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GENERALIZED RELATIVE NEVANLINNA ORDER (α, β) AND GENERALIZED RELATIVE NEVANLINNA TYPE (α, β) BASED SOME GROWTH PROPERTIES OF COMPOSITE ANALYTIC FUNCTIONS IN THE UNIT DISC

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Abstract. Our aim in this paper is to introduce some idea about generalized relative Nevanlinna order (α, β) and generalized relative Nevanlinna type (α, β) of an analytic function with respect to another analytic function in the unit disc where α and β are continuous non-negative functions on $(-\infty, +\infty)$. So we discuss about some growth properties relating to the composition of two analytic functions in the unit disc on the basis of generalized relative Nevanlinna order (α, β) and generalized relative Nevanlinna type (α, β) as compared to the growth of their corresponding left and right factors.

1. Introduction

A function g which is analytic in the unit disc $U = \{z : |z| < 1\}$ is said to have finite Nevanlinna order [1] if there exists a number μ for which the Nevanlinna characteristic function $T_g(r)$ of g satisfies $T_g(r) < (1-r)^{-\mu}$ for all r in $0 < r_0(\mu) < r < 1$ where $T_g(r)$ is defined as

$$T(r, g) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| g\left(re^{i\theta}\right) \right| d\theta$$

where $\log^+ r = \max(0, \log r)$.

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The infimum of all such numbers μ is called the Nevanlinna order of g. Hence the Nevanlinna order $\rho(g)$ of g is formulated as

$$\rho(g)=\limsup_{r\to 1}\frac{\log T_g(r)}{-\log(1-r)}.$$

Similarly, the Nevanlinna lower order $\lambda(g)$ of g is formulated as

$$\lambda(g)=\liminf_{r\to 1}\frac{\log T_g(r)}{-\log(1-r)}.$$

Now let L be a class of continuous non-negative functions α defined on $(-\infty,\infty)$ such that $\alpha(x)=\alpha(x_0)\geq 0$ for $x\leq x_0$ with $\alpha(x)\uparrow\infty$ as $x\to\infty$. Also throughout the present paper we take $\alpha,\ \beta\in L$. Considering the above, Sheremeta [5] introduced the concept of generalized order (α,β) of an entire function. During the past decades, several authors made close investigations on the properties of entire functions related to generalized order (α,β) in some different directions. For the purpose of further applications, Biswas et al. [2] have introduced the definitions of the generalized Nevanlinna order (α,β) and generalized Nevanlinna lower order (α,β) of an analytic function g in the unit disc U which are as follows:

Definition 1. *[2] The generalized Nevanlinna order (α,β) denoted by $\rho_{(\alpha,\beta)}[g]$ and generalized Nevanlinna lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[g]$ of an analytic function g in the unit disc U are defined as:*

$$\rho_{(\alpha,\beta)}[g]=\limsup_{r\to 1}\frac{\alpha(\exp(T_g(r)))}{\beta\left(\frac{1}{1-r}\right)}\ \textit{and}\ \lambda_{(\alpha,\beta)}[g]=\liminf_{r\to 1}\frac{\alpha(\exp(T_g(r)))}{\beta\left(\frac{1}{1-r}\right)}.$$

Clearly $\rho_{(\log\log r,\log r)}[g]=\rho(g)$ and $\lambda_{(\log\log r,\log r)}[g]=\lambda(g)$.

