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THE RELATIONS AMONG INSTANTANEOUS ROTATION VECTORS OF A PARALLEL TIMELIKE RULED SURFACE

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Abstract. In this paper, geometric invariants of a parallel timelike ruled surface with a timelike ruling have been given in terms of those of the main surface. The instantaneous velocities for Frenet, Darboux and Blaschke trihedrons of a parallel timelike ruled surface have been calculated by using their derivative formulas. The relations among dual Lorentzian instantaneous rotation vectors have been obtained for these trihedrons.

1. Introduction

Modern surface modeling systems include ruled surfaces. The geometry of ruled surfaces is essential for studying kinematical and positional mechanisms in $\mathbb{R}^3$. Dual numbers were introduced by W.K. Clifford [4] as a tool for his geometrical investigations. E. Study [15] used dual numbers and dual vectors in his research on the geometry of lines and kinematics. M. Skreiner in 1966 [14] studied on the geometry and kinematics of instantaneous spatial motion, using new geometric explanations gave some theorems and results for the invariants of a closed ruled surface generated by an oriented line of a moving rigid body in $\mathbb{R}^3$. Nizamoğlu [9] studied parallel ruled surfaces in Euclidean space in which he considered them as one-parameter dual curves on the dual unit sphere. Recently, ruled surfaces in Lorentz space have been studied in [5], [16], [17]. Moreover, Uğurlu and Çalışkan [18] proved that one-(real)parameter motions on the dual Lorentzian sphere and dual hyperbolic sphere correspond uniquely to spacelike and timelike ruled surfaces in three dimensional Lorentz space, respectively. In 2008, authors [5] wrote a paper on parallel timelike ruled surfaces with timelike rulings. They compared geometric invariants of the two parallel ruled surfaces.

In this paper, we have found the relations among instantaneous rotation vectors of a parallel timelike ruled surface. For clarity and notation, we recall some fundamental concepts of the subject.

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2. Preliminaries

To meet the requirements in the next sections, here, the basic elements of the theory of dual curves in the space $\mathbb{D}_1^3$ are briefly presented (A more complete elementary treatment can be found in [1], [6], [8]).

W. K. Clifford, in [4], introduced dual numbers with the set

$$\mathbb{D} = \{X = x + \varepsilon \overline{x} : x, \overline{x} \in \mathbb{R}\}.$$

The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$ for $\varepsilon \neq 0$. Thereafter, a good amount of research work has been done on dual numbers, dual functions and as well as dual curves [1], [6], [9]. Then, dual angle is introduced, which is defined as $\widehat{\theta} = \theta + \varepsilon\overline{\theta}$, where θ is the projected angle between two spears and $\overline{\theta}$ is the shortest distance between them. $\langle X, Y\rangle$ is the cosine of dual hyperbolic angle $\widehat{\theta} = \theta + \varepsilon\overline{\theta}$ of two timelike lines: $\langle X, Y\rangle = -\cosh\widehat{\theta}$ *in* [18]. The set $\mathbb{D}$ of dual numbers is a commutative ring with the operations (+) and (.). The set

$$\mathbb{D}^3 = \mathbb{D} \times \mathbb{D} \times \mathbb{D} = \{X : X = x + \varepsilon\overline{x},\ x \in \mathbb{R}^3,\ \overline{x} \in \mathbb{R}^3\}$$

is a module over the ring $\mathbb{D}$ [8].

Let us denote $X = x + \varepsilon\overline{x} = (x_1, x_2, x_3) + \varepsilon(\overline{x_1}, \overline{x_2}, \overline{x_3})$ and $Y = y + \varepsilon\overline{y} = (y_1, y_2, y_3) + \varepsilon(\overline{y_1}, \overline{y_2}, \overline{y_3})$. The Lorentzian inner product of X and Y is defined by

$$\langle X, Y\rangle = \langle x, y\rangle + \varepsilon(\langle x, \overline{y}\rangle + \langle\overline{x}, y\rangle).$$

