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On The Euclidean and the Spectral Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices

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Abstract

In this study, we have established upper and lower bounds for the Euclidean and the Spectral norms of Quaternion Cauchy-Toeplitz (T) and Quaternion Cauchy-Hankel (H) matrices respectively. Besides, by assuming complex matrix $T_n = A_1 + A_2 j$, we have defined the matrix $[T_n] = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$ and similarly, by assuming complex matrix $H_n = B_1 + B_2 j$, we have also defined another matrix $[H_n] = \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix}$. Then we have obtained the bounds of the spectral norms for these matrices.

Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrislerinin Euclidean ve Spektral Normları Üzerine

Anahtar Kelimeler
Quaternion Cauchy-Toeplitz Matrices, Quaternion Cauchy-Hankel Matrices, Euclidean Norm, Spectral Norm. 15A45, 15A60

Özet

Bu çalışmada, sırasıyla Quaternion Cauchy-Toeplitz (T), Quaternion Cauchy-Hankel (H) Matrislerinin Spektral ve Euclidean normlar için alt ve üst sınarlar elde ettik. Ayrıca, $T_n = A_1 + A_2 j$, kompleks matrisi yardımıyla $[T_n] = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix}$ matrisini ve benzer şekilde $H_n = B_1 + B_2 j$, kompleks matrisi yardımıyla $[H_n] = \begin{bmatrix} B_1 & B_2 \\ -\overline{B_2} & \overline{B_1} \end{bmatrix}$ matrisini tanımlayıp bu matrislerin spektral normlar için sınırlar elde ettik.

1. Introduction and Preliminaries

In quantum physic, the family of quaternions plays an important role. But in mathematics they generally play a role in algebraic systems, skew fields or noncommutative division algebras, matrices in commutative rings take attention but, matrices with quaternion entries has not been investigated very much yet. But in recent times quaternions are in order of day.
The main obstacles in the study of quaternion matrices, as expected come from the noncommutative multiplication of quaternions. One will find that working on a quaternion matrix problem is often equivalent to dealing with a pair of complex matrices [Zhang(1997), Lee(1949)]. Recently, the studies concern with matrix norms, has been given by several authors, see for instance [Moenck(1977),Mathias(1990),Visick(2000),Zielke (1988),Horn and Johnson(1991), Bozkurt(1996), Solak and Bozkurt(2003),Türkmen and Bozkurt (2002)] and references cited therein. In this paper, we have obtained some a lower and an upper bounds for the Euclidean and spectral of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices. Now, we need the following definitions and preliminaries.

Definition1. *Let $\mathbb{C}$ and $\mathbb{R}$ denote the fields of the complex and real numbers respectively. Let $\mathbb{Q}$ be a* *four-dimensional vector space over $\mathbb{R}$ with an ordered basis, denoted by $\boldsymbol{e}, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$. A real quaternion, simply called quaternion, is a vector*

$$x = x_0\boldsymbol{e} + x_1\boldsymbol{i} + x_2\boldsymbol{j} + x_3\boldsymbol{k} \in \mathbb{Q}$$

with real coefficients x_0, x_1, x_2 and x_3 .

Besides the addition and the scalar multiplication of the vector space $\mathbb{Q}$ over $\mathbb{R}$, the product of any two quaternions $\boldsymbol{e}, \boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ are defined by the requirement that $\boldsymbol{e}$ act as a identity and by the table

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$$

$$\boldsymbol{ij} = -\boldsymbol{ji} = \boldsymbol{k}, \quad \boldsymbol{jk} = -\boldsymbol{kj} = \boldsymbol{i}, \quad \boldsymbol{ki} = -\boldsymbol{ik} = \boldsymbol{j}.$$

Let $M_{m\times n}(\mathbb{Q})$, simply $M_n(\mathbb{Q})$ when $m = n$, denote the collection of all $m\times n$ matrices with quaternion entries.

Definition 2. *Let $A = A_1 + A_2\boldsymbol{j} \in M_n(\mathbb{Q})$, where A_1, A_2 are $n\times n$ complex matrices. We shall call the $2n\times 2n$ complex matrix*

$$\begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix},$$

uniquely determined by A, the complex adjoint matrix or adjoint of the quaternion matrix A [Lee(1949)].

