Universal Journal of Mathematics and Applications, 4 (3) (2021) 114-124
Research paper

Universal Journal of Mathematics and Applications

Journal Homepage: www.dergipark.gov.tr/ujma
ISSN 2619-9653
DOI: https://doi.org/10.32323/ujma.953684

Refinements of Hermite-Hadamard Type Inequalities for s-Convex Functions with Applications to Special Means

Uğur S. Kırmacı[1]

[1]*Atatürk University, K.K.Education Faculty, Department of Mathematics, 25240, Erzurum, Turkey*

Article Info

Keywords: *Hermite-Hadamard inequality, convex functions, s-concave functions, s-convex fuctions, Hölder's inequality.*
2010 AMS: *26D15, 26B25*
Received: *17 June 2021*
Accepted: *11 October 2021*
Available online: *11 October 2021*

Abstract

In this paper, we establish some Hermite-Hadamard type inequalities for s-convex functions in the first and second sense. Some applications to special means for real numbers are also given.

1. Introduction

Let $f : I \subset R \to R$be a convex function on the interval I of real numbers and $a,b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

is known as Hermite-Hadamard's inequality for convex functions [4].
In [13] and [4, pp.278]), the following concept was introduced by Orlicz.
A function $f : R^+ \to R$, where $R^+ = [0,\infty)$, is said to be s-convex in the first sense if:

$$f(\alpha_1 u + \beta_1 v) \leq {\alpha_1}^s f(u) + {\beta_1}^s f(v),$$

for all $u,v \in R^+$, $\alpha_1, \beta_1 \geq 0$and $s \in (0,1]$ with ${\alpha_1}^s + {\beta_1}^s = 1$. The class of s-convex functions in the first sense is usually denoted with K_s^1.
In [9] and [4, pp.288]), Hudzik and Maligranda considered, among others, the class of functions which is s-convex in the second sense. This class is defined in the following way:
$f : [0,\infty) \to R$ is called s-convex in the second sense if

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y)$$

holds for all $x,y \in [0,\infty), \lambda \in [0,1]$ and for some fixed $s \in (0,1]$. The class of s-convex functions in the second sense is usually denoted with K_s^2.
In [5], S.S. Dragomir and S. Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex functions in the second sense:

Theorem 1.1: Suppose that $\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq (b-a)\left(\frac{1}{8}\right)^{1-\frac{1}{q}}(M^{1/q}+N^{1/q})(|f'(a)|+|f'(b)|)$ is an s-convex function in the second sense, where $s \in (0,1)$and let $a_1 = |f'(a)|^p, b_1 = 2|f'(b)|^p, a_2 = 2|f'(a)|^p, b_2 = |f'(b)|^p$. If $f \in L_1([a,b])$, then the following inequalities hold:

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.1}$$

Email addresses and ORCID numbers: **kirmaci@atauni.edu.tr, 0000-0002-8177-6649 (U.S. Kırmacı),**

The constant $k = 1/(s+1)$ is the best possible in the second inequalitiy in (1.1).
In [6], S.S. Dragomir presented the following result:
Theorem 1.2: Let $f:[a,b]\to R$ be a L-Lipschitzian mapping on $[a,b]$. Then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\le\frac{5}{36}L(b-a)^2. \tag{1.2}$$

In [7], S.S. Dragomir et al. gave the following result:
Theorem 1.3: Suppose $f:[a,b]\to R$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L_1([a,b])$. Then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\le\frac{(b-a)}{3}\left\|f'\right\|_1, \tag{1.3}$$

where $\|f'\|_1=\int_a^b|f'(x)|\,dx$.
Note that the bound of (1.3) for L-Lipschitzian is $\frac{5}{36}L(b-a)$ [7].
In [16], Y. Shuang and F. Qi established the following results:
Theorem 1.4 ([16,Theorem 3.5]): Let $f:R_o=(0,\infty]\to R$ be a differentiable function on R_o, a, $b\in R_o$ with $a<b$ and $f'\in L_1([a,b])$. If $|f'|^q$is (α,m)-convex on $[0,\frac{b}{m}]$ for $(\alpha,m)\in(0,1]^2$ and $q>1$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\right|$$

$$\le\frac{b-a}{4}\left[\frac{(q-1)(3^{\frac{2q-1}{q-1}}+1)}{2^{\frac{2(2q-1)}{q-1}}(2q-1)}\right]^{1-1/q}\left[\frac{1}{\alpha+1}\left|f'(a)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right]^{1/q}+\left[\frac{1}{\alpha+1}\left|f'\left(\frac{a+b}{2}\right)\right|^q+\frac{m\alpha}{\alpha+1}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{1/q}$$

Theorem 1.5 ([16,Corollary 3.6]): Under the assumptions of Theorem 1.4, if $\alpha=m=1$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\right|$$

$$\le\frac{(b-a)}{4}\left[\frac{(q-1)(3^{\frac{2q-1}{q-1}}+1)}{2^{\frac{2(2q-1)}{q-1}}(2q-1)}\right]^{1-\frac{1}{q}}\left\{\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+|f'(b)|^q}{2}\right]^{1/q}\right\}. \tag{1.4}$$

In [17], Y. Shuang et al. gave the following results:
Theorem 1.6 ([17,Theorem 3.2]): Let $f:I\subset R_o\to R$ be a differentiable function on I^o, $a,b\in I$ with $a<b$ and $f'\in L_1([a,b])$. If $|f'|^q$is s-convex function on $[a,b]$ for some fixed $s\in(0,1]$ and $q>1$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right|$$

$$\le\frac{(b-a)}{4}\left[\frac{(q-1)(4^{\frac{2q-1}{q-1}}+1)}{5^{\frac{(2q-1)}{q-1}}(2q-1)}\right]^{1-\frac{1}{q}}\left\{\left[\frac{|f'(a)|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{s+1}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{s+1}\right]^{1/q}\right\}.$$

Theorem 1.7 ([17,Corollary 3.3]): Under the assumptions of Theorem 1.6, for s=1, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right|$$

$$\le\frac{(b-a)}{4}\left[\frac{(q-1)(4^{\frac{2q-1}{q-1}}+1)}{5^{\frac{(2q-1)}{q-1}}(2q-1)}\right]^{1-\frac{1}{q}}\left\{\left[\frac{\left|f'(a)\right|^q+\left|f'\left(\frac{a+b}{2}\right)\right|^q}{2}\right]^{1/q}+\left[\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^q+\left|f'(b)\right|^q}{2}\right]^{1/q}\right\}. \tag{1.5}$$

