



A New Pre-Order Relation for Set Optimization using ℓ -difference

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Abstract

A new relation on the subset of the space is defined via ℓ -difference in this work. This is a pre-order relation on the family of nonempty sets. Some relations between this pre-order relation and well-known order relations are investigated. Also, the solution points of a set-valued optimization problem via set and vector approaches are examined.

Keywords: Pre-order relation, Set optimization, Vector approach

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1. Introduction

Optimization problems appear in all parts of our lives. These problems are classified according to the type of objective functions. For example, when the objective function is a set-valued function/mapping/map, the optimization problem is named set-valued optimization problem (shortly, *SVOP*). Recently, *SVOP* has attracted increasing attention because it has many applications such as finance, control theory, game theory, engineering, statistic, etc.

In the *SVOP*, there are several approaches to solve these optimization problems. Vector and set approaches are the most commonly used types. The first used is the vector approach. In this approach, efficient vectors of the image set of the objective map are investigated. In order to be a solution of a point, the image set of this point has to contain an efficient vector of the image set. The set approach, which is given by Kuroiwa [16, 17], depends on the comparison among values of the objective map. So, an order relation must be used to compare sets in this approach. More information about these approaches and the solution concepts are also available in [4, 6, 8–15, 17, 18], and references therein. In this current investigation, vector and set approaches are considered.

Firstly, Kuroiwa et al. [19] mentioned about set relation based on the ordering cone. Then, they defined six order relations. They gave relationships with each other. By using these order relations, the set optimization approach is constructed by Kuroiwa [16, 17]. Kuroiwa obtained the solutions of *SVOP* with respect to (shortly, wrt) set approach. Jahn & Ha [6] obtained some new order relations for *SVOP*. Two new partial order relations are defined by Karaman et al. [13] for *SVOP*. There are still sets that can not be compared with these partial and the other order relations. That's why we define a new order relation to compare such sets in this paper.

In order to solve *SVOP*, some methods are used as vectorization, scalarization, directional derivative, subdifferential, embedding space, and so on [1–4, 7–9, 11–15, 21, 23]. The well-known scalarization functions are Gerstewitz, the oriented function of Hiriart-Urruty [5] and generalizations of them. Hernández & Rodríguez-Marín [4] found some optimality conditions for *SVOP* via derived an extension of Gerstewitz function. Recently, some authors like Khushboo & Lalitha [15], Xu & Li [23], Jiménez et al. [7], Ansari et al. [1] and Chen et al. [2] obtained scalarizations via some extension of the oriented function.

A new relation on the subset of the space is defined via l -difference in this work. We show that this order relation is a pre-order on the family of nonempty sets. Some properties of this pre-order relation are obtained. This pre-order relation is compared with some well-known order relations in the literature. Also, the solutions of *SVOP* wrt set and vector approaches are examined.

The layout of the study is ordered as follows: The basic definitions and concepts of *SVOP* are stated and mentioned in section 2. In section 3, a pre-order relation is introduced and some properties are discovered. In section 4, after the solutions concept of *SVOP* are recalled wrt set approach, the solutions of *SVOP* are compared according to set and vector approaches.

2. Mathematical Preliminaries

In this study, Y is denoted as a normed space and X is a vector space. Let $K \subset Y$ be given. If $\lambda x \in K$ for all $x \in K$, $\lambda > 0$, then K is called a cone. Assume that cone K is a convex, pointed ($K \cap (-K) = \{0_Y\}$) and closed with the nonempty interior, and Y be ordered by cone K . $\mathcal{P}(Y)$ is denoted the family of proper and nonempty subsets of Y , that is, $\mathcal{P}(Y) := \{A \subset Y : A \neq Y \text{ and } A \text{ is nonempty}\}$. Topological interior and convex hull of any set $A \in \mathcal{P}(Y)$ are indicated by $\text{int}(A)$ and $\text{conv}A$, respectively.

It is denoted that the algebraic sum of A and B by $A + B$, the algebraic difference of A and B by $A - B$, Minkowski (Pontryagin) difference of A and B by $A \dot{-} B := \{x \in Y \mid x + B \subset A\}$ and l -difference of A and B by $A \ominus_l B := \{x \in Y \mid x + B \subset A + K\} = (A + K) \dot{-} B$ for any $A, B \in \mathcal{P}(Y)$. Readers can find more information about these in [20–22].

The cone K induces an ordering relations on Y as follow: For $x, x' \in Y$, $x \leq_K x'$ iff $x' - x \in K$, and $x <_K x'$ iff $x' - x \in \text{int}(K)$.

Let $A \in \mathcal{P}(Y)$ be a set and $a_0 \in A$. If $A \cap (a_0 - K) = \{a_0\}$ ($A \cap (a_0 + K) = \{a_0\}$), then a_0 is called a minimal (maximal) point of A . The set of all minimal and maximal points of A is indicated by $\min A$ and $\max A$, respectively. Likewise, if $A \cap (a_0 - \text{int}(K)) = \emptyset$ ($A \cap (a_0 + \text{int}(K)) = \emptyset$), then we say that a_0 is a weak minimal (weak maximal) point of A .