Now we can introduce the definitions of the generalized relative Nevanlinna order (α,β) and generalized relative Nevanlinna lower order (α,β) of an analytic function g with respect to another entire function w in the unit disc U which are as follows:

Definition 2. *The generalized relative Nevanlinna order (α,β) denoted by $\rho_{(\alpha,\beta)}[g]_w$ and generalized relative Nevanlinna lower order (α,β) denoted by $\lambda_{(\alpha,\beta)}[g]_w$ of an analytic function g with respect to another entire function w in the unit disc U are defined as:*

$$\rho_{(\alpha,\beta)}[g]_w=\limsup_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_g(r))\right)}{\beta\left(\frac{1}{1-r}\right)}\ \textit{and}\ \lambda_{(\alpha,\beta)}[g]_w=\liminf_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_g(r))\right)}{\beta\left(\frac{1}{1-r}\right)}.$$

The previous definitions are easily generated as particular cases, e.g. if $w=z$, then Definition 2 reduces to Definition 1, and if $\alpha(r)=\beta(r)=\log r$ and $w(z)=\exp z$, then $\rho_{(\alpha,\beta)}[g]_w=\rho(g)$ and $\lambda_{(\alpha,\beta)}[g]_w=\lambda(g)$.

Now one may give the definitions of generalized relative Nevanlinna hyper order (α,β) and generalized relative Nevanlinna logarithmic order (α,β) of an analytic function g with respect to another entire function w in the unit disc U in the following way:

Definition 3. *The generalized relative Nevanlinna hyper order* (α,β) *denoted by* $\overline{\rho}_{(\alpha,\beta)}[g]_w$ *and generalized relative Nevanlinna hyper lower order* (α,β) *denoted by* $\overline{\lambda}_{(\alpha,\beta)}[g]_w$ *of an analytic function* g *with respect to entire function* w *in the unit disc* U *are defined as:*

$$\overline{\rho}_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1}\frac{\alpha(\log\left(T_w^{-1}\left(T_g(r)\right)\right))}{\beta\left(\frac{1}{1-r}\right)} \text{ and } \overline{\lambda}_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1}\frac{\alpha(\log\left(T_w^{-1}\left(T_g(r)\right)\right))}{\beta\left(\frac{1}{1-r}\right)}.$$

Definition 4. *The generalized relative Nevanlinna logarithmic order* (α,β) *denoted by* $\underline{\rho}_{(\alpha,\beta)}[g]_w$ *and generalized relative Nevanlinna logarithmic lower order* (α,β) *denoted by* $\underline{\lambda}_{(\alpha,\beta)}[g]_w$ *of an analytic function* g *with respect to entire function* w *in the unit disc* U *are defined as:*

$$\underline{\rho}_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1}\frac{\alpha\left(T_w^{-1}\left(T_g(r)\right)\right)}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)} \text{ and } \underline{\lambda}_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1}\frac{\alpha\left(T_w^{-1}\left(T_g(r)\right)\right)}{\beta\left(\log\left(\frac{1}{1-r}\right)\right)}.$$

Now in order to refine the growth scale namely the generalized relative Nevanlinna order (α,β), we introduce the definitions of another growth indicators, called generalized relative Nevanlinna type (α,β) and generalized relative Nevanlinna lower type (α,β) respectively of an analytic function g with respect to entire function w in the unit disc U which are as follows:

Definition 5. *The generalized relative Nevanlinna type* (α,β) *and generalized relative Nevanlinna lower type* (α,β) *of an analytic function* g *with respect to entire function* w *in the unit disc* U *having finite positive generalized relative Nevanlinna order* (α,β) $\left(0<\rho_{(\alpha,\beta)}[g]_w<\infty\right)$ *are defined as :*

$$\begin{aligned}\sigma_{(\alpha,\beta)}[g]_w &= \limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_g\left(r\right)\right)))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho_{(\alpha,\beta)}[g]_w}}\\ \text{and } \overline{\sigma}_{(\alpha,\beta)}[g]_w &= \liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_g\left(r\right)\right)))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\rho_{(\alpha,\beta)}[g]_w}}.\end{aligned}$$

It is obvious that $0\le\overline{\sigma}_{(\alpha,\beta)}[g]_w\le\sigma_{(\alpha,\beta)}[g]_w\le\infty$.