In [8], T. Du et al. gave the following results:
Theorem 1.8 [8,Corollary 2.8]): Let $f:I\subset R_o\to R$ be a differentiable function on I^o, where $a,b\in I^o$ such that $0<a<b$. If $t=k=\frac{1}{2}$, $-1<s\le1$ *and* $m=1$, the inequality holds for (s,m)-convex functions:

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{f(a)+f(b)}{2}\right|\le\frac{b-a}{8^{1-\frac{1}{q}}}\left(\frac{1}{2^{s+2}(s+1)(s+2)}\right)^{\frac{1}{q}}$$

$$\times\{\left[\left|f'(b)\right|^q+\left(s2^{s+1}+1\right)\left|f'(a)\right|^q\right]^{\frac{1}{q}}+\left[\left|f'(a)\right|^q+\left(s2^{s+1}+1\right)\left|f'(b)\right|^q\right]^{\frac{1}{q}}\}. \tag{1.6}$$

Theorem 1.9 ([8,Corollary 2.5]): Let $f : I \subset R_o \to R$ be a differentiable function on I^o, where $a,b \in I^o$ such that $0 < a < b$. If the mapping $|f'|^{p/(p-1)}$ is (s,m)-convex on $[a,b]$, then we get, for $t = k = \frac{1}{2}$ *and* $m = 1$,

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{(p+1)^{\frac{1}{p}}}\left(\frac{1}{2(s+1)}\right)^{\frac{1}{q}}\left(\frac{1}{2}\right)^{1+\frac{1}{p}}$$

$$\times\left\{\left[\left|f^{'(a)}\right|^q + \left|f^{'(\frac{a+b}{2})}\right|^q\right]^{1/q} + \left[\left|f^{'(\frac{a+b}{2})}\right|^q + \left|f^{'(b)}\right|^q\right]^{1/q}\right\}. \tag{1.7}$$

In [12], U.S. Kırmacı et al. gave the following result:
Theorem 1.10 ([12,Theorem 3]): Let $f : I \to R,\ I \subset [0,\infty)$ be a differentiable function on I^o such that $f' \in L_1([a,b])$, where $a,b \in I,\ a < b$. If $|f'|^q$is s-convex function on $[a,b]$ for some fixed $s \in (0,1)$ and $q > 1$, then

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{2}$$

$$\times\left\{\left[\left|f^{'(a)}\right|^q + \left|f^{'(\frac{a+b}{2})}\right|^q\right]^{1/q} + \left[\left|f^{'(\frac{a+b}{2})}\right|^q + \left|f^{'(b)}\right|^q\right]^{1/q}\right\}. \tag{1.8}$$

In [10], author gave some inequalities for differentiable convex and concave mappings with applications to special means of real numbers.
The aim of this paper is to establish refinements inequalities of Hermite-Hadamard type for s-convex functions in the second sense.
In the development of pure and applied mathematics, convexity has played a key role. In linear programing, combinatory, orthogonal polynomials, quantum theory, number theory, optimization theory, dynamics and in the theory of relativity, integral inequalities have various applications.
For several recent results concerning integral inequalities for convex, quasi-convex, s-convex and $(\alpha,m)-$convex functions, we refer the reader to [1-18].
Throughout we suppose I is an interval on R and $a,b,c,A,B \in I^0$ with $a \le A \le c \le B \le b$. $(c \ne a,b), p,q \in R$ and $f : I^0 \to R$ is differentiable. (I^0denotes the interior of I.)

2. Main Results

First, we give the following Lemma.
Lemma 2.1 [10]: Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0, $a,b \in I^0$ with $a < b$. If $f' \in L_1([a,b])$, then we have

$$\begin{aligned}&f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A - \frac{1}{b-a}\int_a^b f(x)dx \\ &= (a-b)\left[\int_0^c (t-A)f'(ta+(1-t)b)dt + \int_c^1 (t-B)f'(ta+(1-t)b)dt\right],\end{aligned}$$

where $a,b,c,A,B \in I^0$ *with* $a \le A \le c \le B \le b$.

Proof: Let S:[a,b]$\to R$ be defined by

$$S(t) = \begin{cases} t-A\ , & t \in [0,c] \quad , \\ t-B\ , & t \in (c,1] \quad . \end{cases}$$

Integrating by parts and using the change of the variable $x = ta+(1-t)b$, we have

$$\begin{aligned}&\int_0^1 S(t)f'(ta+(1-t)b)dt = \int_0^c (t-A)f'(ta+(1-t)b)dt + \int_c^1 (t-B)f'(ta+(1-t)b)dt \\ &= \frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A - \frac{1}{b-a}\int_a^b f(x)dx\right].\end{aligned}$$

Hence we have the conclusion.

Remark 2.2: i). Applying Lemma 2.1 for $c = 1/2$, then we obtain the Lemma 2.1 given by T. Du et al. in [8, (for m=1)].
ii). Applying Lemma 2.1 for $A = \frac{1}{6},\ B = \frac{5}{6}$ *and* $c = 1/2$, then we obtain the Lemma 2.1 given by Qaisar and He in [15,(for m=1)].
iii). Applying Lemma 2.1 for $A = B = c = \frac{1}{2}$, then we get the Lemma 2.1 given by S.S.Dragomir and R.P. Agarwal in [3].
iv). Applying Lemma 2.1 for $A = 0,\ B = 1$ *and* $c = \frac{1}{2}$, then we get the Lemma 2.1 given by author in [11].