We call the dual space $\mathbb{D}^3$ together with Lorentzian inner product as dual Lorentzian space and show by $\mathbb{D}_1^3$. We call the elements of $\mathbb{D}_1^3$ as the dual vectors. A dual vector $X \in \mathbb{D}_1^3$ is said to be spacelike, timelike or lightlike (null) if the real vector x is spacelike, timelike or lightlike (null), respectively. If $X \neq 0$, then the norm of the dual vector $X \in \mathbb{D}_1^3$ is defined by $\|X\| = \sqrt{|\langle X, X\rangle|}$. Therefore, an arbitrary dual curve, which is a differentiable mapping onto $\mathbb{D}_1^3$, can locally be dual spacelike, dual timelike or dual null, if its velocity vector is respectively, dual spacelike, dual timelike or dual null. Besides, for the dual vectors X, $Y \in \mathbb{D}_1^3$, Lorentzian vector product of dual vectors is defined by

$$X \wedge Y = x \wedge y + \varepsilon(\overline{x} \wedge y + x \wedge \overline{y}),$$

where $x \wedge y$ is the classical Lorentzian cross product according to signature $(+, +, -)$ [18]. Also $\overrightarrow{E_1} \wedge \overrightarrow{E_2} = \overrightarrow{E_3}$, $\overrightarrow{E_2} \wedge \overrightarrow{E_3} = -\overrightarrow{E_1}$ and $\overrightarrow{E_3} \wedge \overrightarrow{E_1} = -\overrightarrow{E_2}$ are obtained for the base $\{\overrightarrow{E_1}, \overrightarrow{E_2}, \overrightarrow{E_3}\}$. The well known Study theorem's expression is that there exists one-to-one correspondence between directed timelike (resp. spacelike) lines of $\mathbb{R}_1^3$ and an ordered pair of vectors $(x, \overline{x})$ such that $< x, x >= -1$ (resp. $< x, x >= +1$) and $< x, \overline{x} >= 0$ [18].

3. The Instantaneous velocity vectors of trihedrons depending on timelike ruled surface

Let us show the dual unit vectors of a solid perpendicular trihedron in $\mathbb{D}_1^3$ depending on a parameter t as $X_i = x_i + \varepsilon\overline{x_i}$, $1 \leq i \leq 3$, where X_1 is dual timelike vector and X_2, X_3 are dual spacelike vectors. In this case, we can write

$$\langle X_1, X_1\rangle = -1,\ \ \langle X_2, X_2\rangle = \langle X_3, X_3\rangle = 1 \tag{3.1}$$

and

$$\langle X_1, X_2\rangle = \langle X_2, X_3\rangle = \langle X_3, X_1\rangle = 0. \tag{3.2}$$

We call the dual vector

$$W = w + \varepsilon\overline{w} = -\langle X_2', X_3\rangle X_1 + \langle X_1', X_3\rangle X_2 + \langle X_1', X_2\rangle X_3 \tag{3.3}$$

as *instantaneous rotation vector* of the moving solid trihedron $(X;\ i,j,k)$. The vectors w and $\overline{w}$ denote *instantaneous rotation vector* and the *velocity of moving trihedron* at point X. Thus, we can write

$$X_i' = W \wedge X_i;\ i = 1,2,3, \tag{3.4}$$

where $\wedge$ denotes the Lorentzian cross product.

Definition 3.1. If t is fixed, $R_1(t)$, $R_2(t)$, $R_3(t)$ are straight lines in $\mathbb{R}^3_1$, and their point of intersection is the point of striction $x(t)$ on the ruling $R_1(t)$. The locus of $x(t)$ is the curve of striction on the ruled surface [6].