Analogously, to determine the relative growth of two analytic functions in the unit disc U having same non zero finite generalized relative Nevanlinna lower order (α,β), one can introduced the definition of generalized relative Nevanlinna weak type (α,β) and generalized relative Nevanlinna upper weak type (α,β) of an analytic function g with respect to entire function w in the unit disc U of finite positive generalized relative Nevanlinna lower order (α,β) , $\lambda_{(\alpha,\beta)}[g]_w$ in the following way:

Definition 6. *The generalized Nevanlinna upper weak type* (α,β) *and generalized Nevanlinna weak type* (α,β) *of an analytic function* g *with respect to entire function* *w in the unit disc U having finite positive generalized relative Nevanlinna lower order* (α,β) $\left(0<\lambda_{(\alpha,\beta)}[g]_w<\infty\right)$ *are defined as :*

$$\overline{\tau}_{(\alpha,\beta)}[g]_w = \limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}(T_g(r))))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda_{(\alpha,\beta)}[g]_w}}$$

$$\text{and } \tau_{(\alpha,\beta)}[g]_w = \liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}(T_g(r))))}{\left(\exp\left(\beta\left(\frac{1}{1-r}\right)\right)\right)^{\lambda_{(\alpha,\beta)}[g]_w}}.$$

It is obvious that $0\le\tau_{(\alpha,\beta)}[g]_w\le\overline{\tau}_{(\alpha,\beta)}[g]_w\le\infty$.

In this paper we study some growth properties relating to the composition of two analytic functions in the unit disc on the basis of generalized relative Nevanlinna order (α,β), generalized relative Nevanlinna hyper order (α,β), generalized relative Nevanlinna logarithmic order (α,β), generalized relative Nevanlinna type (α,β) and generalized relative Nevanlinna weak type (α,β) as compared to the growth of their corresponding left and right factors. Also the standard definitions and notations relating to the theory of entire functions are not explained here, as those are available in [1], [3] and [4].

2. Main Results

In this section, the main results of the paper are presented.

Theorem 1. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that* $0<\lambda_{(\alpha,\beta)}[g(h)]_w\le\rho_{(\alpha,\beta)}[g(h)]_w<\infty$ *and* $0<\lambda_{(\alpha,\beta)}[g]_k\le\rho_{(\alpha,\beta)}[g]_k<\infty$. *Then*

$$\frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}\le\liminf_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)}\le\min\left\{\frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k},\frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}\right\}$$

$$\le\max\left\{\frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k},\frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}\right\}\le\limsup_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)}\le\frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}.$$

Proof. From the definitions of $\lambda_{(\alpha,\beta)}[g(h)]_w$, $\rho_{(\alpha,\beta)}[g(h)]_w$, $\lambda_{(\alpha,\beta)}[g]_k$, $\rho_{(\alpha,\beta)}[g]_k$ and we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)\geqslant\left(\lambda_{(\alpha,\beta)}[g(h)]_w-\varepsilon\right)\beta((1-r)^{-1}), \tag{1}$$

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)\le\left(\rho_{(\alpha,\beta)}[g(h)]_w+\varepsilon\right)\beta((1-r)^{-1}), \tag{2}$$

$$\alpha\left(T_k^{-1}(T_g(r))\right)\geqslant\left(\lambda_{(\alpha,\beta)}[g]_k-\varepsilon\right)\beta((1-r)^{-1}) \tag{3}$$

$$\text{and}\quad\alpha\left(T_k^{-1}(T_g(r))\right)\le\left(\rho_{(\alpha,\beta)}[g]_k+\varepsilon\right)\beta((1-r)^{-1}). \tag{4}$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity,

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)\le\left(\lambda_{(\alpha,\beta)}[g(h)]_w+\varepsilon\right)\beta((1-r)^{-1}), \tag{5}$$

$$\alpha\left(T_w^{-1}(T_{g(h)}(r))\right) \geqslant \Big(\rho_{(\alpha,\beta)}[g(h)]_w - \varepsilon\Big)\,\beta((1-r)^{-1}), \tag{6}$$

$$\alpha\left(T_k^{-1}(T_g(r))\right) \leq \left(\lambda_{(\alpha,\beta)}[g]_k + \varepsilon\right)\beta((1-r)^{-1}) \tag{7}$$

$$\text{and } \alpha\left(T_k^{-1}(T_g(r))\right) \geqslant \Big(\rho_{(\alpha,\beta)}[g]_k - \varepsilon\Big)\,\beta((1-r)^{-1}). \tag{8}$$

Now from (1) and (4) it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\rho_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}, \tag{9}$$

which is the first part of the theorem.