In the following theorems, we present generalized integral inequalities via s-convex mappings in the first and second sense.
Theorem 2.3: Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0 and let $p > 1$. If $|f'|^{p/(p-1)}$ is s-convex mapping in the second sense on $[a,b]$ for some fixed $s \in (0,1]$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right|$$

$$\leq\left[\frac{A^{p+1}+(c-A)^{p+1}}{p+1}\right]^{1/p}\left(\frac{c^{s+1}|f'(a)|^q+(1-(1-c)^{s+1})|f'(b)|^q}{s+1}\right)^{1/q}+\left[\frac{(B-c)^{p+1}+(1-B)^{p+1}}{p+1}\right]^{1/p}\left(\frac{(1-c^{s+1})|f'(a)|^q+(1-c)^{s+1}|f'(b)|^q}{s+1}\right)^{1/q}. \tag{2.1}$$

Proof: From Lemma 2.1, we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \leq \int_0^c |t-A|\,|f'(ta+(1-t)b)|\,dt+\int_c^1 |t-B|\,|f'(ta+(1-t)b|\,dt. \tag{2.2}$$

Using the Hölder's inequality for $p>1$, we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \tag{2.3}$$
$$\leq\left(\int_0^c |t-A|^p\,dt\right)^{1/p}\left(\int_0^c \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q}+\left(\int_c^1 |t-B|^p\,dt\right)^{1/p}\left(\int_c^1 \left|f'(ta+(1-t)b\right|^q dt\right)^{1/q},$$

where $\frac{1}{p}+\frac{1}{q}=1$.Since $|f'|^q$ is s- convex mapping in the second sense on $[a,b]$, we obtain

$$\int_0^c \left|f'(ta+(1-t)b)\right|^q dt\leq\int_0^c\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt=\frac{c^{s+1}\,|f'(a)|^q+(1-(1-c)^{s+1})\,|f'(b)|^q}{s+1} \tag{2.4}$$

and

$$\int_c^1 \left|f'(ta+(1-t)b)\right|^q dt\leq\int_c^1\left[t^s\left|f'(a)\right|^q+(1-t)^s\left|f'(b)\right|^q\right]dt=\frac{(1-c^{s+1})\,|f'(a)|^q+(1-c)^{s+1}\,|f'(b)|^q}{s+1}. \tag{2.5}$$

Where,

$$\int_c^1 t^s dt=\frac{1-c^{s+1}}{s+1},\int_c^1(1-t)^s dt=\frac{(1-c)^{s+1}}{s+1},\quad \int_0^c(1-t)^s dt=\frac{1-(1-c)^{s+1}}{s+1},\quad \int_0^c t^s dt=\frac{c^{s+1}}{s+1}.$$

Also, we have

$$P_p=\int_0^c |t-A|^p\,dt=\int_0^A(A-t)^p dt+\int_A^c(t-A)^p dt=\frac{A^{p+1}+(c-A)^{p+1}}{p+1}, \tag{2.6}$$

$$M_p=\int_c^1 |t-B|^p\,dt=\int_c^B(B-t)^p dt+\int_B^1(t-B)^p dt=\frac{(B-c)^{p+1}+(1-B)^{p+1}}{p+1}. \tag{2.7}$$

A combination of (2.3)-(2.7) gives the required inequality (2.1).

Corollary 2.4: Under the assumptions of Theorem 2.3,
i). When $A=0,\ B=1,\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(\frac{a+b}{2})\right|$$
$$\leq\frac{b-a}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}(s+1)^{\frac{1}{q}}}\left[\left[\left(\frac{1}{2^{s+1}}\left|f^{\prime(a)}\right|^q+\left(1-\frac{1}{2^{s+1}}\right)\left|f^{\prime(b)}\right|^q\right)^{\frac{1}{q}}\right]+\left[\left(1-\frac{1}{2^{s+1}}\right)\left|f^{\prime(a)}\right|^q+\left(\frac{1}{2^{s+1}}\left|f^{\prime(b)}\right|^q\right)^{\frac{1}{q}}\right]\right].$$

Using the fact that

$$\sum_{k=1}^n (a_k+b_k)^s\leq\sum_{k=1}^n a_k^s+\sum_{k=1}^n b_k^s, \tag{2.8}$$

for $0\leq s<1$, we obtain

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}(s+1)^{\frac{1}{q}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f^{\prime(a)}\right|+\left|f^{\prime(b)}\right|\right).$$

$$\text{For p}>1,\ \text{then p}+1>2\ \text{and so}\ \frac{1}{(p+1)^{\frac{1}{p}}}<\frac{1}{2^{\frac{1}{p}}}\ \textit{and also}\ \frac{1}{(s+1)^{\frac{1}{q}}}\leq 1,$$

for $s\in(0,1)$, $q\in(1,\infty)$. Hence, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - f(\frac{a+b}{2})\right| \le \frac{b-a}{2.4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})(\left|f'^{(a)}\right|+\left|f'(b)\right|).$$

ii). When $A=B=\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{2.4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})(\left|f'^{(a)}\right|+\left|f'(b)\right|).$$

iii). When $A=\frac{1}{4}, B=\frac{3}{4}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{1}{2}(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2})\right|$$

$$\le \frac{b-a}{4.4^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

iv). When $A=\frac{1}{6}, B=\frac{5}{6}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right| \le \frac{(b-a)\left(2^{p+1}+1\right)^{\frac{1}{p}}}{6.(12)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

v). When $A=\frac{1}{8},\ B=\frac{7}{8}, c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\right| \le \frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8.(16)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

vi). When $A=\frac{1}{10}, B=\frac{9}{10}, c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right| \le \frac{(b-a)\left(4^{p+1}+1\right)^{\frac{1}{p}}}{10.(20)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

vii). When $A=\frac{1}{12}, B=\frac{11}{12}, c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{5}{6}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{12}\right| \le \frac{(b-a)\left(5^{p+1}+1\right)^{\frac{1}{p}}}{12.(24)^{\frac{1}{p}}}\left(\frac{1}{2^{\frac{(s+1)}{q}}}+\left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\right)\left(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

Theorem 2.5: Let $f: I^0\subset R\to R$ be a differentiable mapping on I^0 and let $p>1$. If $|f'|^{p/(p-1)}$ is s-convex mapping in the first sense on$[a,b]$ for some fixed $s\in(0,1]$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \tag{2.9}$$

$$\le\ {P_p}^{1/p}\left(\frac{c^{s+1}\,|f'(a)|^q+(c(s+1)-c^{s+1})\,|f'(b)|^q}{s+1}\right)^{1/q} \tag{2.10}$$

$$+\ {M_p}^{1/p}\left(\frac{(1-c^{s+1})\,|f'(a)|^q+((1-c)(s+1)-(1-c^{s+1}))\,|f'(b)|^q}{s+1}\right)^{1/q} \tag{2.11}$$

Where P_p *and* M_p are as in (2.6) and (2.7) respectively.