Now we give some preliminaries related to our study. Let A be any $n\times n$ matrix. The ℓ_p norms of the matrix A are defined as

$$\|A\|_p = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^p\right)^{1/p} \quad (1 \le p < \infty).$$

If $p = \infty$, then

$$\|A\|_\infty = \lim_{n\to\infty}\|A\|_p = \max_{i,j}|a_{ij}|.$$

The well-known Euclidean norm of matrix A is

$$\|A\|_E = \left(\sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2}$$

and also the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1\le i\le n}\lambda_i(A^H A)}$$

where A is $m\times n$ and A^H is the conjugate transpose of the matrix A. The following inequality holds:

$$(1.1) \qquad \frac{1}{\sqrt{n}}\|A\|_E \le \|A\|_2 \le \|A\|_E$$

[Zielke (1988)]. A function Ψ is called a psi (or digamma) function if

$$(1.2) \qquad \Psi(x) = \frac{d}{dx}\{\ln|\Gamma(x)|\}$$

where

$$\Gamma(x) = \int_0^x e^{-t}t^{x-1}dt.$$

The n th derivatives of a Ψ function is called a polygamma function

$$(1.3) \qquad \Psi(n,x) = \frac{d}{dx^n}\Psi(x) = \frac{d}{dx^n}\left\{\frac{d}{dx}\ln|\Gamma(x)|\right\}.$$

If $n = 0$ then $\Psi(0,x) = \Psi(x) = \left\{\frac{d}{dx}\ln|\Gamma(x)|\right\}$. On the other hand, if $a > 0$, b is any number and n is positive integer, then

$$(1.4) \qquad \lim_{n\to\infty}\Psi(a, n+b) = 0$$

[Moenck(1977)]. Throughout the paper $\mathbb{Z}^+$ and $\mathbb{R}^+$ will represent the sets of positive integers and positive real numbers, respectively.

Let A and B be $n\times n$ matrices. The Hadamard product of A and B is defined by

$$(1.5) \qquad A \circ B = [a_{ij}b_{ij}].$$

If $\|\cdot\|$ is any norm on $n\times n$ matrices, then

$$(1.6) \qquad \|A\circ B\| \le \|A\|\|B\|$$

[Visick(2000)].

Define the maximum column length norm $c_j(\cdot)$ and the maximum row length norm $r_i(\cdot)$ of any matrix A by

$$(1.7) \qquad c_j(A) = \max_j\left(\sum_{i=1}^{m}|a_{ij}|^2\right)^{1/2} \quad i = 1, 2, \cdots, m$$

and

$$(1.8) \qquad r_i(A) = \max_i\left(\sum_{j=1}^{n}|a_{ij}|^2\right)^{1/2} \quad j = 1, 2, \cdots, n$$

respectively [Horn and Johnson(1991)]. Let A, B and C be $m\times n$ matrices. If $A\circ B = C$ then

$$(1.9) \qquad \|A\|_2 \le r_i(B)\, c_j(C)$$

[Mathias(1990)].

2. Matrices of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel

Definition 3. *The matrices in x quaternion from Definition 1, for $2 \le t, l, m \in \mathbb{Z}^+$ and $p = 1, 2, \cdots, n$, $r = 1, 2, \cdots, n$ and*

$x_0 = 0,\ x_1 = \frac{1}{\frac{1}{t}+p-r},\ x_2 = \frac{1}{\frac{1}{l}+p-r},\ x_3 = \frac{1}{\frac{1}{m}+p-r}$

$$(2.1)\quad T = \left[\frac{\boldsymbol{i}}{\frac{1}{t}+p-r} + \frac{\boldsymbol{j}}{\frac{1}{l}+p-r} + \frac{\boldsymbol{k}}{\frac{1}{m}+p-r} \right]$$

is called Quaternion Cauchy-Toeplitz matrix.

$$(2.2)\quad H = \left[\frac{\boldsymbol{i}}{\frac{1}{t}+p+r} + \frac{\boldsymbol{j}}{\frac{1}{l}+p+r} + \frac{\boldsymbol{k}}{\frac{1}{m}+p+r} \right]$$

is called Quaternion Cauchy-Hankel matrix.

In this section we are going to find an upper and lower bounds for the Euclidean norm and the spectral norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel matrices in (2.1) and (2.2).

2.1. Euclidean and Spectral Norms of Quaternion Cauchy-Toeplitz and Quaternion Cauchy-Hankel Matrices

Theorem 1. *For Euclidean norm of T Quaternion Cauchy-Toeplitz matrix which is defined in (2.1),*

$$(2.3)\quad \|T\|_E \le \pi\sqrt{n(\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m})}$$

is valid upper bound, where $2 \le t,l,m \in \mathbb{Z}^+$.