$$E_1{=}(X;\frac{dx}{ds}{=}x_1,g,n),\ E_2{=}(X;x_1{=}a_1,a_2,a_3),\ E_3{=}(X;r_1,r_2=n,r_3) \tag{3.5}$$

trihedrons state the Darboux, Frenet and Blaschke trihedrons line of a timelike ruled surface by dual timelike unit vector $R_1 = r_1(s) + \varepsilon\overline{r_1}(s)$ in the striction point, where x_1 and r_1 are timelike vectors, the others are spacelike vectors. Also, respectively, the derivative formulas of Darboux, Frenet and Blaschke trihedrons are as follows:

$$\begin{array}{lll} x_1' = \rho_g g + \rho_n n & a_1' = \kappa a_2 & r_1' = p r_2 \\ g' = \rho_g x_1 + \tau_g n & a_2' = \kappa a_1 + \tau a_3 & r_2' = p r_1 + q r_3 \\ n' = \rho_n x_1 - \tau_g g & a_3' = -\tau a_2 & r_3' = -q r_2 \end{array} \tag{3.6}$$

ρ_n, τ_g and ρ_g are differentiable functions given at the Darboux derivative formula in (3.6). Set $x' = ar_1 + br_3$ where $(')$ stands for the derivative with respect to the arc length parameter (or the pseudo-arc parameter) s of x, then we find $a{=}\overline{q}$ and $b{=}\overline{p}$ since $x \wedge r_i{=}\overline{r}_i$, $1 \le i \le 3$. If x is timelike then we have for the Darboux trihedron

$$(x_1, n = r_2, g = x_1 \wedge n) \tag{3.7}$$

as follows:

$$\begin{array}{c} x_1 = \overline{q}r_1 + \overline{p}r_3 \\ n = r_2 \\ g = \overline{p}r_1 + \overline{q}r_3 \end{array} \quad \text{and} \quad \begin{array}{c} r_1 = \overline{q}x_1 - \overline{p}g \\ r_2 = n \\ r_3 = -\overline{p}x_1 + \overline{q}g \end{array} \tag{3.8}$$

By a direct computation, we have

$$\rho_n{=}p\overline{q} - q\overline{p},\quad \tau_g{=}p\overline{p} - q\overline{q},\quad p^2 - q^2{=}\rho_n^2 - \tau_g^2 \tag{3.9}$$

$$\rho_g{=}\overline{q}'/\overline{p}{=}\overline{p}'/\overline{q},\quad \rho_g{=}\overline{q}\overline{p}' - \overline{p}\overline{q}',\ \ \rho_g^2{=}\left(\overline{p}'\right)^2 - \left(\overline{q}'\right)^2. \tag{3.10}$$

The results in (3.9) and (3.10) have been obtained in [5]. The elements of Frenet apparatus are found as follows:

$$a_1{=}x_1 \qquad a_2{=}\frac{a_1'}{\|a_1'\|}{=}\frac{\rho_g g + \rho_n n}{(\rho_g^2 + \rho_n^2)^{\frac{1}{2}}} \qquad a_3{=}a_1 \wedge a_2{=}\frac{\rho_n g - \rho_g n}{(\rho_g^2 + \rho_n^2)^{\frac{1}{2}}}. \tag{3.11}$$

From (3.11), the functions κ and τ are found as follows:

$$\kappa = (\rho_g^2 + \rho_n^2)^{\frac{1}{2}} \quad \text{and} \quad \tau = \frac{\rho_n\rho_g' - \rho_g\rho_n'}{\kappa^2} - \tau_g. \tag{3.12}$$

If we consider the derivative formulae of the Darboux trihedron in $\mathbb{R}_1^3$ given in (3.6) and the following equalities

$$X_1 = x_1 + \varepsilon(x \wedge x_1),\ G = g + \varepsilon(x \wedge g),\ N = n + \varepsilon(x \wedge n), \tag{3.13}$$

then we find the derivative formulae of Darboux trihedron as follows:

$$\begin{bmatrix} X_1' \\ G' \\ N' \end{bmatrix} = \begin{bmatrix} 0 & \rho_g & \rho_n \\ \rho_g & 0 & \tau_g - \varepsilon \\ \rho_n & \varepsilon - \tau_g & 0 \end{bmatrix} \begin{bmatrix} X_1 \\ G \\ N \end{bmatrix} \tag{3.14}$$