Proof: From Lemma 2.1 and using the Hölder's inequality for $p>1$, we get inequality (2.3). Since $|f'|^q$is s- convex mapping in the first sense on $[a,b]$, we obtain

$$\int_0^c \left|f'(ta+(1-t)b)\right|^q dt \le \int_0^c \left[t^s\left|f'(a)\right|^q+(1-t^s)\left|f'(b)\right|^q\right]dt = \frac{c^{s+1}\,|f'(a)|^q+(c(s+1)-c^{s+1})\,|f'(b)|^q}{s+1} \tag{2.12}$$

and

$$\int_c^1 \left|f'(ta+(1-t)b)\right|^q dt \le \int_c^1 \left[t^s\left|f'(a)\right|^q+(1-t^s)\left|f'(b)\right|^q\right]dt$$

$$=\frac{(1-c^{s+1})\,|f'(a)|^q+((1-c)(s+1)-(1-c^{s+1}))\,|f'(b)|^q}{s+1}, \tag{2.13}$$

where,
$\int_c^1 t^s dt = \frac{1-c^{s+1}}{s+1}, \int_c^1 (1-t^s)dt = 1-c-\frac{1-c^{s+1}}{s+1}, \quad \int_0^c (1-t^s)dt = c-\frac{c^{s+1}}{s+1}, \quad \int_0^c t^s dt = \frac{c^{s+1}}{s+1}$. From (2.3),(2.6),(2.7),(2.12) and (2.13), we deduce required inequality (2.9).

Theorem 2.6: Let $f : I^0 \subset R \to R$ be a differentiable mapping on I^0 and let $p > 1$. If $|f'|^{p/(p-1)}$ is s-convex mapping in the second sense on $[a,b]$ for some fixed $s \in (0,1)$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \le P_p{}^{1/p}\left(c\frac{|f'(ca+(1-c)b)|^q+|f'(b)|^q}{s+1}\right)^{1/q} + M_p{}^{1/p}\left((1-c)\frac{|f'(a)|^q+|f'(ca+(1-c)b)|^q}{s+1}\right)^{1/q}. \tag{2.14}$$

Where P_p *and* M_p are as in (2.6) and (2.7) respectively.

Proof: : From Lemma 2.1 and using the Hölder's inequality for $p > 1$, we get inequality (2.3). Let us substitute $x = ta+(1-t)b$ *and* $dx = (a-b)dt$, we get

$$\int_0^c |f'(ta+(1-t)b)|^q dt \le \frac{1}{a-b}\int_b^{ac+(1-c)b} |f'(x)|^q dx = \frac{c}{(a-b)c}\int_b^{ac+(1-c)b} |f'(x)|^q dx$$

and

$$\int_c^1 |f'(ta+(1-t)b)|^q dt \le \frac{1}{a-b}\int_{ac+(1-c)b}^a |f'(x)|^q dx = \frac{1-c}{(a-b)(1-c)}\int_{ac+(1-c)b}^a |f'(x)|^q dx.$$

Since $|f'|^q$is s-convex mapping in the second sense on $[a,b]$, using the above inequalities and by inequality (1.1), we have

$$\int_0^c |f'(ta+(1-t)b)|^q dt \le c\frac{|f'(ca+(1-c)b)|^q+|f'(b)|^q}{s+1} \tag{2.15}$$

and

$$\int_c^1 |f'(ta+(1-t)b)|^q dt \le (1-c)\frac{|f'(a)|^q+|f'(ca+(1-c)b)|^q}{s+1}. \tag{2.16}$$

From (2.3),(2.6),(2.7),(2.15) and (2.16), we obtain required inequality (2.14).

Corollary 2.7: Under the assumptions of Theorem 2.6, using the inequality (2.8) and since $\frac{1}{(p+1)^{\frac{1}{p}}} < \frac{1}{2^{\frac{1}{p}}}$ *and* $\frac{1}{(s+1)^{\frac{1}{q}}} \le 1$,

i). When $A = 0,\ B = 1,\ c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f(\frac{a+b}{2})\right| \le \frac{b-a}{4.2^{\frac{1}{p}}}\left(2\left|f'^{(\frac{a+b}{2})}\right| + \left|f'^{(a)}\right| + \left|f'^{(b)}\right|\right).$$

ii). When $A = \ B = \ c = 1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{f(a)+f(b)}{2}\right| \le \frac{b-a}{4.2^{\frac{1}{p}}}\left(2\left|f'^{(\frac{a+b}{2})}\right| + \left|f'^{(a)}\right| + \left|f'^{(b)}\right|\right). \tag{2.17}$$

iii). When $A = \frac{1}{4}, B = \frac{3}{4}, c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2}(f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2})\right| \le \frac{b-a}{8.4^{\frac{1}{p}}}\left(2\left|f'^{(\frac{a+b}{2})}\right| + \left|f'^{(a)}\right| + \left|f'^{(b)}\right|\right).$$

iv). When $A = \frac{1}{6}, B = \frac{5}{6}, c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{2}{3}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{6}\right| \le \frac{(b-a)\left(2^{p+1}+1\right)^{\frac{1}{p}}}{12.6^{\frac{1}{p}}}(2\left|f'^{(\frac{a+b}{2})}\right| + \left|f'^{(a)}\right| + + |f'(b)|).$$

v). When $A = \frac{1}{8}, B = \frac{7}{8}, c = 1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{3}{4}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{8}\right| \le \frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{16.8^{\frac{1}{p}}}(2\left|f'^{(\frac{a+b}{2})}\right| + \left|f'^{(a)}\right| + \quad + |f'(b)|). \tag{2.18}$$

vi). When $A = \frac{1}{10}, B = \frac{9}{10}, c = 1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - \frac{4}{5}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{10}\right| \le \frac{(b-a)\left(4^{p+1}+1\right)^{\frac{1}{p}}}{20.10^{\frac{1}{p}}}(2\left|f'^{(\frac{a+b}{2})}\right| + \left|f'^{(a)}\right| + + |f'(b)|). \tag{2.19}$$

vii). When $A=\frac{1}{12}, B=\frac{11}{12}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{5}{6}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{12}\right|\leq\frac{(b-a)\left(5^{p+1}+1\right)^{\frac{1}{p}}}{24.12^{\frac{1}{p}}}\left(2\left|f'^{\left(\frac{a+b}{2}\right)}\right|+\left|f'^{(a)}\right|+\left|f'^{(b)}\right|\right).$$

Remark 2.8: The followings are observed that:
i) The inequality (2.18) is a refinement of inequality (1.4) presented by Y. Shuang and F. Qi in [16]
ii) The inequality (2.19)) is a refinement of inequality (1.5) presented by Y. Shuang et al. in [17]
iii) The inequality (2.17) is both a refinement of inequality (1.7) given by T. Du et al. in [8] and the inequality (1.8) given by Kirmaci et al. in [12].