Proof. From Euclidean norm the following

$$\|T\|_E^2 = n(t^2+l^2+m^2) + \sum_{s=1}^{n-1}(n-s)\left\{\left[\frac{t^2}{(1-st)^2} + \frac{l^2}{(1-sl)^2} + \frac{m^2}{(1-sm)^2}\right] + \left[\frac{t^2}{(1+st)^2} + \frac{l^2}{(1+sl)^2} + \frac{m^2}{(1+sm)^2}\right]\right\}$$

is obtained. If we divide both of sides by n and if we take upper bound of right hand side to infinity, we obtain

$$\frac{1}{n}\|T\|_E^2 \le t^2+l^2+m^2 + \lim_{n\to\infty}\left\{\sum_{s=1}^{n-1}(1-\frac{s}{n})\left(\left[\frac{t^2}{(1-st)^2} + \frac{l^2}{(1-sl)^2} + \frac{m^2}{(1-sm)^2}\right] + \left[\frac{t^2}{(1+st)^2} + \frac{l^2}{(1+sl)^2} + \frac{m^2}{(1+sm)^2}\right]\right)\right\}$$
$$= \pi^2\left(\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m}\right)$$

or

$$\frac{1}{n}\|T\|_E^2 \le \pi^2\left[\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m}\right],$$

if we take

$$\frac{1}{\sqrt{n}}\|T\|_E \le \pi\sqrt{\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m}}$$

then we obtain

$$\|T\|_E \le \pi\sqrt{n\left[\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m}\right]}$$

which is an upper bound for Euclidean norm of Quaternion Cauchy-Toeplitz matrix.

Corollary 1. *For spectral norm of T Quaternion Cauchy-Toeplitz matrices defined in (2.1),*

$$(2.4)\quad \pi\sqrt{\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m}} \le \|T\|_2$$

is valid lower bound, where $2 \le t,l,m \in \mathbb{Z}^+$.

Proof. Following relation, for spectral norm of T matrix will be

$$\frac{1}{\sqrt{n}}\|T\|_E \le \|T\|_2.$$

Then we obtain

$$\pi\sqrt{\csc^2\tfrac{\pi}{t} + \csc^2\tfrac{\pi}{l} + \csc^2\tfrac{\pi}{m}} \le \|T\|_2.$$

Now for in definition (2.1) of spectral norm of T Quaternion Cauchy-Toeplitz matrix, we have obtained upper bound to give as a theorem before lets give some definition end concepts.

Definition 4. *$J_i = [1]_{n\times n}$ $(i = 1,2,\cdots,n)$ let it be a square J matrix with all entries 1.*

Now, $A_1 = \left[\frac{\boldsymbol{i}}{\frac{1}{t}+p-r}\right]_{p,r=1}^{n}$ and $A_2 = \left[\frac{1}{\frac{1}{l}+p-r} + \frac{\boldsymbol{i}}{\frac{1}{m}+p-r}\right]_{p,r=1}^{n}$ write $T_n = A_1 + A_2\boldsymbol{j}$. From Definition 2,

$$[T_n] = \begin{bmatrix} A_1 & A_2 \\ -\overline{A_2} & \overline{A_1} \end{bmatrix} \tag{2.5}$$

is being occured as complex matrix in $2n\times 2n$. From definition (2.5) lets give the following theorem.

Theorem 2. *Let $t = l = m = 2$ in* (2.5). *Then the upper bound for the spectral norm of Quaternion Cauchy-Toeplitz matrix is*

$$\left\|[T_n]\right\|_2 \le \left[\left(2+2\sqrt{2}\right)\left(1+\ln 2+\frac{\gamma}{2}+\frac{1}{2}\Psi\left(n-\frac{1}{2}\right)\right) \times\left(\pi^2+2\ln 2+\gamma-2\Psi\left(1,n+\frac{1}{2}\right)+\Psi\left(n+\frac{1}{2}\right)\right)\right]^{\frac{1}{2}}. \tag{2.6}$$

Proof. Consider $2n\times 2n$ complex matrix in (2.5), with $A_1 = \left[\frac{\boldsymbol{i}}{\frac{1}{2}+p-r}\right]_{p,r=1}^{n}$,

$$A_1 = \begin{bmatrix} 2\boldsymbol{i} & -2\boldsymbol{i} & \cdots & -\frac{2}{2n-5}\boldsymbol{i} & -\frac{2}{2n-3}\boldsymbol{i} \\ \frac{2}{3}\boldsymbol{i} & 2\boldsymbol{i} & \cdots & -\frac{2}{2n-7}\boldsymbol{i} & -\frac{2}{2n-5}\boldsymbol{i} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{2}{2n-3}\boldsymbol{i} & \frac{2}{2n-5}\boldsymbol{i} & \cdots & 2\boldsymbol{i} & -2\boldsymbol{i} \\ \frac{2}{2n-1}\boldsymbol{i} & \frac{2}{2n-3}\boldsymbol{i} & \cdots & \frac{2}{3}\boldsymbol{i} & 2\boldsymbol{i} \end{bmatrix}_{n\times n}$$

