = \frac{1}{c_r}\left(a_r + \frac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *or* $r = \frac{1}{c_r}\left(a_r - \frac{2}{3}\sqrt{3(a_r^2 + b_r c_r)}\right)$ *then* M *is a parabolic LW-surface.*

Theorem 2.12. *Let* M_r *be a parabolic LW-surface with* $c_r > 0$ *and* $a_r > 0$ *or* $c_r < 0$ *and* $a_r < 0$ *in* E^3

a) If $r < 0$ *or* $r > \frac{2a_r}{c_r}$ *then* M *is a hyperbolic LW-surface.*

b) If $0 < r < \frac{2a_r}{c_r}$ *then* M *is an elliptic LW-surface.*

c) If $r = 0$ *or* $r = \frac{2a_r}{c_r}$ *then* M *is a parabolic LW-surface.*

Theorem 2.13. *Let* M_r *be a parabolic LW-surface with* $c_r > 0$ *and* $a_r < 0$ *or* $c_r < 0$ *and* $a_r > 0$ *in* E^3.

a) If $r < \frac{2a_r}{c_r}$ *or* $r > 0$ *then* M *is a hyperbolic LW-surface.*

b) If $\frac{2a_r}{c_r} < r < 0$ *then* M *is an elliptic LW-surface.*

c) If $r = \frac{2a_r}{c_r}$ *or* $r = 0$ *then* M *is a parabolic LW-surface.*

Example 2.1. Let M be a sphere surface in E^3 given with the equation $y_1^2 + y_2^2 + y_3^2 = 1$. The Gaussian curvature and the mean curvature of M are respectively $K = 1$ and $H = 2$. If we take $a = 1$ and $b = 1$ then we obtain from the relation (2.1) $c = 3 > 0$. So that $\Delta_r = -27r^2 - 18r + 13$ and the roots of this equation are $r_1 = \frac{-6 - 8\sqrt{3}}{18}$ and $r_2 = \frac{-6 + 8\sqrt{3}}{18}$. Therefore

a) If $\frac{-3 - 4\sqrt{3}}{9} < r < \frac{-3 + 4\sqrt{3}}{9}$ then M_r is elliptic.

b) If $r < \frac{-3 - 4\sqrt{3}}{9}$ or $r > \frac{-3 + 4\sqrt{3}}{9}$ then M_r is hyperbolic.

c) If $r = \frac{-3 - 2\sqrt{3}}{9}$ or $r = \frac{-3 + 2\sqrt{3}}{9}$ then M_r is parabolic.

3. Parallel Surfaces in E_1^3

Definition 3.1. Let M be a pseudo-Euclidean surface in E_1^3 and D be the Levi-Civita connection on E_1^3. Then,

$$S : \chi(M) \to \chi(M), \qquad X \to S(X) = D_X N$$

is called the shape operator (Weingarten map), where N is the unit normal vector on M [4].

Definition 3.2. Let M be a pseudo-Euclidean surface in E_1^3 and S be shape operator on M, for $p \in M$, K denotes Gauss curvature of M and defined as

$$\begin{array}{lccl} K : & M & \to & R \\ & p & \to & K(p) = \varepsilon \det S_p \end{array}$$

where $\varepsilon = \langle N, N\rangle = \pm 1$ and N is the unit normal vector field on M [5].

Definition 3.3. Let M be a pseudo-Euclidean surface in E_1^3 and H denotes mean curvature of M and defined as $H = \varepsilon iz S_p$ where $\varepsilon = \langle N, N\rangle = \pm 1$ and N is the unit normal vector field on M [5].

Note that the principal curvatures of the Weingarten map on M can be obtained easily

$$2k_1 = H + \sqrt{H^2 - 4\varepsilon K}$$

and

$$2k_2 = H - \sqrt{H^2 - 4\varepsilon K}.$$

Let M be a pseudo-Euclidean surface with $N = (a_1, a_2, a_3)$ where each a_i is a real valued C^∞ function on M and $-a_1^2 + a_2^2 + a_3^2 = \pm 1$. For any constant r in R, let $M_r = \{P + rN_p : P \in M\}$. Thus if $P = (p_1, p_2, p_3)$ is on M, then $f(P) = P + rN_p = (p_1 + ra_1(p), p_2 + ra_2(p), p_3 + ra_3(p))$ defines a new surface M_r. The map f is called the natural map on M into M_r, and if f is univalent, then M_r is a parallel surface of M with unit normal N, i.e., $N_{f(p)} = N_p$ for all P in M.