Theorem 2.9: Let $f: I^0\subset R\to R$ be a differentiable mapping on I^0 and let $p>1$. If $|f'|^{p/(p-1)}$ is s-concave mapping on$[a,b]$ for some fixed $s\in(0,1)$, then we have

$$\left|\frac{1}{a-b}\left[f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\frac{1}{b-a}\int_a^b f(x)dx\right]\right| \tag{2.20}$$

$$\leq {P_p}^{1/p}\left(2^{s-1}\left|f'\left(\tfrac{c}{2}a+\tfrac{2-c}{2}b\right)\right|^q\right)^{1/q}+{M_p}^{1/p}\left(2^{s-1}\left|f'\left(\tfrac{1+c}{2}a+\tfrac{1-c}{2}b\right)\right|^q\right)^{1/q}.$$

Where P_p *and* M_p are as in (2.6) and (2.7) respectively.

Proof: : From Lemma 2.1 and using the Hölder's inequality for $p>1$, we get inequality (2.3). Since $|f'|^q$is s-concave mapping on $[a,b]$ and using inequality (1.1), we have

$$\int_0^c\left|f'(ta+(1-t)b)\right|^q dt\leq 2^{s-1}\left|f'\left(\frac{c}{2}a+\frac{2-c}{2}b\right)\right|^q \tag{2.21}$$

and

$$\int_c^1\left|f'(ta+(1-t)b)\right|^q dt\leq 2^{s-1}\left|f'\left(\frac{1+c}{2}a+\frac{1-c}{2}b\right)\right|^q. \tag{2.22}$$

From (2.3),(2.6),(2.7),(2.21) and (2.22), we obtain required inequality (2.20).

Corollary 2.10: Under the assumptions of Theorem 2.9, using the inequality (2.8) and since $2^{(s-1)/q}<1$ *and* $\frac{1}{(p+1)^{\frac{1}{p}}}<\frac{1}{2^{\frac{1}{p}}}$ *for* $s\in(0,1)$ *and* $q\in(1,\infty)$,

i). When $A=0,\ B=1,\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{(p+1)^{\frac{1}{p}}2^{\frac{p+1}{p}}}2^{\frac{s-1}{q}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right]$$

$$\leq\frac{1}{2.4^{\frac{1}{p}}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

ii). When $A=\ B=\ c=\frac{1}{2}$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{1}{2.4^{\frac{1}{p}}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

iii). When $A=\frac{1}{4}, B=\frac{3}{4}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{1}{2}(f\left(\frac{a+b}{2}\right)+\frac{f(a)+f(b)}{2})\right|$$

$$\leq\frac{b-a}{4.4^{\frac{1}{p}}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

iv) When $A=\frac{1}{6}, B=\frac{5}{6}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{2}{3}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{6}\right|\leq\frac{(b-a)\left(2^{p+1}+1\right)^{\frac{1}{p}}}{6^{\frac{1}{p}+1}}\left[\left|f'^{\left(\frac{a+3b}{4}\right)}\right|+\left|f'^{\left(\frac{3a+b}{4}\right)}\right|\right].$$

v). When $A=\frac{1}{8}, B=\frac{7}{8}, c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{3}{4}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{8}\right|\leq\frac{(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8^{\frac{1}{p}+1}}\left[\left|f'\left(\tfrac{a+3b}{4}\right)\right|+\left|f'\left(\tfrac{3a+b}{4}\right)\right|\right].$$

vi). When $A=\frac{1}{10}, B=\frac{9}{10}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{4}{5}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{10}\right|\leq\frac{(b-a)\left(4^{p+1}+1\right)^{\frac{1}{p}}}{10^{\frac{1}{p}+1}}\left[\left|f'\left(\tfrac{a+3b}{4}\right)\right|+\left|f'\left(\tfrac{3a+b}{4}\right)\right|\right].$$

vii). When $A=\frac{1}{12}, B=\frac{11}{12}, c=1/2$, we get

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx-\frac{5}{6}f\left(\frac{a+b}{2}\right)-\frac{f(a)+f(b)}{12}\right|\leq\frac{(b-a)\left(5^{p+1}+1\right)^{\frac{1}{p}}}{12^{\frac{1}{p}+1}}\left[\left|f'\left(\tfrac{a+3b}{4}\right)\right|+\left|f'\left(\tfrac{3a+b}{4}\right)\right|\right].$$

Theorem 2.11: Let $f:I^0\subset R\to R$ be a differentiable mapping on I^0, $a,b\in I^0$ with $a<b$ and let $p\geq 1$. If the mapping $|f'|^p$ is s-convex in the first sense on $[a,b]$ for some fixed $s\in(0,1]$, then we have

$$\begin{aligned}&\left|f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\tfrac{1}{b-a}\textstyle\int_a^b f(x)dx\right|\\&\leq(a-b)\left\{P_2^{1-\frac{1}{p}}\left[T_1\,|f'(a)|^p+(P_2-T_1)\,|f'(b)|^p\right]^{1/p}+M_2^{1-\frac{1}{p}}\left[N_1\,|f'(a)|^p+(M_2-N_1)\,|f'(b)|^p\right]^{1/p}\right\},\end{aligned}\tag{2.23}$$

where,

$$\begin{aligned}P_2&=\frac{A^2+(c-A)^2}{2}, T_1=\frac{2}{(s+1)(s+2)}A^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{A}{s+1}\right],\\M_2&=\frac{(B-c)^2+(1-B)^2}{2}, N_1=\frac{2}{(s+1)(s+2)}B^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{B}{s+1}\right]+\frac{1}{s+2}-\frac{B}{s+1}.\end{aligned}$$