and with $A_2 = \left[\frac{1}{\frac{1}{2}+p-r} + \frac{\boldsymbol{i}}{\frac{1}{2}+p-r}\right]_{p,r=1}^{n}$,

$$A_2 = \begin{bmatrix} 2(1+\boldsymbol{i}) & -2(1+\boldsymbol{i}) & \cdots & -\frac{2}{2n-5}(1+\boldsymbol{i}) & -\frac{2}{2n-3}(1+\boldsymbol{i}) \\ \frac{2}{3}(1+\boldsymbol{i}) & 2(1+\boldsymbol{i}) & \cdots & -\frac{2}{2n-7}(1+\boldsymbol{i}) & -\frac{2}{2n-5}(1+\boldsymbol{i}) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{2}{2n-3}(1+\boldsymbol{i}) & \frac{2}{2n-5}(1+\boldsymbol{i}) & \cdots & 2(1+\boldsymbol{i}) & -2(1+\boldsymbol{i}) \\ \frac{2}{2n-1}(1+\boldsymbol{i}) & \frac{2}{2n-3}(1+\boldsymbol{i}) & \cdots & \frac{2}{3}(1+\boldsymbol{i}) & 2(1+\boldsymbol{i}) \end{bmatrix}_{n\times n}$$

in this case, we write $2n\times 2n$ matrix as.

$$[T_n] = \begin{bmatrix} 2i & -2i & \cdots & -\frac{2}{2n-3}i & 2(1+\boldsymbol{i}) & -2(1+\boldsymbol{i}) & \cdots & -\frac{2}{2n-3}(1+\boldsymbol{i}) \\ \frac{2}{3}i & 2i & \cdots & -\frac{2}{2n-5}i & \frac{2}{3}(1+\boldsymbol{i}) & 2(1+\boldsymbol{i}) & \cdots & -\frac{2}{2n-5}(1+\boldsymbol{i}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{2}{2n-1}i & \frac{2}{2n-3}i & \cdots & 2i & \frac{2}{2n-1}(1+\boldsymbol{i}) & \frac{2}{2n-3}(1+\boldsymbol{i}) & \cdots & 2(1+\boldsymbol{i}) \\ -2(1-\boldsymbol{i}) & 2(1-\boldsymbol{i}) & \cdots & \frac{2}{2n-3}(1-\boldsymbol{i}) & -2i & 2i & \cdots & \frac{2}{2n-3}i \\ -\frac{2}{3}(1-\boldsymbol{i}) & -2(1-\boldsymbol{i}) & \cdots & \frac{2}{2n-5}(1-\boldsymbol{i}) & -\frac{2}{3}i & -2i & \cdots & -\frac{2}{2n-5}i \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{2}{2n-1}(1-\boldsymbol{i}) & -\frac{2}{2n-3}(1-\boldsymbol{i}) & \cdots & -2(1-\boldsymbol{i}) & -\frac{2}{2n-1}i & -\frac{2}{2n-3}i & \cdots & -2i \end{bmatrix}$$

Then, the Hadamard Product (1.5) of

$$[T_n] = [B_n \circ D_n]$$

for

$$[B_n] = \begin{bmatrix} \sqrt{A_1} & \sqrt{A_2} \\ J_i & J_i \end{bmatrix}$$

and

$$[D_n] = \begin{bmatrix} \frac{\sqrt{A_1}}{-A_2} & \frac{\sqrt{A_2}}{A_1} \end{bmatrix}$$

$$[T_n] = \begin{bmatrix} \sqrt{A_1} & \sqrt{A_2} \\ J_i & J_i \end{bmatrix} \circ \begin{bmatrix} \frac{\sqrt{A_1}}{-A_2} & \frac{\sqrt{A_2}}{A_1} \end{bmatrix}.$$

