Konuralp Journal of Mathematics, *9 (2) (2021) 245-249*

Konuralp Journal of Mathematics

Research Paper

https://dergipark.org.tr/en/pub/konuralpjournalmath

e-ISSN: 2147-625X

Weaker Forms of Some Star Selection Properties

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Abstract

In this work, we consider weaker forms of some new types of star covering properties, star versions of quasi-Mengerness and related spaces. We discuss the relations of these properties with some other selective covering properties and investigate some topological properties of these spaces.

Keywords: *Almost set star-Menger; faintly set star-Menger; quasi star-Menger; star selection principles; weakly set star-Menger.*

2010 Mathematics Subject Classification: *Primary 54D20; Secondary 54B05; 54C10.*

1. Introduction and Preliminaries

Kočinac and Konca [13] initiated investigation of new types of selective covering properties called set covering properties such as set Menger, set Rothberger, set Hurewicz and weaker versions of these properties. Kocinac [12] and Singh [20] noticed that the properties set-Menger, set-Rothberger, set-Hurewicz are another view of Menger [17], Rothberger [19], Hurewicz [6, 7] properties. Konca and Kočinac [16] formerly presented the so-called set versions of the star covering properties in a very general form. In [14, 15], Kočinac et al. continued the investigation of those star covering properties. Here, we consider weakly set star-Menger, almost set star-Menger, faintly set star-Menger and quasi star-Menger and related spaces. We discuss the relations of these properties with some other selective covering properties and investigate some topological properties of these spaces.

We use the usual topological notation and terminology as in [5]. By "a space" we mean "a topological space". $\mathbb{N}$ denotes the set of natural numbers.

A space X is said to have the *Menger* (M) [17] (respectively, *almost Menger* (aM) [11], *weakly Menger* (wM) [3]) property if for each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $X=\bigcup_{n\in\mathbb{N}}\bigcup\mathscr{V}_n$ (respectively, $X=\bigcup_{n\in\mathbb{N}}\cup\{\overline{V}:V\in\mathscr{V}_n\}$, $\overline{\bigcup_{n\in\mathbb{N}}\cup\{V:V\in\mathscr{V}_n\}}=X$). For more results related to the weaker forms of the classical selective covering properties, see, for instance, [11, 3, 8, 9, 18, 1, 2].

A space X is said to be *star-Menger* (SM) (respectively, *weakly star-Menger* (weakly-SM), *almost star-Menger* (almost-SM)) if for each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of =open covers of X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that for each $n\in\mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $X=\bigcup_{n\in\mathbb{N}}St(\bigcup\mathscr{V}_n,\mathscr{U}_n)$ [10] (respectively, $X=\overline{\bigcup_{n\in\mathbb{N}}\mathrm{St}(\bigcup\mathscr{V}_n,\mathscr{U}_n)}$ [18], $X=\bigcup_{n\in}\overline{\mathrm{St}(\bigcup\mathscr{V}_n,\mathscr{U}_n)}$ [8, 11]).

A space X is said to have the *set-Menger* (set-M) (respectively, *almost set-Menger* (almost set-M), *weakly set-Menger* (weakly set-M)) property if for each nonempty set $A\subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A}\subset\cup\mathscr{U}_n$, $n\in\mathbb{N}$, there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of finite sets such that $\mathscr{V}_n\subset\mathscr{U}_n$ for each n, and $A\subset\bigcup_{n\in\mathbb{N}}(\bigcup\mathscr{V}_n)$ (respectively, $A\subset\bigcup_{n\in\mathbb{N}}\overline{(\bigcup\mathscr{V}_n)}$), $A\subset\overline{\bigcup_{n\in\mathbb{N}}(\bigcup\mathscr{V}_n)}$ [13].