Proof: From Lemma 2.1, we get the inequality (2.2). By the power-mean inequality, we obtain

$$\int_0^c|t-A|\left|f'(ta+(1-t)b\right|dt\leq\left(\int_0^c|t-A|\,dt\right)^{1-\frac{1}{p}}\left(\int_0^c|t-A|\left|f'(ta+(1-t)b\right|^p dt\right)^{1/p}\tag{2.24}$$

and

$$\int_c^1|t-B|\left|f'(ta+(1-t)b\right|dt\leq\left(\int_c^1|t-B|\,dt\right)^{1-\frac{1}{p}}\left(\int_c^1|t-B|\left|f'(ta+(1-t)b)\right|^p dt\right)^{1/p}.\tag{2.25}$$

Since $|f'|^p$is s-convex in the first sense, we have

$$\begin{aligned}\int_0^c|t-A|\left|f'(ta+(1-t)b\right|^p dt&\leq\int_0^c|t-A|\left(t^s\left|f'(a)\right|^p+(1-t^s)\left|f'(b)\right|^p\right)dt\\&\leq T_1\left|f'^{(a)}\right|^p+(P_2-T_1)\left|f'(b)\right|^p\end{aligned}\tag{2.26}$$

and

$$\begin{aligned}\int_c^1|t-B|\left|f'(ta+(1-t)b\right|^p dt&\leq\int_c^1|t-B|\left(t^s\left|f'(a)\right|^p+(1-t^s)\left|f'(b)\right|^p\right)dt\\&\leq N_1\left|f'(a)\right|^p+(M_2-N_1)\left|f'(b)\right|^p.\end{aligned}\tag{2.27}$$

where,

$$\begin{aligned}P_2&=\int_0^c|t-A|\,dt=\int_0^A(A-t)dt+\int_A^c(t-A)dt=\frac{A^2+(c-A)^2}{2},\\T_1&=\int_0^c|t-A|\,t^s dt=\int_0^A(A-t)t^s dt+\int_A^c(t-A)t^s dt=\frac{2}{(s+1)(s+2)}A^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{A}{s+1}\right],\\M_2&=\int_c^1|t-B|\,dt=\int_c^B(B-t)dt+\int_B^1(t-B)dt=\frac{(B-c)^2+(1-B)^2}{2},\\N_1&=\int_c^1|t-B|\,t^s dt=\int_c^B(B-t)t^s dt+\int_B^1(t-B)t^s dt=\frac{2}{(s+1)(s+2)}B^{s+2}+c^{s+1}\left[\frac{c}{s+2}-\frac{B}{s+1}\right]++\frac{1}{s+2}-\frac{B}{s+1}\end{aligned}\tag{2.28}$$

and

$$P_2-T_1=\int_0^c|t-A|(1-t)dt=\int_0^A(A-t)(1-t)dt+\int_A^c(t-A)(1-t)dt,$$

$$M_2-N_1=\int_c^1|t-B|(1-t)dt=\int_c^B(B-t)(1-t)dt+\int_B^1(t-B)(1-t)dt.$$

A combination of (2.2) and (2.24)-(2.28) gives the required inequality (2.23).

Theorem 2.12: Let $f:I^0\subset R\to R$ be a differentiable mapping on I^0, $a,b\in I^0$ with $a<b$ and let $p\geq 1$. If the mapping $|f'|^p$ is s-convex in the second sense on $[a,b]$ for some fixed $s\in(0,1]$, then we have

$$\begin{aligned}&\left|f(ca+(1-c)b)(B-A)+f(a)(1-B)+f(b)A-\tfrac{1}{b-a}\int_a^b f(x)dx\right|\\&\leq(a-b)\left\{P_2^{1-\frac{1}{p}}\left[T_1|f'(a)|^p+T_2|f'(b)|^p\right]^{1/p}+M_2^{1-\frac{1}{p}}\left[N_1|f'(a)|^p+N_2|f'(b)|^p\right]^{1/p}\right\},\end{aligned}\tag{2.29}$$

where,

$$T_2=\frac{A}{s+1}+\frac{2(1-A)^{s+2}}{(s+1)(s+2)}-(1-c)^{s+1}\left[\frac{c-A}{s+1}+\frac{1-c}{(s+1)(s+2)}\right]-\frac{1}{(s+1)(s+2)},$$

$N_2=\frac{2(1-B)^{s+2}}{(s+1)(s+2)}+(1-c)^{s+1}\left[\frac{B-c}{s+1}-\frac{1-c}{(s+1)(s+2)}\right]$ and P_2,M_2,T_1,N_1 are as in (2.28).

Proof: From Lemma 2.1, we have the inequality (2.2). By the power-mean inequality, we get inequalities (2.24) and (2.25). Since $|f'|^p$is s-convex mapping in the second sense on $[a,b]$, we have

$$\int_0^c|t-A|\left|f'(ta+(1-t)b\right|^p dt\leq\int_0^c|t-A|\left(t^s\left|f'(a)\right|^p+(1-t)^s\left|f'(b)\right|^p\right)dt\leq T_1\left|f'(a)\right|^p+T_2\left|f'(b)\right|^p\tag{2.30}$$

and

$$\int_c^1|t-B|\left|f'(ta+(1-t)b\right|^p dt\leq\int_c^1|t-B|\left(t^s\left|f'(a)\right|^p+(1-t)^s\left|f'(b)\right|^p\right)dt\leq N_1\left|f'(a)\right|^p+N_2\left|f'(b)\right|^p.\tag{2.31}$$

Where,

$$\begin{aligned}T_2&=\int_0^c|t-A|(1-t)^sdt=\int_0^A(A-t)(1-t)^sdt+\int_A^c(t-A)(1-t)^sdt\\&=\frac{A}{s+1}+\frac{2(1-A)^{s+2}}{(s+1)(s+2)}-(1-c)^{s+1}\left[\frac{c-A}{s+1}+\frac{1-c}{(s+1)(s+2)}\right]-\frac{1}{(s+1)(s+2)},\end{aligned}$$

$$\begin{aligned}N_2&=\int_c^1|t-B|(1-t)^sdt=\int_c^B(B-t)(1-t)^sdt+\int_B^1(t-B)(1-t)^sdt\\&=\frac{2(1-B)^{s+2}}{(s+1)(s+2)}+(1-c)^{s+1}\left[\frac{B-c}{s+1}-\frac{1-c}{(s+1)(s+2)}\right]\end{aligned}\tag{2.32}$$

and T_1,N_1 are as in (2.28). A combination of (2.2),(2.24),(2.25), (2.30),(2.31) and (2.32) gives the required inequality (2.29).