Combining (5) and (3) , we have for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w + \varepsilon}{\lambda_{(\alpha,\beta)}[g]_k - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\liminf_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}. \tag{10}$$

Again from (1) and (7), for a sequence of values of $\frac{1}{1-r}$ tending to infinity, we get

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\lambda_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geq \frac{\lambda_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}. \tag{11}$$

Now, it follows from (3) and (2) , for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w + \varepsilon}{\lambda_{(\alpha,\beta)}[g]_k - \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\lambda_{(\alpha,\beta)}[g]_k}. \tag{12}$$

Which is the last part of the theorem.

Now from (2) and (8), it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w + \varepsilon}{\rho_{(\alpha,\beta)}[g]_k - \varepsilon}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \leq \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}. \tag{13}$$

So combining (4) and (6), we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\rho_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\rho_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows that

$$\limsup_{r\to 1} \frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \geqslant \frac{\rho_{(\alpha,\beta)}[g(h)]_w}{\rho_{(\alpha,\beta)}[g]_k}. \tag{14}$$

So, the second part of the theorem follows from (10) and (13) ,the third part is trivial and fourth part follows from (11) and (14).

Thus the theorem follows from (9), (10), (11), (12), (13) and (14). □

Remark 1. *If we take "*$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$*" instead of "*$0 < \lambda_{(\alpha,\beta)}[g]_k \leq \rho_{(\alpha,\beta)}[g]_k < \infty$*" and other conditions remain same, the conclusion of Theorem 1 remains true with "*$\lambda_{(\alpha,\beta)}[g]_k$*", "*$\rho_{(\alpha,\beta)}[g]_k$*" and "*$\alpha\left(T_k^{-1}(T_g(r))\right)$*" replaced by "*$\lambda_{(\alpha,\beta)}[h]_k$*", "*$\rho_{(\alpha,\beta)}[h]_k$*" and "*$\alpha\left(T_k^{-1}(T_h(r))\right)$*" respectively in the denominator.*

Theorem 2. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \lambda_{(\alpha,\beta)}[g]_k \leq \rho_{(\alpha,\beta)}[g]_k < \infty$ and $\lambda_{(\alpha,\beta)}[g(h)]_w = \infty$. Then*

$$\lim_{r\to 1} \frac{\alpha(T_w^{-1}(T_{g(h)}(r)))}{\alpha(T_k^{-1}(T_g(r)))} = \infty.$$

Proof. If possible, let the conclusion of the theorem does not hold. Then we can find a constant $\Delta > 0$ such that for a sequence of values of $\frac{1}{1-r}$ tending to infinity

$$\alpha(T_w^{-1}(T_{g(h)}(r))) \leq \Delta \cdot \alpha(T_k^{-1}(T_g(r))). \tag{15}$$

Again from the definition of $\rho_{(\alpha,\beta)}[g]_k$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\alpha(T_k^{-1}(T_g(r))) \leq (\rho_{(\alpha,\beta)}[g]_k + \epsilon)\beta(\frac{1}{1-r}). \tag{16}$$

From (15) and (16), for a sequence of values of r tending to 1,we have

$$\alpha(T_w^{-1}(T_{g(h)}(r))) \leq \Delta(\rho_{(\alpha,\beta)}[g]_k + \epsilon)\beta(\frac{1}{1-r})$$

$$i.e., \ \frac{\alpha(T_w^{-1}(T_{g(h)}(r)))}{\beta(\frac{1}{1-r})} \leq \Delta(\rho_{(\alpha,\beta)}[g]_k + \epsilon)$$

$$i.e., \ \liminf_{r\to 1} \frac{\alpha(T_w^{-1}(T_{g(h)}(r)))}{\beta(\frac{1}{1-r})} = \lambda_{(\alpha,\beta)}[g(h)]_w < \infty.$$

This is a contradiction. □

Thus the theorem follows.