If the matrix form (3.14) is considered, the instantaneous rotation vector of motion of Darboux trihedron E_1 with respect to the fixed trihedron E_0 can be given by

$$D = (\varepsilon - \tau_g)X_1 + \rho_n G - \rho_g N \tag{3.15}$$

such that

$$X_1' = D \wedge X_1;\ G' = D \wedge G,\ N' = D \wedge N. \tag{3.16}$$

Frenet trihedron is

$$X_1 = A_1 = a_1 + \varepsilon(x \wedge a_1),\ A_2 = a_2 + \varepsilon(x \wedge a_2),\ A_3 = a_3 + \varepsilon(x \wedge a_3) \tag{3.17}$$

The Frenet derivative formulae at striction point can also be written as follows:

$$\begin{bmatrix} A_1' \\ A_2' \\ A_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ \kappa & 0 & \tau - \varepsilon \\ 0 & \varepsilon - \tau & 0 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}. \tag{3.18}$$

The *instantaneous rotation vector* of the motion of E_2 with respect to E_0 is

$$F = (\varepsilon - \tau)A_1 - \kappa A_3 \tag{3.19}$$

such that

$$A_i' = F \wedge A_i;\ i = 1, 2, 3. \tag{3.20}$$

The derivative formulae of Blashcke trihedron E_3 can be obtained with the same method as follows:

$$\begin{bmatrix} R_1' \\ R_2' \\ R_3' \end{bmatrix} = \begin{bmatrix} 0 & P & 0 \\ P & 0 & Q \\ 0 & -Q & 0 \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ R_3 \end{bmatrix} \tag{3.21}$$

where P and Q show dual curvature and dual torsion, respectively. If we consider the relation (3.21), the instantaneous rotation vector of the motion of the Blaschke trihedron E_3 with the respect to E_0 is given by

$$B = QR_1 - PR_3 \tag{3.22}$$

such that

$$R_i' = B \wedge R_i;\ i = 1, 2, 3. \tag{3.23}$$

3.1. **The relations among instantaneous velocity vectors.** Let us consider the relations among the instantaneous rotation vectors D, F and B that are the results of motions $(E_1/E_0), (E_2/E_0)$ and (E_3/E_0) of Darboux, Frenet and Blaschke trihedrons E_1, E_2 and E_3 of a timelike ruled surface at the striction point with respect to trihedron E_0.

From the relations (3.16) and (3.20) we have

$$(D - F) \wedge X_1 = 0 \tag{3.24}$$

and

$$D = F + LX_1,\ L \in R. \tag{3.25}$$

By (3.15) and (3.19) we obtain

$$(\varepsilon - \tau_g)X_1 + \rho_n G - \rho_g N = (\varepsilon - \tau)A_1 - \kappa A_3 + LX_1. \tag{3.26}$$

From (3.26), we get $L = \tau - \tau_g$. Then we have

$$D = F + (\tau - \tau_g)X_1 \tag{3.27}$$

and

$$\kappa A_3 = -\rho_n G + \rho_g N. \tag{3.28}$$

Let $\Gamma = \gamma + \varepsilon\overline{\gamma}$ be dual angle between the spacelike vectors A_2 and N. Then we can write

$$G = \sin\Gamma . A_2 - \cos\Gamma . A_3 \tag{3.29}$$

$$N = \cos\Gamma . A_2 + \sin\Gamma . A_3 \tag{3.30}$$

By (3.29) we have

$$\langle A_3, G\rangle = -\cos\Gamma \qquad \cos\gamma = \frac{\rho_n}{\kappa} \qquad \overline{\gamma} = 0 \tag{3.31}$$

$$\langle A_3, N\rangle = \sin\Gamma \qquad \sin\gamma = \frac{\rho_g}{\kappa} \qquad \overline{\gamma} = 0 \tag{3.32}$$

If we consider (3.16) and (3.23), we obtain

$$(D - B) \wedge N = 0$$

and

$$D = B + MN,\ M \in R. \tag{3.33}$$

From (3.15) and (3.22), we find

$$(\varepsilon - \tau_g)X_1 + \rho_n G - \rho_g N = QR_1 - PR_3 + MN. \tag{3.34}$$