Corollary 2.13: Under the assumptions of Theorem 2.12 and using the inequality (2.8),

i) When $A=0,\ B=1,\ c=1/2$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(T_{1'}^{\frac{1}{p}}+N_{1'}^{\frac{1}{p}})\left|f'^{(a)}\right|+(T_{2'}^{\frac{1}{p}}+N_{2'}^{\frac{1}{p}})\left|f'^{(b)}\right|],\tag{2.33}$$

where,

$$T_{1'}=\frac{1}{2^{s+2}(s+2)},\ N_{1'}=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)},\ T_{2'}=\frac{2^{s+2}-s-3}{2^{s+2}(s+1)(s+2)},\ N_{2'}=\frac{s+1}{2^{s+2}(s+1)(s+2)}.$$

Taking s=1 and p=1 in (2.33) yields

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b f(x)dx-f(\frac{a+b}{2})\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(\frac{1}{24}^{\frac{1}{p}}+\frac{1}{12}^{\frac{1}{p}})(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|)]\\&\leq\frac{b-a}{8}(\left|f'^{(a)}\right|+\left|f'^{(b)}\right|).\end{aligned}$$

ii) When $A=\ B=\ c=\frac{1}{2}$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)dx-\frac{f(a)+f(b)}{2}\right|\leq\frac{b-a}{8^{1-\frac{1}{p}}}[(T_{1''}^{\frac{1}{p}}+N_{1''}^{\frac{1}{p}})\left|f'^{(a)}\right|+(T_{2''}^{\frac{1}{p}}+N_{2''}^{\frac{1}{p}})\left|f'^{(b)}\right|],\tag{2.34}$$

where,

$$T_{1''} = \frac{1}{2^{s+2}(s+1)(s+2)},\ N_{1''} = \frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)},\ T_{2''} = \frac{s2^{s+1}+1}{2^{s+2}(s+1)(s+2)},\ N_{2''} = \frac{1}{2^{s+2}(s+1)(s+2)}.$$

Taking s=1 and p=1 in (2.34) yields

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{f(a)+f(b)}{2}\right| \leq \frac{b-a}{8^{1-\frac{1}{p}}}[(\frac{1}{48}^{\frac{1}{p}} + \frac{5}{48}^{\frac{1}{p}})(\left|f^{'(a)}\right| + \left|f^{'(b)}\right|)]$$

$$\leq \frac{b-a}{8}\left(\left|f^{'(a)}\right| + \left|f^{'(b)}\right|\right).$$

iii) When $A=\frac{1}{6},\ B=\frac{5}{6},\ c=\frac{1}{2}$ *and* $s=1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{2}{3}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{6}\right| \leq (b-a)\left(\frac{5}{72}\right)^{1-\frac{1}{p}}(\frac{90}{1296})^{1/p}(\left|f^{'(a)}\right| + |f'(b)|)$$

$$\leq \frac{5(b-a)}{72}(\left|f^{'(a)}\right| + |f'(b)|).$$

If $|f'(x)| \leq L$, then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{2}{3}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{6}\right| \leq \frac{5(b-a)}{36}L. \tag{2.35}$$

iv) When $A=\frac{1}{10},\ B=\frac{9}{10},\ c=\frac{1}{2}$ *and* $s=1$, we have

$$\left|\frac{1}{b-a}\int_a^b f(x)\,dx - \frac{8}{10}f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{10}\right| \leq (b-a)\left(\frac{17}{200}\right)^{1-\frac{1}{p}}(\frac{17}{200})^{1/p}(\left|f^{'(a)}\right| + |f'(b)|)$$

$$\leq \frac{17(b-a)}{200}(\left|f^{'(a)}\right| + |f'(b)|).$$

Remark 2.14: The followings are observed that
i)The inequality (2.34) is a refinement of inequality (1.6) presented by Du et al. in [8]
ii) The inequality (2.35) is a refinement of inequality (1.2) established by Dragomir et al. in [6] and the same as inequality (1.3) presented by Dragomir in [7].

3. Applications To Special Means

We shall consider the means for arbitrary real numbers $\alpha,\beta,\alpha\neq\beta$. We take
A$(\alpha,\beta)=\frac{\alpha+\beta}{2}$, $\alpha,\beta\in$R, (arithmetic mean)
L$_n(\alpha,\beta)=\left[\frac{\beta^{n+1}-\alpha^{n+1}}{(n+1)(\beta-\alpha)}\right]^{1/n}$, n$\in$Z$\backslash${-1,0}, $\alpha,\beta\in$R, $\alpha\neq\beta$, (generalized log-mean)

In [9] and [4, pp 288], the following example is given:
Let $s\in(0,1)$and $a,b,c\in R$. We define function $f:[0,\infty)\to R$ as

$$f(t) = \left\{\begin{array}{cc} a, & t=0 \\ bt^s + c, & t>0 \end{array}\right.$$

if $b\geq 0$ and $0\leq c\leq a$, then $f\in K_s^2$. Hence, for $a=c=0, b=1$, we have $f(t)=t^s, f:[0,1]\to[0,1], f\in K_s^2$.

Now, using the results of Section 2, we give some applications to special means of real numbers.