Remark 2. *If we take "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \lambda_{(\alpha,\beta)}[h]_k \leq \rho_{(\alpha,\beta)}[h]_k < \infty$" and other conditions remain same, the conclusion of Theorem 2 remains true with "$\alpha(T_k^{-1}(T_g(r)))$" replaced by "$\alpha(T_k^{-1}(T_h(r)))$" in the denominator.*

Remark 3. *Theorem 2 and Remark 2 are also valid with "limit superior" instead of "limit" if "$\lambda_{(\alpha,\beta)}[g(h)] = \infty$" is replaced by "$\rho_{(\alpha,\beta)}[g(h)] = \infty$" and the other conditions remain the same.*

We may now state the following theorem without proof based on Definition 3.

Theorem 3. *Let g be an analytic function and h, w and k be non-constant entire functions in U such that $0 < \overline{\lambda}_{(\alpha,\beta)}[g(h)]_w \leq \overline{\rho}_{(\alpha,\beta)}[g(h)]_w < \infty$ and $0 < \overline{\lambda}_{(\alpha,\beta)}[g]_k \leq \overline{\rho}_{(\alpha,\beta)}[g]_k < \infty$. Then*

$$\frac{\overline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\overline{\rho}_{(\alpha,\beta)}[g]_k} \leq \liminf_{r\to 1} \frac{\alpha(\log\left(T_w^{-1}\left(T_{g(h)}(r)\right)\right))}{\alpha(\log\left(T_k^{-1}(T_g(r))\right))} \leq \min\left\{\frac{\overline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\overline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\overline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\overline{\rho}_{(\alpha,\beta)}[g]_k}\right\}$$

$$\leq \max\left\{\frac{\overline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\overline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\overline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\overline{\rho}_{(\alpha,\beta)}[g]_k}\right\} \leq \limsup_{r\to 1} \frac{\alpha(\log\left(T_w^{-1}\left(T_{g(h)}(r)\right)\right))}{\alpha(\log\left(T_k^{-1}(T_g(r))\right))} \leq \frac{\overline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\overline{\lambda}_{(\alpha,\beta)}[g]_k}.$$

Remark 4. *If we take "$0 < \overline{\lambda}_{(\alpha,\beta)}[h]_k \leq \overline{\rho}_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \overline{\lambda}_{(\alpha,\beta)}[g]_k \leq \overline{\rho}_{(\alpha,\beta)}[g]_k < \infty$" and other conditions remain same, the conclusion of Theorem 3 remains true with "$\overline{\lambda}_{(\alpha,\beta)}[g]_k$", "$\overline{\rho}_{(\alpha,\beta)}[g]_k$" and "$\alpha(\log\left(T_k^{-1}(T_g(r))\right))$" replaced by "$\overline{\lambda}_{(\alpha,\beta)}[h]_k$", "$\overline{\rho}_{(\alpha,\beta)}[h]_k$" and "$\alpha(\log\left(T_k^{-1}(T_h(r))\right))$" respectively in the denominator.*

We may now state the following theorem without proof based on Definition 4.