From (1.7) and (1.8), we obtain

$$\begin{aligned} r_1^2(B_n) &= 2 + 2\left(1 + \tfrac{1}{3} + \cdots + \tfrac{1}{2s-1}\right) + 2\sqrt{2} + 2\sqrt{2}\left(1 + \tfrac{1}{3} + \cdots + \tfrac{1}{2s-1}\right) \\ &= \left(2 + 2\sqrt{2}\right)\left(1 + \sum_{s=1}^{n-1} \tfrac{1}{2s-1}\right) \\ &= \left(2 + 2\sqrt{2}\right)\left(1 + \ln 2 + \tfrac{\gamma}{2} + \tfrac{1}{2}\Psi\left(n - \tfrac{1}{2}\right)\right) \end{aligned}$$

in this case we write

$$\text{(2.7)} \qquad r_1(B_n) = \left[\left(2 + 2\sqrt{2}\right)\left(1 + \ln 2 + \tfrac{\gamma}{2} + \tfrac{1}{2}\Psi\left(n - \tfrac{1}{2}\right)\right)\right]^{\frac{1}{2}}$$

similarly

$$\begin{aligned} c_1^2(D_n) &= 2\left(1 + \tfrac{1}{3} + \cdots + \tfrac{1}{2s-1}\right) + 2^3\left(1 + \tfrac{1}{9} + \cdots + \tfrac{1}{(2s-1)^2}\right) \\ &= 2\sum_{s=1}^{n} \tfrac{1}{2s-1} + 2^3 \sum_{s=1}^{n} \tfrac{1}{(2s-1)^2} \\ &= \pi^2 + 2\ln 2 + \gamma - 2\Psi\left(1, n + \tfrac{1}{2}\right) + \Psi\left(n + \tfrac{1}{2}\right) \end{aligned}$$

Then

$$\text{(2.8)} \qquad c_1(D_n) = \left[\pi^2 + 2\ln 2 + \gamma - 2\Psi\left(1, n + \tfrac{1}{2}\right) + \Psi\left(n + \tfrac{1}{2}\right)\right]^{\frac{1}{2}}$$

we can see the following relation and we utilize from (1.9) that from (2.7) and (2.8).

$$\|[T_n]\|_2 \le \left[\left(2 + 2\sqrt{2}\right)\left(1 + \ln 2 + \frac{\gamma}{2} + \frac{1}{2}\Psi\left(n - \frac{1}{2}\right)\right) \left(\pi^2 + 2\ln 2 + \gamma - 2\Psi\left(1, n + \frac{1}{2}\right) + \Psi\left(n + \frac{1}{2}\right)\right)\right]^{\frac{1}{2}}$$

is being obtained as upper bound.

Theorem 3. *Let* $2 \le t, l, m \in \mathbb{Z}^+$ *hold in* (2.2). *Then for the Euclidean norm of Quaternion Cauchy-Hankel matrix* H*, we have*

$$\text{(2.9)} \qquad \|H\|_E \to \infty.$$

Proof. From the definition of Euclidean

$$\|H\|_E^2 = \sum_{s=1}^{n} s\left[\frac{t^2}{(1+(s+1)t)^2} + \frac{l^2}{(1+(s+1)l)^2} + \frac{m^2}{(1+(s+1)m)^2}\right] + \sum_{s=1}^{n-1}(n-s)\left[\frac{t^2}{(1+(n+s+1)t)^2} + \frac{l^2}{(1+(n+s+1)l)^2} + \frac{m^2}{(1+(n+s+1)m)^2}\right]$$

is clear. If we calculate right hand of this equality, then

$$\begin{aligned}\|H\|_E^2 = &\frac{(1+t)}{t}\Psi\left(1,n+1+\frac{1+t}{t}\right)+\Psi\left(n+1+\frac{1+t}{t}\right)+\frac{(1+l)}{l}\Psi\left(1,n+1+\frac{1+l}{l}\right)\\ &+\Psi\left(n+1+\frac{1+l}{l}\right)+\frac{(1+m)}{m}\Psi\left(1,n+1+\frac{1+m}{m}\right)+\Psi\left(n+1+\frac{1+m}{m}\right)\\ &-\frac{(1+t)}{t}\Psi\left(1,1+\frac{1+t}{t}\right)-\Psi\left(1+\frac{1+t}{t}\right)-\frac{(1+l)}{l}\Psi\left(1,1+\frac{1+l}{l}\right)-\Psi\left(1+\frac{1+l}{l}\right)\\ &-\frac{(1+m)}{m}\Psi\left(1,1+\frac{1+m}{m}\right)-\Psi\left(1+\frac{1+m}{m}\right)-\frac{(1+t+2tn)}{t}\Psi\left(1,n+1+\frac{1+t+tn}{t}\right)\\ &-\Psi\left(1,n+1+\frac{1+t+tn}{t}\right)-\frac{(1+l+2nl)}{l}\Psi\left(1,n+1+\frac{1+l+nl}{l}\right)-\Psi\left(1,n+1+\frac{1+l+nl}{l}\right)\\ &-\frac{(1+m+2nm)}{m}\Psi\left(1,n+1+\frac{1+m+nm}{m}\right)-\Psi\left(1,n+1+\frac{1+m+nm}{m}\right)\\ &+\frac{(1+t+2tn)}{t}\Psi\left(1,1+\frac{1+t+tn}{t}\right)+\Psi\left(1+\frac{1+t+tn}{t}\right)+\frac{(1+l+2nl)}{l}\Psi\left(1,1+\frac{1+l+nl}{l}\right)\\ &+\Psi\left(1+\frac{1+l+nl}{l}\right)+\frac{(1+m+2nm)}{m}\Psi\left(1,1+\frac{1+m+nm}{m}\right)+\Psi\left(1+\frac{1+m+nm}{m}\right)\end{aligned}$$