Theorem 3.1. *Let M and M_r be two parallel pseudo-Euclidean surface in E_1^3 and S_r be the Weingarten map on M_r. Let*

$$f : M \to M_r$$

be a parallellization function. Then for $X \in X(M)$,

1. *$f_*(X) = X + r\overline{S(X)}$*
2. *$S_r(f_*(X) = \overline{S(X)}$*
3. *f preserves principal directions of curvature, that is*

$$S_r(f_*(X)) = \frac{k}{1+rk} f_*(X)$$

where k is a principal curvature of M at p in direction X [4].

Theorem 3.2. *Let M and M_r be two parallel pseudo-Euclidean surface in E_1^3. Then we have*

$$K_r = \frac{K}{1 + \varepsilon r H + \varepsilon r^2 K} \tag{3.1}$$

and

$$H_r = \frac{H + 2rK}{1 + \varepsilon r H + \varepsilon r^2 K} \tag{3.2}$$

where $\langle N_r, N_r \rangle = \varepsilon$ and Gaussian curvature and mean curvature of M (and M_r) be denoted by K (and K_r) and H (and H_r) [6].

Theorem 3.3. *Let M is a regular surface with no umbilic points and such that its Gaussian curvature does not vanish.*

If M has constant mean curvature $H > 0$, then there exist two surfaces parallel to M such that one has constant positive Gaussian curvature $K_r = \varepsilon H^2$ and the other one has constant mean curvature equal to $-H$.

If εK is positive constant, then there exist two surfaces parallel to M at the distance $r = \pm\sqrt{\varepsilon K}$ whose mean curvatures are constant and equal to $H = \pm\varepsilon\sqrt{\varepsilon K}$.

Proof. Suppose M has constant mean curvature $H > 0$. Substituting $r = -\dfrac{\varepsilon}{H}$ into (3.1) and (3.2) we get,

$$K_r = \frac{K}{1-\varepsilon\frac{\varepsilon}{H}H+\varepsilon\frac{1}{H^2}K} = \varepsilon H^2$$

$$H_r = \frac{H-2\frac{\varepsilon}{H}K}{1-\varepsilon\frac{\varepsilon}{H}H+\varepsilon\frac{1}{H^2}K} = \frac{\varepsilon H^3-2H^2K}{K}$$

By assumption, we have $K \neq 0$. So the parallel surface at distance $-\frac{\varepsilon}{H}$ has constant Gaussian curvature εH^2.

Substituting $r = -\frac{2\varepsilon}{H}$ into (3.1) and (3.2) we get,

$$K_r = \frac{K}{1-\varepsilon\frac{2\varepsilon}{H}H+\frac{4\varepsilon}{H^2}K} = \frac{H^2K}{-H^2+4\varepsilon K}$$

$$H_r = \frac{H-4\frac{\varepsilon}{H}K}{1-\varepsilon\frac{2\varepsilon}{H}H+\varepsilon\frac{4}{H^2}K} = -H$$

We have

$$-H^2+4\varepsilon K = 0 \Longleftrightarrow -(k_1+k_2)^2+4\varepsilon(\varepsilon k_1k_2) = (k_1-k_2)^2 = 0 \Longleftrightarrow k_1 = k_2$$

By assumption M has no umbilic points, so $-H^2+4\varepsilon K \neq 0$. So the parallel surface at distance $-\frac{2\varepsilon}{H}$ has constant mean curvature $-H$. The rest of the theorem can be proven with similar arguments. □

Theorem 3.4. *Let $M \subset E^3_1$ be a regular surface.*

i) If M has non-zero Gaussian curvature and constant mean curvature $H = -\frac{\varepsilon}{r}$, then the parallel surface M_r has constant Gaussian curvature $K_r = \frac{\varepsilon}{r^2}$.

ii) If M has Gaussian curvature $K \neq \frac{\varepsilon}{4r^2}$ and constant mean curvature $H = -\frac{\varepsilon}{r}$, then the parallel surface M_{2r} has constant mean curvature $H_{2r} = \frac{\varepsilon}{r}$.

iii) If M has Gaussian curvature $K = \frac{\varepsilon}{r^2}$ and constant mean curvature $H \neq \mp\frac{2\varepsilon}{r}$, then the parallel surface $M_{\pm r}$ has constant mean curvature $H_{\pm r} = \pm\frac{\varepsilon}{r}$.