Theorem 4. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \underline{\lambda}_{(\alpha,\beta)}[g(h)]_w \leq \underline{\rho}_{(\alpha,\beta)}[g(h)]_w < \infty$ and $0 < \underline{\lambda}_{(\alpha,\beta)}[g]_k \leq \underline{\rho}_{(\alpha,\beta)}[g]_k < \infty$. Then*

$$\frac{\underline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\underline{\rho}_{(\alpha,\beta)}[g]_k} \le \liminf_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \le \min\left\{\frac{\underline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\underline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\underline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\underline{\rho}_{(\alpha,\beta)}[g]_k}\right\}$$
$$\le \max\left\{\frac{\underline{\lambda}_{(\alpha,\beta)}[g(h)]_w}{\underline{\lambda}_{(\alpha,\beta)}[g]_k}, \frac{\underline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\underline{\rho}_{(\alpha,\beta)}[g]_k}\right\} \le \limsup_{r\to 1}\frac{\alpha\left(T_w^{-1}(T_{g(h)}(r))\right)}{\alpha\left(T_k^{-1}(T_g(r))\right)} \le \frac{\underline{\rho}_{(\alpha,\beta)}[g(h)]_w}{\underline{\lambda}_{(\alpha,\beta)}[g]_k}.$$

Remark 5. *If we take "$0 < \underline{\lambda}_{(\alpha,\beta)}[h]_k \le \underline{\rho}_{(\alpha,\beta)}[h]_k < \infty$" instead of "$0 < \underline{\lambda}_{(\alpha,\beta)}[g]_k \le \underline{\rho}_{(\alpha,\beta)}[g]_k < \infty$" and other conditions remain same, the results of Theorem 4 remain true with "$\underline{\lambda}_{(\alpha,\beta)}[g]_k$", "$\underline{\rho}_{(\alpha,\beta)}[g]_k$" and "$\alpha\left(T_k^{-1}(T_g(r))\right)$" replaced by "$\underline{\lambda}_{(\alpha,\beta)}[h]_k$", "$\underline{\rho}_{(\alpha,\beta)}[h]_k$" and "$\alpha\left(T_k^{-1}(T_h(r))\right)$" respectively in the denominator.*

Theorem 5. *Let g be an analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \overline{\sigma}_{(\alpha,\beta)}[g(h)]_w \le \sigma_{(\alpha,\beta)}[g(h)]_w < \infty$, $0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \le \sigma_{(\alpha,\beta)}[g]_k < \infty$ and $\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$. Then*

$$\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k} \le \liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \le \min\left\{\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}, \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}, \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}\right\} \le \limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \le \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}.$$

Proof. From the definitions of $\sigma_{(\alpha,\beta)}[g]_k, \overline{\sigma}_{(\alpha,\beta)}[g]_k, \sigma_{(\alpha,\beta)}[g(h)]_w$ and $\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w$, we have for arbitrary positive ε and for all sufficiently large values of $\frac{1}{1-r}$ that

$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \ge \left(\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w - \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g(h)]_w}, \quad (17)$$

$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \le \left(\sigma_{(\alpha,\beta)}[g]_k + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (18)$$

$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \ge \left(\overline{\sigma}_{(\alpha,\beta)}[g]_k - \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (19)$$

$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \le \left(\sigma_{(\alpha,\beta)}[g(h)]_w + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g(h)]_w}. \quad (20)$$

Again for a sequence of values of $\frac{1}{1-r}$ tending to infinity, we get that

$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \le \left(\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g(h)]_w}, \quad (21)$$

$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \le \left(\overline{\sigma}_{(\alpha,\beta)}[g]_k + \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (22)$$

$$\exp(\alpha(T_k^{-1}\left(T_g(r)\right))) \ge \left(\sigma_{(\alpha,\beta)}[g]_k - \varepsilon\right)\left(\exp(\beta((1-r)^{-1}))\right)^{\rho_{(\alpha,\beta)}[g]_k}, \quad (23)$$

$$\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right))) \geqslant (\sigma_{(\alpha,\beta)}[g(h)]_w - \varepsilon)(\exp(\beta((1-r)^{-1})))^{\rho_{(\alpha,\beta)}[g(h)]_w}. \quad (24)$$