If we consider $M = -\rho_g$, the equations (3.33) and (3.34) can be written as, respectively,

$$D = B - \rho_g N \tag{3.35}$$

and

$$(\varepsilon - \tau_g)X_1 + \rho_n G = QR_1 - PR_3. \tag{3.36}$$

If the angle between the timelike unit vectors R_1 and X_1 is $\Phi = \varphi + \varepsilon\overline{\varphi}$, by using (3.35) we obtain

$$P = (\tau_g - \varepsilon)\sinh\varphi - \rho_n\cosh\varphi \tag{3.37}$$

$$Q = (\varepsilon - \tau_g)\cosh\varphi + \rho_n\sinh\varphi. \tag{3.38}$$

From (3.27) and (3.36), the relation among the instantaneous rotation vectors F and B is given as

$$F = B + (\tau_g - \tau)X_1 - \rho_g N. \tag{3.39}$$

4. Parallel timelike ruled surfaces with timelike ruling

We define parallel timelike ruled surfaces with timelike rulings by a rotation about the axis R_2, that is,

Definition 4.1. Let $[R_1]$ be timelike ruled surface with a timelike ruling and $\Theta = \theta + \varepsilon\overline{\theta}$ be a constant dual hyperbolic angle (cf.[18]). Then a parallel ruled surface $[R_1^*]$ to $[R_1]$ is defined by

$$\begin{aligned} R_1^* &= \cosh\Theta R_1 + \sinh\Theta R_3 \\ R_2^* &= R_2 \\ R_3^* &= \sinh\Theta R_1 + \cosh\Theta R_3 \end{aligned} \tag{4.1}$$

where $\sinh\Theta = \sinh\theta + \varepsilon\overline{\theta}\cosh\theta$, $\cosh\Theta = \cosh\theta + \varepsilon\overline{\theta}\sinh$. Let's assume that the change of the new trihedron with respect to s is

$$\begin{aligned} R_1^{*'} &= P^* R_2^*, \\ R_2^{*'} &= P^* R_1^* + Q^* R_3^*, \\ R_3^{*'} &= -Q^* R_2^*, \end{aligned} \tag{4.2}$$

where $P^* = p^* + \varepsilon\overline{p}^*$, $Q^* = q^* + \varepsilon\overline{q}^*$ [5].

By using (4.1)-(4.2) we have

$$P^* = P\cosh\Theta - Q\sinh\Theta, \; Q^* = -P\sinh\Theta + Q\cosh\Theta, \tag{4.3}$$

and then splitting these equations into real and dual parts gives the following result:

Proposition 4.1. *Let $[R_1]$ be timelike ruled surface with a timelike ruling. Then we have*

$$p^* = p\cosh\theta - q\sinh\theta, \; q^* = -p\sinh\theta + q\cosh\theta \tag{4.4}$$

and

$$\begin{aligned} \overline{p}^* &= \overline{p}\cosh\theta - \overline{q}\sinh\theta + \overline{\theta}(p\sinh\theta - q\cosh\theta) \\ \overline{q}^* &= -\overline{p}\sinh\theta + \overline{q}\cosh\theta + \overline{\theta}(-p\cosh\theta + q\sinh\theta). \end{aligned} \tag{4.5}$$

These results are similar to those in [5]. Let's express Darboux trihedron of parallel timelike ruled surface, we know that

$$x_1^* = \frac{dx^*}{ds^*}.\frac{ds^*}{ds} = \overline{q}^* r_1^* + \overline{p}^* r_3^* \tag{4.6}$$

where $\dfrac{ds^*}{ds} = \sqrt{\overline{q}^{*2} - \overline{p}^{*2}} = \sqrt{A}$. If we write the elements of Darboux trihedron for parallel timelike ruled surface x_1^*, g^* and n^* as respect to s, we have

$$\begin{aligned} x_1^* &= \frac{\overline{q}^*}{\sqrt{A}} r_1^* + \frac{\overline{p}^*}{\sqrt{A}} r_3^* \\ g^* &= \frac{\overline{q}^*}{\sqrt{A}} r_3^* + \frac{\overline{p}^*}{\sqrt{A}} r_1^* \\ n^* &= r_2^* \end{aligned} \tag{4.7}$$