Definition 1.1. A space X is said to be:

1. *quasi-Menger* (qM) if for each closed set $F\subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of covers of F by sets open in X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of finite sets such that for each n, $\mathscr{V}_n\subset\mathscr{U}_n$ and $\overline{\bigcup_{n\in\mathbb{N}}\cup\mathscr{V}_n}\supset F$ [4].
2. *quasi-Rothberger* (qR) if for each closed set $F\subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of covers of F by sets open in X there is a sequence $(U_n)_{n\in\mathbb{N}}$ such that for each n, $U_n\in\mathscr{U}_n$ and $\overline{\bigcup_{n\in\mathbb{N}}U_n}\supset F$ [4].
3. *quasi-Hurewicz* (qH) if for each closed set $F\subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of covers of F by sets open in X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of finite sets such that for each n, $\mathscr{V}_n\subset\mathscr{U}_n$ and each $x\in F$ belongs to $\overline{\cup\mathscr{V}_n}$ for all but finitely many $n\in\mathbb{N}$ [4].
4. *quasi-Gerlits-Nagy* (qGN) if for each closed set $F\subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of ω-covers of F by sets open in X there is a sequence $(U_n)_{n\in\mathbb{N}}$ such that for each n, $U_n\in\mathscr{U}_n$ and the set $\mathscr{V}=\{U_n:n\in\mathbb{N}\}$ can be partitioned into finite pairwise disjoint sets $\mathscr{V}_n, n\in\mathbb{N}$, so that each $x\in F$ belongs to $\overline{\cup\mathscr{V}_n}$ for all but finitely many $n\in\mathbb{N}$ [4].

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2. Set star-Menger, related spaces and properties

In this section we consider set versions of star selection principles and these new covering properties first was presented in [16].

Definition 2.1. A space X is said to be:

(1) *set star-Menger* (set-SM) (respectively, *weakly set star-Menger* (weakly set-SM), *almost set star-Menger* (almost set-SM), *faintly set star-Menger* (faintly set-SM)) property if for each nonempty set $A \subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \cup\mathscr{U}_n$, $n \in \mathbb{N}$, there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ of finite sets such that $\mathscr{V}_n \subset \mathscr{U}_n$ for each n, and $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{\bigcup\mathscr{V}_n}, \mathscr{U}_n)$);

(2) *set star-Rothberger* (set-SR) (respectively, *weakly set star-Rothberger* (weakly set-SR), *almost set star-Rothberger* (almost set-SR), *faintly set star-Rothberger* (faintly set-SR)) property if for each nonempty set $A \subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \cup\mathscr{U}_n$, $n \in \mathbb{N}$, there is a sequence $(U_n)_{n\in\mathbb{N}}$ such that $U_n \in \mathscr{U}_n$ for each $n \in \mathbb{N}$ and $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathscr{U}_n)$ (respectively, $A \subset \overline{\bigcup_{n\in\mathbb{N}} \mathrm{St}(U_n, \mathscr{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \overline{\mathrm{St}(U_n, \mathscr{U}_n)}$, $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{U_n}, \mathscr{U}_n)$);

(3) *set star-Hurewicz* (set-SH) (respectively, *almost set star-Hurewicz* (almost set-SH), *faintly set star-Hurewicz* (faintly set-SH)) property if for each nonempty set $A \subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of collections of sets open in X such that $\overline{A} \subset \bigcup\mathscr{U}_n$, $n \in \mathbb{N}$, there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each $n \in \mathbb{N}$ and each $x \in A$ belongs to all but finitely many sets $\mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)$ (respectively, to all but finitely many sets $\overline{\mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)}$, to all but finitely many $\bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{\bigcup\mathscr{V}_n}, \mathscr{U}_n)$).

We will mainly be concentrated on investigation of weaker forms of set star-Menger spaces and their relatives.

We begin the section with a simple fact.

Theorem 2.2. *Set star-Mengerness implies star-Mengerness, and almost set star-Mengerness implies weak set star-Mengerness. Clearly, set star-Mengerness implies faintly set star-Mengerness.*

Theorem 2.3. *Let X be a regular space. If X is faintly set star-Menger, then X is set star-Menger.*