Proof. If $H = -\frac{\varepsilon}{r}$, then it follows from (3.1), that

$$K_r = \frac{K}{1-\frac{r}{r}+\varepsilon r^2K} = \frac{\varepsilon}{r^2}$$

(ii) and (iii) follow from (3.1) and (3.2) in similar fashion. □

Theorem 3.5. *Let $M \subset E^3$ be a regular surface with constant positive curvature εa^{-2} where $a > 0$. Let M_r denote the surface parallel to M at a distance r. Suppose* *that the umbilic points of M are isolated. If M_r has constant mean curvature, then $r = \pm a$.*

Proof. Fix r, and suppose that H_r is constant on M_r. Then (3.2) implies that

$$\varepsilon k_1(1 + rk_2) + \varepsilon k_2(1 + rk_1) = H_r(1 + rk_1)(1 + rk_2)$$

$$\varepsilon(k_1 + k_2) + 2ra^{-2} = H_r(1 + r(k_1 + k_2) + r^2a^{-2}).$$

Hence

$$(k_1 + k_2)(\varepsilon - rH_r) = H_r + H_r r^2 a^{-2} - 2r\varepsilon a^{-2}. \tag{3.3}$$

By hypothesis, the right hand of (3.3) is constant. But if the left hand of (3.3) constant, it must vanish at the nonumbilic points of M_r. Hence

$$\varepsilon - rH_r = 0 \tag{3.4}$$

at the nonumbilic points of M_r. Then (3.3) and (3.4) imply that

$$\begin{aligned} 0 &= H_r + H_r r^2 a^{-2} - 2\varepsilon r a^{-2} \\ &= H_r(1 + r^2a^{-2}) - 2\varepsilon r a^{-2} \\ &= \frac{\varepsilon}{r}\left(1 + \frac{r^2}{a^2}\right) - \frac{2\varepsilon r}{a^2} \\ &= \frac{\varepsilon}{r} - \frac{\varepsilon r}{a^2}. \end{aligned}$$

Therefore, $r = \pm a$. □

4. Parallel Linear Weingarten Surfaces in E_1^3

Theorem 4.1. *M is a linear Weingarten surface if and only if M_r is a linear Weingarten surface in E_1^3.*

Proof. It can be proved easily following the same procedure as in the Teorem 2.1. □

Let M (or M_r) be a timelike surface. Since $\varepsilon = 1$ the Gaussian and the mean curvature of M (or M_r) are

$$K = \frac{K_r}{1 - rH_r + r^2K_r} \quad \textit{and} \quad H = \frac{H_r - 2rK_r}{1 - rH_r + r^2K_r}$$

or

$$K_r = \frac{K}{1 + rH + r^2K} \quad \textit{and} \quad H_r = \frac{H + 2rK}{1 + rH + r^2K}.$$

These formulas are the same for any surface in E^3. Therefore Theorem 2.2, 2.3, 2.4, 2.5, 2.6, 2.7 are valid for M and Theorem 2.8, 2.9, 2.10, 2.11, 2.12, 2.13 are valid for M_r in E_1^3.

Because of that in this section we give the teorems for only spacelike surfaces.