Proposition 3.1: Let $a,b\in I^o,\ 0<a<b$ *and* $0<s<1$. Then we have, for all $p>1$
i) $|L_s^s(a,b) - A^s(a,b)| \leq \frac{s(b-a)}{4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}} + \left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})A(|a|^{s-1},|b|^{s-1})$.
ii) $|L_s^s(a,b) - A(a^s,b^s)| \leq \frac{s(b-a)}{4^{\frac{1}{p}}}(\frac{1}{2^{\frac{(s+1)}{q}}} + \left(1-\frac{1}{2^{s+1}}\right)^{\frac{1}{q}})A(|a|^{s-1},|b|^{s-1})$.
Proof: The assertions follow from Corollaries 2.4-i and 2.4-ii applied to the mapping $f(x)=x^s, f:[0,1]\to[0,1]$, respectively.
Proposition 3.2: Let $a,b\in I^o,\ 0<a<b$ *and* $0<s<1$. Then we have, for all $p>1$
i)$|L_s^s(a,b) - A^s(a,b)| \leq \frac{s(b-a)}{2^{\frac{1}{p}+1}}(|A(a,b)|^{s-1} + A(|a|^{s-1},|b|^{s-1}))$.
ii) $|L_s^s(a,b) - A(a^s,b^s)| \leq \frac{s(b-a)}{2^{\frac{1}{p}+1}}(|A(a,b)|^{s-1} + A(|a|^{s-1},|b|^{s-1}))$.
iii) $\left| L_s^s(a,b) - \frac{3A^s(a,b)}{4} - \frac{A(a^s,b^s)}{8}\right| \leq \frac{s(b-a)\left(3^{p+1}+1\right)^{\frac{1}{p}}}{8^{\frac{1}{p}+1}}(\left(|A(a,b)|^{s-1} + A\left(|a|^{s-1},|b|^{s-1}\right)\right)$.
Proof: The assertions follow from Corollaries 2.7-i, 2.7-ii and 2.7-v applied to the mapping$f(x)=x^s, f:[0,1]\to[0,1]$, respectively.

Proposition 3.3: Let $a,b \in I^o$, $0 < a < b$ *and* $0 < s < 1$. Then we have, for all $p \geq 1$

i)$|L_s^s(a,b) - A^s(a,b)| \leq \frac{s(b-a)}{4.8^{-\frac{1}{p}}} A(\left(T^{\frac{1}{p}}_{1'+N^{\frac{1}{p}}_{1'}} \right) |a|^{s-1}, \left(T^{\frac{1}{p}}_{2'+N^{\frac{1}{p}}_{2'}} \right) |b|^{s-1}).$

ii) $|L_s^s(a,b) - A(a^s,b^s)| \leq \frac{s(b-a)}{4.8^{-\frac{1}{p}}} A(\left(T^{\frac{1}{p}}_{1''} + N^{\frac{1}{p}}_{1''} \right) |a|^{s-1}, \left(T^{\frac{1}{p}}_{2''} + N^{\frac{1}{p}}_{2''} \right) |b|^{s-1}).$

iii) $\left| L_s^s(a,b) - \frac{2A^s(a,b)}{3} - \frac{A(a^s,b^s)}{6} \right| \leq \frac{5s(b-a)}{36} A\left(|a|^{s-1}, |b|^{s-1} \right).$

Proof: The assertions follow from Corollaries 2.13-i, 2.13-ii and 2.13-iii applied to the mapping $f(x) = x^s, f : [0,1] \to [0,1]$, respectively.

Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

There is no funding for this work.

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

References

[1] M.W. Alomari, M. Darus, U.S. Kirmaci, *Refinements of Hadamard type inequalities for quasi-convex functions with applictions to trapezoidal formula and to special means,* Comput. Math. Appl. 59,1 (2010) 225-232.
[2] M. W. Alomari, M. Darus, U. S. Kirmaci, *Some inequalities of Hermite-Hadamard type for s-convex functions*, Acta Math. Sci. Ser. B Eng. Ed. **31** (2011), 1643-1652.
[3] S.S. Dragomir, R.P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, Appl. Math. Lett. 11 (1998) 91–95.
[4] S.S.Dragomir, *Selected Topics on Hermite-Hadamard Inequalities and Applications*, http://rgmia.vu.edu.au/SSDragomirWeb.html
[5] S.S. Dragomir, S.Fitzpatrick, *The Hadamard's inequality for s-convex functions in the second sense,* Demonstratio Math. 32 (1999) 687-696.
[6] S.S. Dragomir, *On Simpson's quadrature formula for Lipschitzian mappings and applications*, Soochow J. Math. 25 (1999) 175-180.
[7] S.S. Dragomir, R.P. Agarwal, P.Cerone, *On Simpson's inequality and applications*, J. Ineq. Appl. 5 (2000) 533-579.
[8] T. Du, Y. Li, Z. Yang, *A generalization of simpson's inequality via differentiable mapping using extended (s,m)-convex functions*, Appl. Math. Comput. 293 (2017) 358-369.
[9] H. Hudzik, L. Maligranda, *Some remarks on s-convex functions*, Aequationes Math. 48 (1994) 100-111.
[10] U.S. Kirmaci, *Improvement and further generalization of inequalities for differentiable mappings and applications,* Comput. Math. Appl. 55 (2008) 485-493.
[11] U.S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula*, Appl. Math. Comput. 147(2004), 137-146.
[12] U.S. Kirmaci, M. K. Bakula, M.E. Özdemir, J.E. Pečarić , *Hadamard-type inequalities for s-convex functions,* Appl. Math. Comput. 193 (2007) 26-35.
[13] W. Orlicz, *A note on modular spaces* I, Bull. Acad. Polon. Sci. Ser. Math. Astronom. Phys., 9 (1961), 157-162.
[14] J.E. Pečarić, F. Proschan, Y.L. Tong, Convex Functions, Partial Ordering and Statistical Applications, Academic Press, New York, (1991).
[15] S. Qaisar, C. He, S. Hussain, *A generalizations of Simpson's type inequality for differentiable functions using (α,m)-convex functions and applications*, Journal of Inequalities and Applications 2013;158: 13 pages.
[16] Y. Shuang, F. Qi, *Some integral inequalities for s-convex functions*, Gazi University Journal of Science, (2018), 31, 1192-1200.
[17] Y. Shuang, Y. Wang, F. Qi, *Integral iequalities of Simpson's type for (α,m)-convex functions,* J. Nonlinear Sci. Appl. 9 (2016) 6364-6370.
[18] M. Tunç, U.S.Kirmaci, *New integral inequalities for s-convex functions with applications*, International Electronic Journal of Pure and Applied Mathematics, 1(2) (2010), 131-141.