Now from (17), (18) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$, it follows for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \geqslant \frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\sigma_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_w^{-1}\left(T_g(r)\right)))}\geqslant\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_w}. \tag{25}$$

Combining (21) and (19) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w+\varepsilon}{\overline{\sigma}_{(\alpha,\beta)}[g]_k-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}. \tag{26}$$

Now from (17), (22) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, we obtain for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\geq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w-\varepsilon}{\overline{\sigma}_{(\alpha,\beta)}[g]_k+\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\geq\frac{\overline{\sigma}_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}. \tag{27}$$

In view of the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, it follows from (19) and (20) for all sufficiently large values of $\frac{1}{1-r}$ that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w+\varepsilon}{\overline{\sigma}_{(\alpha,\beta)}[g]_k-\varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\overline{\sigma}_{(\alpha,\beta)}[g]_k}. \tag{28}$$

Now from (20), (23) and the condition $\rho_{(\alpha,\beta)}[g(h)]_w=\rho_{(\alpha,\beta)}[g]_k$, it follows for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w+\varepsilon}{\sigma_{(\alpha,\beta)}[g]_k-\varepsilon}.$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))}\leq\frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}. \tag{29}$$

So combining (18) and (24) and in view of the condition $\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$, we get for a sequence of values of $\frac{1}{1-r}$ tending to infinity that

$$\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \geqslant \frac{\sigma_{(\alpha,\beta)}[g(h)]_w - \varepsilon}{\sigma_{(\alpha,\beta)}[g]_k + \varepsilon}.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \geqslant \frac{\sigma_{(\alpha,\beta)}[g(h)]_w}{\sigma_{(\alpha,\beta)}[g]_k}. \tag{30}$$

Thus the theorem follows from (25) , (26) , (27), (28) , (29) and (30) . □

Remark 6. *If we take "$0 < \overline{\sigma}_{(\alpha,\beta)}[h]_k \leq \sigma_{(\alpha,\beta)}[h]_k < \infty$"and "$\rho_{(\alpha,\beta)}[g(h)]_w = \rho^{(\alpha,\beta)}[h]_k$" instead of "$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$" and "$\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$" and other conditions remain same, the results of Theorem 5 remain true with "$\sigma_{(\alpha,\beta)}[g]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))$" replaced by "$\sigma_{(\alpha,\beta)}[h]_k$", "$\overline{\sigma}_{(\alpha,\beta)}[h]_k$" and "$\exp(\alpha(T_k^{-1}\left(T_h(r)\right)))$" respectively in the denominator.*

Remark 7. *If we take "$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$" and "$\rho_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$" instead of "$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$" and "$\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$" and other conditions remain same, the results of Theorem 5 remain true with "$\sigma_{(\alpha,\beta)}[g]_k$" and "$\overline{\sigma}_{(\alpha,\beta)}[g]_k$" replaced by "$\overline{\tau}_{(\alpha,\beta)}[g]_k$" and "$\tau_{(\alpha,\beta)}[g]_k$" respectively in the denominator.*

Remark 8. *If we take "$0 < \tau_{(\alpha,\beta)}[h]_k \leq \overline{\tau}_{(\alpha,\beta)}[h]_k < \infty$" and "$\rho_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[h]_k$" instead of "$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$" and "$\rho_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$" and other conditions remain same, the results of Theorem 5 remain true with "$\overline{\sigma}_{(\alpha,\beta)}[g]_k$", "$\sigma_{(\alpha,\beta)}[g]_k$" and "$\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))$" replaced by "$\tau_{(\alpha,\beta)}[h]_k$", "$\overline{\tau}_{(\alpha,\beta)}[h]_k$" and "$\exp(\alpha(T_k^{-1}\left(T_h(r)\right)))$" respectively in the denominator.*

Now in the line of Theorem 5 , one can easily prove the following theorem using the notion of generalized Nevanlinna weak type and therefore the proof is omitted.