Theorem 4.2. *Let M be a spacelike LW-surface with $c = 0$ in E_1^3. Then M and M_r are elliptic LW-surface.*

Theorem 4.3. *Let M be a spacelike elliptic LW-surface with $c > 0$ in E_1^3*

a) If $a^2 < bc$ then M is an elliptic LW-surface.

b) Let $a^2 = bc$.

b.i) If $r \neq \frac{a}{c}$ or then M_r is an elliptic LW-surface.

b.ii) If $r = \frac{a}{c}$ then M_r is a spacelike parabolic LW-surface.

c) Let $a^2 > bc$ and $c > 0$.

c.i) If $\frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right) < r < \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is an hyperbolic LW-surface.

c.ii) If $r < \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ or $r > \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is an elliptic LW-surface.

c.iii) If $r = \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ or $r = \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is a parabolic LW-surface.

d) Let $a^2 > bc$ and $c < 0$.

d.i) If $\frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right) < r < \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is an hyperbolic LW-surface.

d.ii) If $r < \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ or $r > \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is an elliptic LW-surface.

d.iii) If $r = \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ or $r = \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is a parabolic LW-surface.

Theorem 4.4. *Let M be a spacelike hyperbolic LW-surface with $c \neq 0$ in E_1^3.*

a) If $a^2 < bc$ then M_r is an elliptic LW-surface.

b) Let $a^2 = bc$.

b.i) If $r \neq \frac{a}{c}$ or then M_r is a elliptic LW-surface.

b.ii) If $r = \frac{a}{c}$ then M_r is a parabolic LW-surface.

c) Let $bc < a^2 < -4bc$ and $c > 0$.

c.i) If $\frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right) < r < \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is an hyperbolic LW-surface.

c.ii) If $r < \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ or $r > \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is an elliptic LW-surface.

c.iii) If $r = \frac{1}{c}\left(a - \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ or $r = \frac{1}{c}\left(a + \frac{2}{5}\sqrt{5(a^2 - bc)}\right)$ then M_r is a parabolic LW-surface.

d) Let $bc < a^2 < -4bc$ and $c < 0$.

d.i) If $\frac{1}{c}\left(a+\frac{2}{5}\sqrt{5(a^2-bc)}\right) < r < \frac{1}{c}\left(a-\frac{2}{5}\sqrt{5(a^2-bc)}\right)$ *then* M_r *is an hyperbolic LW-surface.*

d.ii) If $r < \frac{1}{c}\left(a+\frac{2}{5}\sqrt{5(a^2-bc)}\right)$ *or* $r > \frac{1}{c}\left(a-\frac{2}{5}\sqrt{5(a^2-bc)}\right)$ *then* M_r *is an elliptic LW-surface.*

d.iii) If $r = \frac{1}{c}\left(a+\frac{2}{5}\sqrt{5(a^2-bc)}\right)$ *or* $r = \frac{1}{c}\left(a-\frac{2}{5}\sqrt{5(a^2-bc)}\right)$ *then* M_r *is a parabolic LW-surface.*

Theorem 4.5. *Let* M *be a spacelike parabolic LW-surface with* $c > 0$ *and* $a > 0$ *or* $c < 0$ *and* $a < 0$ *in* E_1^3*.*

a) If $r < 0$ *or* $r > \frac{2a}{c}$ *then* M_r *is an elliptic LW-surface.*

b) If $0 < r < \frac{2a}{c}$ *then* M_r *is a hyperbolic LW-surface.*

c) If $r = 0$ *or* $r = \frac{2a}{c}$ *then* M_r *is a parabolic LW-surface.*

Theorem 4.6. *Let* M *be a spacelike parabolic LW-surface with* $c > 0$ *and* $a < 0$ *or* $c < 0$ *and* $a > 0$ *in* E_1^3*.*

a) If $r < \frac{2a}{c}$ *or* $r > 0$ *then* M_r *is an elliptic LW-surface.*

b) If $\frac{2a}{c} < r < 0$ *then* M_r *is a hyperbolicW-surface.*

c) If $r = 0$ *or* $r = \frac{2a}{c}$ *then* M_r *is a parabolic LW-surface.*

Theorem 4.7. *Let* M_r *be a spacelike LW-surface with* $c_r = 0$ *in* E_1^3*. Then* M *is an elliptic LW-surface.*