Proof. Let $A \subset X$ and let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of families of open sets in X such that $\overline{A} \subset \bigcup\mathscr{U}_n$ for each $n \in \mathbb{N}$. Since X is a regular space, for each $n \in \mathbb{N}$ there exists an open cover $\mathscr{W}_n$ of $\overline{A}$, by sets open in X, such that $\overline{\mathscr{W}_n} := \{\overline{W} : W \in \mathscr{W}_n\}$ refines $\mathscr{U}_n$ and covers $\overline{A}$. As X is faintly set-star Menger, there exists a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ such that for each n, $\mathscr{H}_n$ is a finite subset of $\mathscr{W}_n$ and $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\overline{\bigcup\mathscr{H}_n}, \mathscr{W}_n)$. For each $n \in \mathbb{N}$, and each $H \in \mathscr{H}_n$ pick $V_H \in \mathscr{U}_n$ such that $\overline{H} \subset V_H$. Let $\mathscr{V}_n := \{V_H : H \in \mathscr{H}_n\}$, $n \in \mathbb{N}$. Then each $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ and $A \subset \bigcup_{n\in\mathbb{N}} \mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)$ which shows that X is a set star-Menger space. □

A topological space X is a *P-space* if the union of countably many closed subsets of X is closed in X.

Theorem 2.4. *If a P-space X is weakly set star-Menger, then X is almost set star-Menger.*

Proof. Let A be a subset of X and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of open covers of $\overline{A}$ by sets open in X. Since X is weakly set star-Menger, there exists a sequence $(\mathscr{W}_n)_{n\in\mathbb{N}}$ such that for each $n \in \mathbb{N}$, $\mathscr{W}_n$ is a finite subset of $\mathscr{U}_n$ and $A \subset \overline{\bigcup_{n\in\mathbb{N}} St\,(\cup\mathscr{W}_n, \mathscr{U}_n)}$. As X is a P-space, we have that $\bigcup_{n\in\mathbb{N}} \overline{St\,(\cup\mathscr{W}_n, \mathscr{U}_n)}$ is closed in X and contains $\bigcup_{n\in\mathbb{N}} St\,(\cup\mathscr{W}_n, \mathscr{U}_n)$. On the other hand, $\overline{\bigcup_{n\in\mathbb{N}} St\,(\cup\mathscr{W}_n, \mathscr{U}_n)}$ is the smallest closed subset of X containing $\bigcup_{n\in\mathbb{N}} St\,(\cup\mathscr{W}_n, \mathscr{U}_n)$. Thus

$$A \subset \overline{\bigcup_{n\in\mathbb{N}} St\,(\cup\mathscr{W}_n, \mathscr{U}_n)} \subset \bigcup_{n\in\mathbb{N}} \overline{St\,(\cup\mathscr{W}_n, \mathscr{U}_n)}$$

which means that X is almost set star-Menger. □

We have the following corollary.

Theorem 2.5. *Let X be a regular P-space. Then the following statements are equivalent:*

1. *X is set star-Menger;*
2. *X is almost set star-Menger;*
3. *X is weakly set star-Menger.*

Theorem 2.6. If (A, τ_A) is a clopen subspace of a weakly set star-Menger space (X, τ), then (A, τ_A) is also weakly set star-Menger.

Proof. Let B be a subset of (A, τ_A) and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $\mathrm{Cl}_{\tau_A}(B)$ by sets open in (A, τ_A). Since A is open, each $\mathscr{U}_n$ is a family of sets open in (X, τ). Since A is closed, $\mathrm{Cl}_{\tau_A}(B) = \mathrm{Cl}_{\tau}(B)$. Apply now the fact that X is weakly set star-Menger, for each $n \in \mathbb{N}$ there exists a sequence of finite subsets $\mathscr{V}_n$ of $\mathscr{U}_n$, such that $B \subset \overline{\bigcup_{n\in} \mathrm{St}(\bigcup\mathscr{V}_n, \mathscr{U}_n)}$. It follows that A is weakly set star-Menger. □

Theorem 2.7. *A continuous image of a weakly (almost, faintly) set star-Menger space is also weakly (almost, faintly) set star-Menger.*