From (4.7), we obtain

$$\begin{aligned} r_1^* &= \frac{\overline{q}^*}{\sqrt{A}} x_1^* - \frac{\overline{p}^*}{\sqrt{A}} g^* \\ r_2^* &= n^* \\ r_3^* &= -\frac{\overline{p}^*}{\sqrt{A}} x_1^* + \frac{\overline{q}^*}{\sqrt{A}} g^* \end{aligned} \tag{4.8}$$

And also we find the derivative formulas of Darboux trihedron for parallel timelike ruled surface with respect to the parameter s as follows:

$$\begin{aligned} x_1^{*\prime} &= \sqrt{A}(\rho_g^* g^* + \rho_n^* n^*) \\ g^{*\prime} &= \sqrt{A}(\rho_g^* x_1^* + \tau_g^* n^*) \\ n^{*\prime} &= \sqrt{A}(\rho_n^* x_1^* - \tau_g^* g^*) \end{aligned} \tag{4.9}$$

Also, by using (4.5), in which the values are obtained for $\theta = 0$, in (4.7) we have

$$\begin{aligned} x_1^* &= \frac{(\overline{q} - \overline{\theta} p)}{\sqrt{A}}.r_1 + \frac{(\overline{p} - \overline{\theta} q)}{\sqrt{A}}.r_3 \\ g^* &= \frac{(\overline{p} - \overline{\theta} q)}{\sqrt{A}}.r_1 + \frac{(\overline{q} - \overline{\theta} p)}{\sqrt{A}}.r_3 \\ n^* &= r_2 \end{aligned} \tag{4.10}$$

If we use (3.8), (3.9) and (3.10) in (4.10), then we obtain

$$\begin{aligned} x_1^* &= \frac{(1 - \overline{\theta}\rho_n)}{\sqrt{A}}.x_1 + \frac{\overline{\theta}\tau_g}{\sqrt{A}}.g \\ g^* &= \frac{\overline{\theta}\tau_g}{\sqrt{A}}.x_1 + \frac{(1 - \overline{\theta}\rho_n)}{\sqrt{A}}.g \\ n^* &= n. \end{aligned} \tag{4.11}$$

Let's express $\rho_g^*, \rho_n^*, \tau_g^*$ on the striction point of parallel timelike ruled surface $[R_1^*]$ in terms of ρ_g, ρ_n, τ_g which belong to timelike ruled surface $[R_1]$. By differentiating in (4.8) and using (4.2) and (4.9), we obtain

$$p^* r_1 + q^* r_3 = \sqrt{A}(\rho_n^* x_1^* - \tau_g^* g^*). \tag{4.12}$$

From (4.10) and (4.12), we obtain

$$\rho_n^* = \frac{q^*\overline{p}^* - p^*\overline{q}^*}{A} \text{ and } \tau_g^* = \frac{p^*\overline{p}^* - q^*\overline{q}^*}{A}. \tag{4.13}$$

By using (3.9), (4.4) and (4.5) in (4.13), we have

$$\rho_n^* = \frac{\rho_n - \overline{\theta}(\rho_n^2 - \tau_g^2)}{A} \text{ and } \tau_g^* = \frac{\tau_g}{A}. \tag{4.14}$$

From (3.9) and (3.10), it is written as follows:

$$\sqrt{A}\rho_g^* = \frac{\overline{q}^{*\prime}}{\overline{p}^*}, \ \sqrt{A}\rho_g^* = \frac{\overline{p}^{*\prime}}{\overline{q}^*}. \tag{4.15}$$

The geodesic curvature of parallel timelike ruled surface is found as

$$\rho_g^* = \frac{1}{\sqrt{A}}[\rho_g + \frac{\overline{\theta}}{A}\left[\tau_g' + \overline{\theta}(\rho_n\tau_g)'\right]]. \tag{4.16}$$