is being found above. Therefore, we arrive

$$\lim_{n\to\infty}\Psi\left(n+1+\frac{1+t}{t}\right)=\lim_{n\to\infty}\Psi\left(n+1+\frac{1+l}{l}\right)=\lim_{n\to\infty}\Psi\left(n+1+\frac{1+m}{m}\right)=\infty$$

and

$$\lim_{n\to\infty}\Psi\left(1,n+1+\frac{1+t}{t}\right)=\lim_{n\to\infty}\Psi\left(1,n+1+\frac{1+l}{l}\right)=\lim_{n\to\infty}\Psi\left(1,n+1+\frac{1+m}{m}\right)=0$$

This implies,

$$\|H\|_E\to\infty$$

which is desired. Now, lets show

$$B_1=\left[\frac{\boldsymbol{i}}{\frac{1}{t}+p+r}\right]_{p,r=1}^{n}$$

and

$$B_2=\left[\frac{1}{\frac{1}{l}+p+r}+\frac{\boldsymbol{i}}{\frac{1}{m}+p+r}\right]_{p,r=1}^{n}.$$

Then, obtain

$$H_n=B_1+B_2\boldsymbol{j}.$$

From Definition 2,

$$[H_n]=\begin{bmatrix}B_1 & B_2\\ -\overline{B_2} & \overline{B_1}\end{bmatrix} \tag{2.10}$$

is being occurred as $2n\times 2n$ complex matrix. Now, we ready to give the upper bound below.

Theorem 4. *Let* $t=l=m=2$ *for spectral norm of* $[H_n]$*, Quaternion Cauchy-Hankel matrix in* (2.10) *than,*

$$\begin{aligned}\|[H_n]\|_2 \quad \le \quad &\left[(2+2\sqrt{2})\left(\ln 2-\tfrac{4}{3}+\tfrac{\gamma}{2}+\tfrac{1}{2}\Psi\left(n+\tfrac{5}{2}\right)\right)\right.\\ &\times\left.\left(\pi^2-\tfrac{104}{9}+2\ln 2+\gamma-2\Psi\left(1,n+\tfrac{5}{2}\right)+\Psi\left(n+\tfrac{5}{2}\right)\right)\right]^{\frac{1}{2}}\end{aligned} \tag{2.11}$$

is valid.

Proof. To use (2.10), define $B_1 = \left[\dfrac{\boldsymbol{i}}{\frac{1}{2}+p+r} \right]_{p,\, r=1}^{n}$,

$$B_1 = \begin{bmatrix} \frac{2}{5}i & \frac{2}{7}i & \cdots & \frac{2}{2n+1}i & \frac{2}{2n+3}i \\ \frac{2}{7}i & \frac{2}{9}i & \cdots & \frac{2}{2n+3}i & \frac{2}{2n+5}i \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{2}{2n+1}i & \frac{2}{2n+3}i & \cdots & \frac{2}{4n-3}i & \frac{2}{4n-1}i \\ \frac{2}{2n+3}i & \frac{2}{2n+5}i & \cdots & \frac{2}{4n-1}i & \frac{2}{4n+1}i \end{bmatrix}_{n\times n}$$

and $B_2 = \left[\dfrac{1}{\frac{1}{2}+p+r} + \dfrac{\boldsymbol{i}}{\frac{1}{2}+p+r} \right]_{p,\, r=1}^{n}$,

$$B_2 = \begin{bmatrix} \frac{2}{5}(1+i) & \frac{2}{7}(1+i) & \cdots & \frac{2}{2n+1}(1+i) & \frac{2}{2n+3}(1+i) \\ \frac{2}{7}(1+i) & \frac{2}{9}(1+i) & \cdots & \frac{2}{2n+3}(1+i) & \frac{2}{2n+5}(1+i) \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{2}{2n+1}(1+i) & \frac{2}{2n+3}(1+i) & \cdots & \frac{2}{4n-3}(1+i) & \frac{2}{4n-1}(1+i) \\ \frac{2}{2n+3}(1+i) & \frac{2}{2n+5}(1+i) & \cdots & \frac{2}{4n-1}(1+i) & \frac{2}{4n+1}(1+i) \end{bmatrix}_{n\times n}$$

in this case, we write $2n\times 2n$ matrix as.