Proof. We prove only the weakly set star-Menger case; the other two cases are proved similarly. Suppose that X is weakly set star-Menger and $f : X \to Y$ is a continuous onto mapping. Let B be any subset of Y and $(\mathscr{V}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $\overline{B}$ by sets open in Y. Let $A = f^{\leftarrow}(B)$. Since f is continuous, for each $n \in \mathbb{N}$, $\mathscr{U}_n = \{f^{\leftarrow}(V) : V \in \mathscr{V}_n\}$ is a collection of sets open in X with $\bar{A} = \overline{f^{\leftarrow}(B)} \subset f^{\leftarrow}(\bar{B}) \subset f^{\leftarrow}(\cup\mathscr{V}_n) = \cup\mathscr{U}_n$. As X is weakly set star-Menger there is a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ of finite sets such that for each $n \in \mathbb{N}$, $\mathscr{H}_n \subset \mathscr{U}_n$ and $A \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{H}_n, \mathscr{U}_n)}$. Let $\mathscr{W}_n = \{V : f^{\leftarrow}(V) \in \mathscr{H}_n\}$. Then for each $n \in \mathbb{N}$, $\mathscr{W}_n$ is a finite subset of $\mathscr{V}_n$ and $f^{\leftarrow}(\cup\mathscr{W}_n) = \cup\mathscr{H}_n$. Hence we have

$$B = f(A) \subset f\left(\overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{H}_n, \mathscr{U}_n)}\right) \subset \overline{f\left(\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{H}_n, \mathscr{U}_n)\right)} = \overline{\bigcup_{n\in\mathbb{N}} f\left(St\left(f^{\leftarrow}(\cup\mathscr{W}_n), \mathscr{U}_n\right)\right)} \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{W}_n, \mathscr{V}_n)}.$$

Thus Y is weakly set star-Menger. □

Theorem 2.8. *If $f : X \to Y$ is an open, perfect mapping from a space X onto a weakly set star-Menger space Y, then for each set $A \subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of $f^{\leftarrow}(\overline{f(A)})$ by sets open in X there is a sequence $(\mathscr{G}_n)_{n\in\mathbb{N}}$ of finite sets such that for each n, $\mathscr{G}_n \subset \mathscr{U}_n$ and $A \subset f^{\leftarrow}(\overline{f(A)}) \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{G}_n, \mathscr{U}_n)}$.*

Proof. Let A be a subset of X and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $f^{\leftarrow}(\overline{f(A)}) = \overline{f^{\leftarrow}(f(A))}$ (because f is open and continuous) by sets open in X. For each $y \in \overline{f(A)}$ the set $C_y := f^{\leftarrow}(y)$ is compact so that for each $n \in \mathbb{N}$ there is a finite set $\mathscr{V}_n(y) \subset \mathscr{U}_n$ which covers C_y. Let $V_n(y) = \bigcup\mathscr{V}_n(y)$. As f is a closed mapping, for each $n \in \mathbb{N}$ and each $y \in \overline{f(A)}$ there is an open set $W_n(y) \subset Y$ such that $y \in W_n(y)$ and $f^{\leftarrow}(W_n(y)) \subset V_n(y)$. For each $n \in \mathbb{N}$ set $\mathscr{W}_n = \{W_n(y) : y \in \overline{f(A)}\}$. Then each $\mathscr{W}_n$ is a cover of $\overline{f(A)}$ by sets open in Y. Since Y is weakly set star-Menger, there is a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ such that $\mathscr{H}_n$ is a finite subset of $\mathscr{W}_n$, $n \in \mathbb{N}$, and $f(A) \subset \overline{\bigcup_{n\in} St(\cup\mathscr{H}_n, \mathscr{W}_n)}$. For each n and each $H \in \mathscr{H}_n$ there is a finite $\mathscr{U}_H \subset \mathscr{U}_n$ with $f^{\leftarrow}(H) \subset \bigcup\mathscr{U}_H$. If $\mathscr{G}_n = \{U \in \mathscr{U}_n : U \in \mathscr{U}_H, H \in \mathscr{H}_n\}$, then $\mathscr{G}_n$ is a finite subset of $\mathscr{U}_n$ for each n. Since f is open, we have

$$A \subset f^{\leftarrow}\left(\overline{f(A)}\right) \subset f^{\leftarrow}\left(\overline{\bigcup_{n\in} St(\cup\mathscr{H}_n, \mathscr{W}_n)}\right) = \overline{f^{\leftarrow}\left(\bigcup_{n\in} St(\cup\mathscr{H}_n, \mathscr{W}_n)\right)} \subset \overline{\bigcup_{n\in} St(\cup\mathscr{G}_n, \mathscr{U}_n)}.$$