Theorem 6. *Let g be a analytic function and h, w and k be non-constant entire functions in the unit disc U such that $0 < \tau_{(\alpha,\beta)}[g(h)]_w \leq \overline{\tau}_{(\alpha,\beta)}[g(h)]_w < \infty$, $0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$ and $\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$. Then*

$$\frac{\tau_{(\alpha,\beta)}[g(h)]_w}{\overline{\tau}_{(\alpha,\beta)}[g]_k} \leq \liminf_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \leq \min\left\{\frac{\tau_{(\alpha,\beta)}[g(h)]_w}{\tau_{(\alpha,\beta)}[g]_k}, \frac{\overline{\tau}_{(\alpha,\beta)}[g(h)]_w}{\overline{\tau}_{(\alpha,\beta)}[g]_k}\right\}$$

$$\leq \max\left\{\frac{\tau_{(\alpha,\beta)}[g(h)]_w}{\tau_{(\alpha,\beta)}[g]_k}, \frac{\overline{\tau}_{(\alpha,\beta)}[g(h)]_w}{\overline{\tau}_{(\alpha,\beta)}[g]_k}\right\} \leq \limsup_{r\to 1}\frac{\exp(\alpha(T_w^{-1}\left(T_{g(h)}(r)\right)))}{\exp(\alpha(T_k^{-1}\left(T_g(r)\right)))} \leq \frac{\overline{\tau}_{(\alpha,\beta)}[g(h)]_w}{\tau_{(\alpha,\beta)}[g]_k}.$$

Remark 9. *If we take "$0 < \tau_{(\alpha,\beta)}[h]_k \leq \overline{\tau}_{(\alpha,\beta)}[h]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[h]_k$" instead of "$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$" and "$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$" and other conditions remain same, the results of Theorem 6 remain true* *with "*$\tau_{(\alpha,\beta)}[g]_k$*", "*$\overline{\tau}_{(\alpha,\beta)}[g]_k$*" and "*$\exp(\alpha(T_k^{-1}(T_g(r))))$*" replaced by "*$\tau_{(\alpha,\beta)}[h]_k$*", "*$\overline{\tau}_{(\alpha,\beta)}[h]_k$*" and "*$\exp(\alpha(T_k^{-1}(T_h(r))))$*" respectively in the denominator.*

Remark 10. *If we take "*$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\lambda_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[g]_k$*" instead of "*$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$*" and other conditions remain same, the results of Theorem 6 remain true with "*$\tau_{(\alpha,\beta)}[g]_k$*" and "*$\overline{\tau}_{(\alpha,\beta)}[g]_k$*" replaced by "*$\overline{\sigma}_{(\alpha,\beta)}[g]_k$*" and "*$\sigma_{(\alpha,\beta)}[g]_k$*" respectively in the denominator.*

Remark 11. *If we take "*$0 < \overline{\sigma}_{(\alpha,\beta)}[g]_k \leq \sigma_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\lambda_{(\alpha,\beta)}[g(h)]_w = \rho_{(\alpha,\beta)}[h]_k$*" instead of "*$0 < \tau_{(\alpha,\beta)}[g]_k \leq \overline{\tau}_{(\alpha,\beta)}[g]_k < \infty$*" and "*$\lambda_{(\alpha,\beta)}[g(h)]_w = \lambda_{(\alpha,\beta)}[g]_k$*" and other conditions remain same, the results of Theorem 6 remain true with "*$\tau_{(\alpha,\beta)}[g]_k$*", "*$\overline{\tau}_{(\alpha,\beta)}[g]_k$*" and "*$\exp(\alpha(T_k^{-1}(T_g(r))))$*" replaced by "*$\overline{\sigma}_{(\alpha,\beta)}[h]_k$*", "*$\sigma_{(\alpha,\beta)}[h]_k$*" and "*$\exp(\alpha(T_k^{-1}(T_h(r))))$*" respectively in the denominator.*

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