$$[H_n] = \begin{bmatrix} \frac{2}{5}i & \frac{2}{7}i & \cdots & \frac{2}{2n+3}i & \frac{2}{5}(1+i) & \frac{2}{7}(1+i) & \cdots & \frac{2}{2n+3}(1+i) \\ \frac{2}{7}i & \frac{2}{9}i & \cdots & \frac{2}{2n+5}i & \frac{2}{7}(1+i) & \frac{2}{9}(1+i) & \cdots & \frac{2}{2n+5}(1+i) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{2}{2n+3}i & \frac{2}{2n+5}i & \cdots & \frac{2}{4n+1}i & \frac{2}{2n+3}(1+i) & \frac{2}{2n+5}(1+i) & \cdots & \frac{2}{4n+1}(1+i) \\ -\frac{2}{5}(1-i) & -\frac{2}{7}(1-i) & \cdots & -\frac{2}{2n+3}(1-i) & -\frac{2}{5}i & -\frac{2}{7}i & \cdots & -\frac{2}{2n+3}(1+i) \\ -\frac{2}{7}(1-i) & -\frac{2}{9}(1-i) & \cdots & -\frac{2}{2n+5}(1-i) & -\frac{2}{7}i & -\frac{2}{9}i & \cdots & -\frac{2}{2n+5}(1+i) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -\frac{2}{2n+3}(1-i) & -\frac{2}{2n+5}(1-i) & \cdots & -\frac{2}{4n+1}(1-i) & -\frac{2}{2n+3}(1-i) & -\frac{2}{2n+5}(1-i) & \cdots & -\frac{2}{4n+1}(1-i) \end{bmatrix}$$

finding Hadamard Product

$$[M_n] = \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ J_i & J_i \end{bmatrix}$$

and

$$[K_n] = \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ -\overline{B_2} & \overline{B_1} \end{bmatrix}$$

Since, these two matrices are $2n\times 2n$, we write

$$[H_n] = [M_n \circ K_n]$$

or

$$[H_n] = \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ J_i & J_i \end{bmatrix} \circ \begin{bmatrix} \sqrt{B_1} & \sqrt{B_2} \\ -\overline{B_2} & \overline{B_1} \end{bmatrix}$$

From (1.7) and (1.8), we obtain

$$\begin{aligned} r_1^2(M_n) &= 2\left(\tfrac{1}{5}+\tfrac{1}{7}+\cdots+\tfrac{1}{2s+3}\right)+2\sqrt{2}\left(\tfrac{1}{5}+\tfrac{1}{7}+\cdots+\tfrac{1}{2s+3}\right) \\ &= \left(2+2\sqrt{2}\right)\left(\sum_{s=1}^{n}\tfrac{1}{2s+3}\right) \\ &= \left(2+2\sqrt{2}\right)\left(\ln 2-\tfrac{4}{3}+\tfrac{\gamma}{2}+\tfrac{1}{2}\Psi\left(n+\tfrac{5}{2}\right)\right) \end{aligned}$$

in this case

(2.12) $$r_1(M_n)=\left[\left(2+2\sqrt{2}\right)\left(\ln 2-\frac{4}{3}+\frac{\gamma}{2}+\frac{1}{2}\Psi\left(n+\frac{5}{2}\right)\right)\right]^{\frac{1}{2}}$$

we found

$$\begin{aligned} c_1^2(K_n) &= 2\left(\tfrac{1}{5}+\tfrac{1}{7}+\cdots+\tfrac{1}{(2s+3)}\right)+2^3\left(\tfrac{1}{25}+\tfrac{1}{49}+\cdots+\tfrac{1}{(2s+3)^2}\right) \\ &= 2\sum_{s=1}^{n}\tfrac{1}{2s+3}+2^3\sum_{s=1}^{n}\tfrac{1}{(2s+3)^2} \\ &= \left(\pi^2-\tfrac{104}{9}+2\ln 2+\gamma-2\Psi\left(1,n+\tfrac{5}{2}\right)+\Psi\left(n+\tfrac{5}{2}\right)\right) \end{aligned}$$