Now we will prove the last inclusion. Suppose that $f^{\leftarrow}(\cup\mathscr{H}_n) \cap f^{\leftarrow}(W) \neq \emptyset$. Then also $f(f^{\leftarrow}(\cup\mathscr{H}_n)) \cap f(f^{\leftarrow}(W)) \neq \emptyset$, so $\cup\mathscr{H}_n \cap W \neq \emptyset$ with $f^{\leftarrow}(\mathscr{W}_n) = \mathscr{U}_n$ and $f^{\leftarrow}(U) = W$ where $W \in \mathscr{W}_n$. This completes the proof. □

3. Star versions of quasi forms of the covering properties

Definition 3.1. A space X is said to have:

1. *quasi-star Menger property* q-SM (respectively, *quasi-star Rothberger property* q-SR) if for each closed set $F \subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of covers of F by sets open in X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ (respectively, a sequence $(U_n)_{n\in\mathbb{N}}$) such that for each n, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ (respectively, $U_n \in \mathscr{U}_n$) and $F \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{V}_n, \mathscr{U}_n)}$ (respectively, $F \subset \overline{\bigcup_{n\in\mathbb{N}} St(U_n, \mathscr{U}_n)}$.
2. the *quasi-star Hurewicz property* q-SH (respectively, *quasi-star Gerlits-Nagy property* q-SGN) if for each closed set $F \subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of covers of F by sets open in X there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ (respectively, a sequence $(U_n)_{n\in\mathbb{N}}$) such that for each $n \in \mathbb{N}$, $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ (respectively, $U_n \in \mathscr{U}_n$) and each $x \in F$ belongs to $\overline{St(\cup\mathscr{V}_n, \mathscr{U}_n)}$ (respectively, to $\overline{St(U_n, \mathscr{U}_n)}$) for all but finitely many n.

Theorem 3.2. *Every hereditarily separable space X is quasi-star Rothberger.*

Proof. Let F be a closed subset of X and let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of families of open sets such that $F \subset \cup\mathscr{U}_n$ for each $n \in \mathbb{N}$. Fix a countable set $A = \{a_n : n \in \mathbb{N}\} \subset F$ which is dense in F. For each n choose an element U_n in $\mathscr{U}_n$ such that $a_n \in U_n$. Then $A \subset \bigcup_{n\in\mathbb{N}} U_n \subset \bigcup_{n\in\mathbb{N}} St(U_n, \mathscr{U}_n)$ so that we have $F = \bar{A} \subset \overline{\bigcup_{n\in\mathbb{N}} St(U_n, \mathscr{U}_n)}$, i.e. X is quasi-star Rothberger. □

It is worth to mention by this proposition that every hereditarily separable space is quasi-star Menger so that it is weakly star-Menger.

Theorem 3.3. *Set star-Mengerness implies quasi-star Mengerness.*

Proof. Let F be a closed subset of a set star-Menger space X and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of F by sets open in X. Since X is set star-Menger there is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each $n \in \mathbb{N}$ and $F \subset \bigcup_{n\in\mathbb{N}} St(\cup\mathscr{V}_n, \mathscr{U}_n)$. Then $\overline{F} = F \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{V}_n, \mathscr{U}_n)}$, i.e. X is quasi-star Menger. □

Theorem 3.4. *Quasi-star Mengerness implies weak set star-Mengerness.*

Proof. Let A be a subset of a quasi-star Menger space X and let $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of open subsets of X such that $\overline{A} \subset \bigcup\mathscr{U}_n$ for each n. Apply the fact that X is quasi-star Menger to the closed set $\overline{A}$ and the sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$. There is a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$ such that $\mathscr{V}_n$ is a finite subset of $\mathscr{U}_n$ for each n and $\bar{A} \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{V}_n, \mathscr{U}_n)}$. Therefore, $A \subset \overline{\bigcup_{n\in\mathbb{N}} St(\cup\mathscr{V}_n, \mathscr{U}_n)}$, i.e. X is weakly set star-Menger. □

In a similar way we prove the corresponding theorems for other considered classes of spaces. Therefore, we have the following diagrams.