The expression of Frenet vectors in terms of Darboux vectors is

$$\begin{aligned} a_1^* &= x_1^* \\ a_2^* &= \frac{a_1^{*\prime}}{\|a_1^{*\prime}\|} = \frac{\rho_g^* g^* + \rho_n^* n^*}{(\rho_g^{*2} + \rho_n^{*2})^{\frac{1}{2}}} \\ a_3^* &= a_1^* \wedge a_2^*. \end{aligned} \tag{4.17}$$

From the equations (4.17), the curvature and torsion of parallel timelike ruled surface are obtained as

$$\kappa^* = (\rho_g^{*2} + \rho_n^{*2})^{\frac{1}{2}} \text{ and } \tau^* = \frac{\rho_n^* \rho_g^{*\prime} - \rho_g^* \rho_n^{*\prime}}{\kappa^{*2}} - \tau_g^*. \tag{4.18}$$

4.1. **The relations among instantaneous rotation vectors of parallel timelike ruled surfaces.** We consider the timelike ruled surface $[R_1^*]$ determined by Definition 4.1 which is parallel to the ruled surface $[R_1]$.

For the vectorial moments of the dual vectors we have the formulae:

$$\begin{array}{lll} \overline{x}_1^* = x^* \Lambda x_1^* & \overline{g}^* = x^* \Lambda g^* & \overline{n}^* = x^* \Lambda n^* \\ \overline{r}_i^* = x^* \Lambda r_i^* & (i = 1, 2, 3) & \end{array} \tag{4.19}$$

Let's find the instantaneous rotation vectors of Frenet, Darboux and Blaschke trihedrons $\{A_1^*, A_2^*, A_3^*\}$, $\{X_1^*, G^*, N^*\}$ and $\{R_1^*, R_2^*, R_3^*\}$, respectively, for parallel timelike ruled surface.

The Frenet derivative formulae are given as

$$\begin{bmatrix} A_1^{*\prime} \\ A_2^{*\prime} \\ A_3^{*\prime} \end{bmatrix} = \sqrt{A} \begin{bmatrix} 0 & \kappa^* & 0 \\ \kappa^* & 0 & \tau^* - \varepsilon \\ 0 & \varepsilon - \tau^* & 0 \end{bmatrix} \begin{bmatrix} A_1^* \\ A_2^* \\ A_3^* \end{bmatrix} \tag{4.20}$$

By using (4.20), the instantaneous rotation vector of Frenet trihedron is obtained as

$$F^* = \sqrt{A}\,[(\varepsilon - \tau^*) A_1^* - \kappa^* A_3^*] \tag{4.21}$$

such that $A_i^{*\prime} = F^* \wedge A_i^*$, $i = 1, 2, 3$.

The Darboux derivative formulas are given as follows:

$$\begin{bmatrix} X_1^{*\prime} \\ G^{*\prime} \\ N^{*\prime} \end{bmatrix} = \sqrt{A} \begin{bmatrix} 0 & \rho_g^* & \rho_n^* \\ \rho_g^* & 0 & \tau_g^* - \varepsilon \\ \rho_n^* & \varepsilon - \tau_g^* & 0 \end{bmatrix} \begin{bmatrix} X_1^* \\ G^* \\ N^* \end{bmatrix} \tag{4.22}$$

By using (4.22), the instantaneous rotation vector of Darboux trihedron is obtained as

$$D^* = \sqrt{A}\left[(\varepsilon - \tau_g^*) X_1^* + \rho_n^* G^* - \rho_g^* N^*\right] \tag{4.23}$$

such as

$$X_i^{*\prime} = D \wedge X_i^*, i = 1, 2, 3.$$

The Blaschke derivative formulas are given as follows:

$$\begin{bmatrix} R_1^{*\prime} \\ R_2^{*\prime} \\ R_3^{*\prime} \end{bmatrix} = \begin{bmatrix} 0 & P^* & 0 \\ P^* & 0 & Q^* \\ 0 & -Q^* & 0 \end{bmatrix} \begin{bmatrix} R_1^* \\ R_2^* \\ R_3^* \end{bmatrix}. \tag{4.24}$$