Finally,

(2.13) $$c_1(K_n)=\left(\pi^2-\frac{104}{9}+2\ln 2+\gamma-2\Psi\left(1,n+\frac{5}{2}\right)+\Psi\left(n+\frac{5}{2}\right)\right)^{\frac{1}{2}}$$

is written. We utilize between these norms following relation and from (1.9). By the way from (2.12) and (2.13)

$$\|[H_n]\|_2 \le \left[(2+2\sqrt{2})\left(\ln 2-\frac{4}{3}+\frac{\gamma}{2}+\frac{1}{2}\Psi\left(n+\frac{5}{2}\right)\right)\right. \\ \left.\times\left(\pi^2-\frac{104}{9}+2\ln 2+\gamma-2\Psi\left(1,n+\frac{5}{2}\right)+\Psi\left(n+\frac{5}{2}\right)\right)\right]^{\frac{1}{2}}$$

is obtained as upper bound which is desired.

3. Numerical Results

In this section, have compared our findings with the known bounds of the norms of matrices in the illustrative examples below. We have found between in theorems we have given real norms of matrices in second section.

Example 1. *For* $t=l=m=2, \alpha=E$ *comparative values in Theorem 1.*

$$\Delta=\pi\sqrt{n(\csc^2\tfrac{\pi}{t}+\csc^2\tfrac{\pi}{l}+\csc^2\tfrac{\pi}{m})}$$

n	$\|T\|_E$	Δ
1	3.464101616	5.441398092
2	6.110100936	7.695298983
3	8.029946959	9.424777962
4	9.609972145	10.88279619
10	16.26327299	17.20721163
20	23.58858696	24.33467206
30	29.15607022	29.80376480
50	37.93662562	38.47649492
70	45.04825130	45.52600274
100	53.99508558	54.41398005
150	66.28330952	66.64324407
200	76.63022697	76.95298983

For $t=2,\ l=3,\ m=4,\ \alpha=E$ *comparative values in Theorem 1.*

$$\Delta=\pi\sqrt{n(\csc^2\tfrac{\pi}{t}+\csc^2\tfrac{\pi}{l}+\csc^2\tfrac{\pi}{m})}$$

n	$\|T\|_E$	Δ
1	5.385164807	6.539746609
2	8.226464748	9.248598352
3	10.393345017	11.32717340
4	12.20932796	13.07949322
10	20.00459317	20.68049461
20	28.70031876	29.24663595
40	40.92572873	41.36098924
50	45.83949482	46.24299178
70	54.35628926	54.71544573
100	65.08083214	65.39746609
150	79.82165254	80.09521125
200	92.23982009	92.48598352

Example 2. *For* $t=l=m=2,\alpha=2$ *comparative values in Theorem 2.*

$$\mathbf{a}=\left[\left(2+2\sqrt{2}\right)\left(1+\ln 2+\frac{\gamma}{2}+\frac{1}{2}\Psi\left(n-\frac{1}{2}\right)\right)\right.$$
$$\left.\left(\pi^2+2\ln 2+\gamma-2\Psi\left(1,n+\frac{1}{2}\right)+\Psi\left(n+\frac{1}{2}\right)\right)\right]^{\frac{1}{2}}$$

n	$\|T_n\|_2$	**a**
2	5.318032538	10.61170569
3	5.436440268	12.61503380
4	5.441222385	13.76718565
8	5.441398093	16.09897965
10	5.441398092	16.77019419
20	"	18.71758844
50	"	21.10453911
100	"	22.83372244
200	"	24.52399451
500	"	26.71920096
1000	"	28.35987611
2000	"	29.98503802

Example 3. *For* $t=l=m=2,\alpha=2$ *comparative values in Theorem 4.*

$$\mathbf{b}=\left[(2+2\sqrt{2})\left(\ln 2-\frac{4}{3}+\frac{\gamma}{2}+\frac{1}{2}\Psi\left(n+\frac{5}{2}\right)\right)\right.$$
$$\left.\times\left(\pi^2-\frac{104}{9}+2\ln 2+\gamma-2\Psi\left(1,n+\frac{5}{2}\right)+\Psi\left(n+\frac{5}{2}\right)\right)\right]^{\frac{1}{2}}$$

n	$\|H_n\|_2$	**b**
2	1.057128196	1.391114591
3	1.296686245	1.807190534
4	1.471951182	2.138280861
8	1.892000418	3.032364624
10	2.022502915	3.342059572
20	"	4.347036326
50	"	5.732291279
100	"	6.799419579
200	"	7.873475309
500	"	9.297649719
1000	"	10.37618120
2000	"	11.45496106

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