$$\begin{array}{ccccccccc}
\text{set–SR} & \Rightarrow & \text{SR} & \Rightarrow & \text{q}-\text{SR} & \Rightarrow & \text{w–set–SR} & \Leftarrow & \text{a–set–SR} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Downarrow & & \Uparrow \\
\text{set–SGN} & \Rightarrow & \text{SGN} & \Rightarrow & \text{q}-\text{SGN} & \Rightarrow & \text{w–set–SGN} & \Leftarrow & \text{a–set–SGN} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{set–SH} & \Rightarrow & \text{SH} & \Rightarrow & \text{q}-\text{SH} & \Rightarrow & \text{w–set–SH} & \Leftarrow & \text{a–set–SH} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{set–SM} & \Rightarrow & \text{SM} & \Rightarrow & \text{q}-\text{SM} & \Rightarrow & \text{w–set–SM} & \Leftarrow & \text{a–set–SM} \\
\Uparrow & & \Uparrow & & \Uparrow & & \Uparrow & & \Uparrow \\
\text{set–SR} & \Rightarrow & \text{SR} & \Rightarrow & \text{q}-\text{SR} & \Rightarrow & \text{w–set–SR} & \Leftarrow & \text{a–set–SR}
\end{array}$$

Diagram 1

$$\begin{array}{ccccccccc}
\text{set–R} & \Leftrightarrow & \text{R} & \Rightarrow & \text{q}-\text{R} & \Rightarrow & \text{w–set–R} & \Leftarrow & \text{a–set–R} \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\
\text{set–SR} & \Rightarrow & \text{SR} & \Rightarrow & \text{q}-\text{SR} & \Rightarrow & \text{w–set–SR} & \Leftarrow & \text{a–set–SR}
\end{array}$$

Diagram 2

Theorem 3.5. *If A is a clopen subset of a quasi-star Menger (quasi-star Rothberger) space X, then A is also quasi-star Menger (quasi-star Rothberger).*

Proof. Let F be a closed subset of (A,τ_A) and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of $F=\mathrm{Cl}_{\tau_A}(F)$ (the closure of F with respect to the topology τ_A) by sets open in (A,τ_A). Since A is open, each $\mathscr{U}_n$ is a family of sets open in (X,τ). Since A is closed $\mathrm{Cl}_{\tau_A}(F)=F=\mathrm{Cl}_\tau(F)$. F is a closed subset of X, and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ is a sequence of covers of $F=\mathrm{Cl}_\tau(F)$ by sets open in X. As X is quasi-star Menger there exists a sequence $(\mathscr{V}_n)_{n\in\mathbb{N}}$, $\mathscr{V}_n$ is finite subset of $\mathscr{U}_n$ for each $n\in\mathbb{N}$ and $F\subset\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{V}_n,\mathscr{U}_n)}$. Since each $\mathscr{V}_n$ is open in (A,τ_A), it follows that A is also quasi-star Menger.
Quite similarly we prove the quasi-Rothberger part of the theorem. □

Theorem 3.6. *Continuous image of a quasi-star Menger (quasi-star Rothberger) space is also quasi-star Menger (quasi-star Rothberger).*

Proof. We consider only the quasi-star Menger case. Suppose that X is quasi-star Menger and $f:X\to Y$ is a continuous onto mapping. Let F be a closed subset of Y and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ be a sequence of covers of F by sets open in Y. Let $f^{\leftarrow}(F)=f^{\leftarrow}(\overline{F})=\overline{H}=H$. Because f is continuous $f^{\leftarrow}(\overline{F})$ is closed in X and $f^{\leftarrow}(\mathscr{U}_n)=\mathscr{U}_n'$ is the collection of open sets in X with

$$H=\overline{H}=f^{\leftarrow}(\overline{F})=f^{\leftarrow}(F)\subset f^{\leftarrow}(\cup\mathscr{U}_n)=\cup\mathscr{U}_n'.$$

As X is quasi-star Menger, there are $\mathscr{W}_n\subset\mathscr{U}_n'$ for each $n\in\mathbb{N}$, $\mathscr{W}_n$ is finite for each n and $H\subset\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{W}_n,\mathscr{U}_n')}$. So $f(H)=F\subset f\left(\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{W}_n,\mathscr{U}_n')}\right)\subset\overline{f\left(\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{W}_n,\mathscr{U}_n')\right)}\subset\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{V}_n,\mathscr{U}_n)$. Now we will prove the last inclusion. Suppose that $f^{\leftarrow}(\cup\mathscr{V}_n)\cap f^{\leftarrow}(U)\neq\emptyset$. Then also $f(f^{\leftarrow}(\cup\mathscr{V}_n))\cap f(f^{\leftarrow}(U))\neq\emptyset$, so $\cup\mathscr{V}_n\cap U\neq\emptyset$. Then we conclude that Y is quasi-star Menger. □