By using (4.22), the instantaneous rotation vector of Blaschke trihedron is obtained as

$$B^* = Q^* R_1^* - P^* R_3^* \tag{4.25}$$

such that

$$R_i^{*\prime} = B^* \wedge R_i^*, i = 1, 2, 3.$$

The relation between instantaneous rotation vectors of Darboux and Frenet trihedrons is

$$D^* = F^* + \sqrt{A}(\tau^* - \tau_g^*)X_1^*. \tag{4.26}$$

By using (4.23) and (4.25), the relation between instantaneous rotation vectors of Darboux and Blaschke trihedrons is

$$D^* = B^* - \sqrt{A}\rho_g^* N^*. \tag{4.27}$$

By using (4.21) and (4.25), the relation between instantaneous rotation vectors of Frenet and Blaschke trihedrons is

$$F^* = B^* + \sqrt{A}(\tau^* - \tau_g^*)X_1^*. \tag{4.28}$$

By using (4.8)-(4.11), Darboux, Frenet and Blaschke elements of parallel timelike ruled surface are obtained as follows:

$$\begin{array}{ccc} x_1^* = x_1 & a_1^* = a_1 & r_1^* = r_1 \\ g^* = g & a_2^* = a_2 & r_2^* = r_2 \\ n^* = n & a_3^* = a_3 & r_3^* = r_3 \end{array} \tag{4.29}$$

respectively.

If the equation $x^* = x - \overline{\theta} r_2$ and (4.29) are substituted in $X_i^* = x_i^* + \varepsilon(x^* \wedge x_i^*)$, $i = 1, 2, 3$, then we have

$$\begin{array}{l} X_1^* = X_1 + \varepsilon\overline{\theta}G \\ G^* = G + \varepsilon\overline{\theta}X_1 \\ N^* = N. \end{array} \tag{4.30}$$

If the equation $a^* = a - \overline{\theta} r_2$ and (4.29) are substituted in $A_i^* = a_i^* + \varepsilon(a^* \wedge a_i^*)$, $i = 1, 2, 3$, then we get

$$\begin{array}{l} A_1^* = A_1 + \varepsilon\dfrac{\overline{\theta}}{\kappa}(\rho_g A_2 - \rho n A_3) \\ A_2^* = A_2 + \varepsilon\dfrac{\overline{\theta}\rho_g}{\kappa}A_1 \\ A_3^* = A_3 - \varepsilon\dfrac{\overline{\theta}\rho_n}{\kappa}A_1. \end{array} \tag{4.31}$$

If the equation $r^* = r - \overline{\theta} r_2$ and (4.29) are substituted in $R_i^* = r_i^* + \varepsilon(r^* \wedge r_i^*)$, $i = 1, 2, 3$, then we obtain

$$\begin{array}{l} R_1^* = R_1 - \varepsilon\overline{\theta}R_3 \\ R_2^* = R_2 \\ R_3^* = R_3 - \varepsilon\overline{\theta}R_1. \end{array} \tag{4.32}$$

Also, from (4.14) and (4.16) for the values $\overline{\theta} = 0$ and $\tau_g = 0$, invariants of parallel timelike ruled surface are found as follows:

$$\tau_g^* = 0, \quad \rho_n^* = \frac{\rho_n}{\sqrt{A}}, \quad \rho_g^* = \frac{\rho_g}{\sqrt{A}}. \tag{4.33}$$

From (4.18) and (4.3), respectively, we have

$$\kappa^* = \frac{\kappa}{\sqrt{A}}, \quad \tau^* = \tau \tag{4.34}$$

$$P^* = P - \varepsilon\overline{\theta}q, \quad Q^* = Q - \varepsilon\overline{\theta}p. \tag{4.35}$$

If the equations (4.30)-(4.35) are used in the formulae of (4.23) and (4.25), then the relation between Darboux and Blaschke instantaneous rotation vectors of the main and parallel surfaces are found as follows:

$$D^* = D \text{ and } B^* = B,$$

respectively.

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