Theorem 4.8. *Let* M_r *be a spacelike elliptic LW-surface with* $c_r \neq 0$ *in* E_1^3*.*

a) If $a_r^2 < b_r c_r$ *then* M *is an elliptic LW-surface.*

b) Let $a_r^2 = b_r c_r$*.*

b.i) If $r \neq -\frac{a_r}{c_r}$ *or* $r > -\frac{a_r}{c_r}$ *then* M *is an elliptic LW-surface.*

b.ii.) If $r = -\frac{a_r}{c_r}$ *then* M *is a parabolic LW-surface.*

c) Let $a_r^2 > b_r c_r$ *and* $c_r > 0$*.*

c i) If $\frac{1}{c_r}\left(-a_r-\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right) < r < \frac{1}{c_r}\left(-a_r+\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right)$ *then* M *is a hyperbolic LW-surface.*

c.ii) If $r < \frac{1}{c_r}\left(-a_r-\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right)$ *or* $r > \frac{1}{c_r}\left(-a_r+\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right)$ *then* M *is an elliptic LW-surface.*

c.iii) If $r = \frac{1}{c_r}\left(-a_r-\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right)$ *or* $r = \frac{1}{c_r}\left(-a_r+\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right)$ *then* M *is a parabolic LW-surface.*

d) Let $a_r^2 > b_r c_r$ *and* $c_r < 0$*.*

d i) If $\frac{1}{c_r}\left(-a_r+\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right) < r < \frac{1}{c_r}\left(-a_r-\frac{2}{5}\sqrt{5(a_r^2-b_rc_r)}\right)$ *then* M *is a hyperbolic LW-surface.*

d.ii) If $r < \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *or* $r > \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is an elliptic LW-surface.*

d.iii) If $r = \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *or* $r = \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is a parabolic LW-surface.*

Theorem 4.9. *Let* M_r *be a spacelike hyperbolic LW-surface with* $c_r \neq 0$ *in* E_1^3.

a) If $a_r^2 < b_r c_r$ *then* M *is an elliptic LW-surface.*

b) Let $a_r^2 = b_r c_r$.

b.i) If $r < -\frac{a_r}{c_r}$ *or* $r > -\frac{a_r}{c_r}$ *then* M *is an elliptic LW-surface.*

b.ii) If $r = -\frac{a_r}{c_r}$ *then* M *is a parabolic LW-surface.*

c) Let $b_r c_r < a_r^2 < -4 b_r c_r$ *and* $c_r > 0$.

c i) If $\frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right) < r < \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is a hyperbolic LW-surface.*

c.ii) If $r < \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *or* $r > \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is an elliptic LW-surface.*

c.iii) If $r = \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *or* $r = \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is a parabolic LW-surface.*

d) Let $a_r^2 > b_r c_r$ *and* $c_r < 0$.

d i) If $\frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right) < r < \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is a hyperbolic LW-surface.*

d.ii) If $r < \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *or* $r > \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M *is an elliptic LW-surface.*

d.iii) If $r = \frac{1}{c_r}\left(-a_r + \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *or* $r = \frac{1}{c_r}\left(-a_r - \frac{2}{5}\sqrt{5(a_r^2 - b_r c_r)}\right)$ *then* M_r *is a parabolic LW-surface.*

Theorem 4.10. *Let* M_r *be a spacelike parabolic LW-surface with* $c_r > 0$ *and* $a_r > 0$ *or* $c_r < 0$ *and* $a_r < 0$ *in* E_1^3.

a)If $r < -\frac{2a_r}{c_r}$ *or* $r > 0$ *then* M *is an elliptic LW-surface.*

b) If $-\frac{2a_r}{c_r} < r < 0$ *then* M *is a hyperbolic LW-surface.*

c) $r = 0$ *or* $r = -\frac{2a_r}{c_r}$ *then* M *is a parabolic LW-surface.*

Theorem 4.11. *Let* M_r *be a spacelike parabolic LW-surface with* $c_r > 0$ *and* $a_r < 0$ *or* $c_r < 0$ *and* $a_r > 0$ *in* E_1^3.

a) If $r < 0$ *or* $r > -\frac{2a_r}{c_r}$ *then* M *is an elliptic LW-surface.*

b) If $0 < r < -\frac{2a_r}{c_r}$ *then* M *is a hyperbolic LW-surface.*

c) If $r = 0$ *or* $r = -\frac{2a_r}{c_r}$ *then* M *is a parabolic LW-surface.*

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