Theorem 3.7. *If $f:X\to Y$ is an open, perfect mapping from a space X onto a quasi-star Menger space Y, then for each closed set $F\subset X$ and each sequence $(\mathscr{U}_n)_{n\in\mathbb{N}}$ of open covers of $f^{\leftarrow}(\overline{f(F)})$ by sets open in X there is a sequence $(\mathscr{G}_n)_{n\in\mathbb{N}}$ of finite sets such that for each n, $\mathscr{G}_n\subset\mathscr{U}_n$ and $f^{\leftarrow}(f(F))\subset\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{G}_n,\mathscr{U}_n)}$.*

Proof. Let F be a closed subset of X and $(\mathscr{U}_n)_{n\in\mathbb{N}}$ a sequence of open covers of $f^{\leftarrow}(f(F))$ by sets open in X. Since f is a closed mapping, the set $f(F)$ is closed in Y. For each $y\in f(F)$ the set $F_y:=f^{\leftarrow}(y)$ is compact so that for each $n\in\mathbb{N}$ there is a finite set $\mathscr{V}_{(y,n)}\subset\mathscr{U}_n$ which covers F_y. Let $V_{(y,n)}=\cup\mathscr{V}_{(y,n)}$. As f is a closed mapping, for each $n\in\mathbb{N}$ and each $y\in f(F)$ there is an open set $W_{(y,n)}\subset Y$ such that $y\in W_{(y,n)}$ and $f^{\leftarrow}(W_{(y,n)})\subset V_{(y,n)}$.
For each $n\in\mathbb{N}$ set $W_n=\{W_{(y,n)}:y\in f(F)\}$. Then each $\mathscr{W}_n$ is a cover of $f(F)$ by sets open in Y. Since Y is quasi-star Menger, there is a sequence $(\mathscr{H}_n)_{n\in\mathbb{N}}$ such that $\mathscr{H}_n$ is a finite subset of $\mathscr{W}_n$, $n\in\mathbb{N}$, and $f(F)\subset\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{H}_n,\mathscr{W}_n)}$. For each n and each $H\in\mathscr{H}_n$ there is a finite $\mathscr{U}_H\subset\mathscr{U}_n$ with $f^{\leftarrow}(H)\subset\bigcup\mathscr{U}_H$. If $\mathscr{G}_n=\{U\in\mathscr{U}_n:U\in\mathscr{U}_H,H\in\mathscr{H}_n\}$, then $\mathscr{G}_n$ is a finite subset of $\mathscr{U}_n$ for each n. Since f is open, we have

$$f^{\leftarrow}(f(F))\subset f^{\leftarrow}\left(\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{H}_n,\mathscr{W}_n)}\right)=\overline{f^{\leftarrow}\left(\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{H}_n,\mathscr{W}_n)\right)}\subset\overline{\bigcup_{n\in\mathbb{N}}St(\cup\mathscr{G}_n,\mathscr{U}_n)}.$$

Now we will prove the last inclusion. Suppose that $f^{\leftarrow}(\cup\mathscr{H}_n)\cap f^{\leftarrow}(W)\neq\emptyset$. Then also $f(f^{\leftarrow}(\cup\mathscr{H}_n))\cap f(f^{\leftarrow}(W))\neq\emptyset$, so $\cup\mathscr{H}_n\cap W\neq\emptyset$.
This completes the proof. □

4. Conclusion

In this paper, we did not consider weaker forms of set star-Rothberger spaces, but all properties from Section 2 concerning the weaker forms of set star-Mengerness properties can be investigated for weaker forms of set star-Rothberger spaces applying similar techniques for their proofs.

Acknowledgement

The author grateful to Prof. Ljubisa D.R. Kocinac for his valuable suggestions and comments to improve the paper. The author also thanks to the referees for their careful readings of the paper and comments to improve it.

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