Note that the binary relation \preceq on $\mathcal{S} \subset \mathcal{P}(Y)$ is called a pre-order on \mathcal{S} if \preceq is reflexive and transitive. Also, if pre-order relation \preceq is antisymmetric then the order relation is called a partial order on \mathcal{S} .

Definition 2.1. Let $\mathcal{S} \subset \mathcal{P}(Y)$ and $A, B, C \in \mathcal{S}$ be any sets. The relation \preceq on \mathcal{S} is said to be

- (i) compatible with the addition if $A \preceq B$ implies $A + C \preceq B + C$,
- (ii) compatible with positive scalar multiplication if $A \preceq B$ implies $\lambda A \preceq \lambda B$ for all scalars $\lambda > 0$.

Let $F : X \rightrightarrows Y$ be a set-valued function such that $F(x) \in \mathcal{P}(Y)$ for all $x \in X$, and $\text{dom}(F) := \{x \in X \mid F(x) \neq \emptyset\}$ be efficient domain set of the set-valued mapping F . $S \subset \text{dom}(F)$ be given. Basic *SVOP* is described by

$$SVOP \begin{cases} \min(\max)F(x) \\ \text{s.t. } x \in S. \end{cases}$$

We denote the problem by (v -*SVOP*) when *SVOP* considers wrt vector approach. Efficient points of the set $F(S) := \bigcup_{x \in S} F(x)$ are investigated to solve (v -*SVOP*), that is, if $F(x_0)$ contains a minimal (maximal) point of $F(S)$, then $x_0 \in S$ is called a solution of (v -*SVOP*). In the same way, if $F(x_0)$ contains a weak minimal (weak maximal) point of $F(S)$, then $x_0 \in S$ is entitled a weak solution of (v -*SVOP*).

In the set approach, a comparison among the values of the set-valued mapping is considered. Namely, efficient sets of $\mathcal{F}(S) := \{F(x) \mid x \in S\}$ are investigated to solve *SVOP*. So, an order relation is needed to solve a *SVOP* by using the set approach. In the following definition, some order relations are given:

Definition 2.2. [6, 13, 17] Let $A, B \in \mathcal{P}(Y)$.

- (i) $A \preceq^1 B$ is described by $\forall a \in A, \forall b \in B, a \leq_K b$,
- (ii) $A \preceq^2 B$ is described by $\exists a \in A$ such $\forall b \in B, a \leq_K b$,
- (iii) $A \preceq^3 B$ is described by $\forall b \in B, \exists a \in A$ such $a \leq_K b$,
- (iv) $A \preceq^4 B$ is described by $\exists b \in B, \text{ such } \forall a \in A, a \leq_K b$,
- (v) $A \preceq^5 B$ is described by $\forall a \in A, \exists b \in B$ such that $a \leq_K b$,
- (vi) $A \preceq^6 B$ is described by $\exists a \in A, \exists b \in B$ such that $a \leq_K b$,
- (vii) $A \preceq^s B$ is described by $A \preceq^3 B$ and $A \preceq^5 B$,

- (viii) $A \preceq^{m_1} B$ is described by $(B \dot{-} A) \cap K \neq \emptyset$,
(ix) $A \preceq^{m_2} B$ is described by $(A \dot{-} B) \cap (-K) \neq \emptyset$.

It is assumed that $*$ \in $\{1, 2, 3, 4, 5, 6, s, m_1, m_2\}$ in the rest of the study. In the set approach, the problem is denoted by $(* - SVOP)$ when $SVOP$ considers wrt order relation \preceq^* . The efficient set of $\mathcal{F}(S)$ is investigated to solve $(* - SVOP)$. That is, if $F(x_0) \in \mathcal{F}(S)$ is a minimal (resp., maximal) set of $\mathcal{F}(S)$, then x_0 is called a solution of $(* - SVOP)$. Similarly, if $F(x_0) \in \mathcal{F}(S)$ is a weak minimal (resp., weak maximal) set of $\mathcal{F}(S)$, then x_0 is named a weak solution of $(* - SVOP)$.

3. A new Order Relation for Set Approach

In this section, a pre-order relation is derived by using l -difference and some properties of this relation are examined.

Definition 3.1. Let $A, B \in \mathcal{P}(Y)$. l_1 relation is defined as

$$A \preceq^{l_1} B : \iff (B \ominus_\ell A) \cap K \neq \emptyset.$$

When A and B are taken as singleton, there is a relation between \preceq^{l_1} and vector order relation \leq_K on Y as:

$$a \leq_K b \implies \{a\} \preceq^{l_1} \{b\}$$

for any $A = \{a\}, B = \{b\}$ and $a, b \in Y$.

When two sets don't compare wrt partial order relation \preceq^{m_1} , they may be compared wrt order relation \preceq^{l_1} . For example, when $A = \{(x, 0) \in \mathbb{R}^2 \mid 1 \leq x \leq 3\}$ and $B = \{(0, y) \in \mathbb{R}^2 \mid 1 \leq y \leq 3\}$, we have $A \preceq^{l_1} B$ and $A \not\preceq^{m_1} B$.

Now, some properties of \preceq^{l_1} are presented.

Proposition 3.2. The order relation \preceq^{l_1} has the following properties;

- (i) \preceq^{l_1} is compatible with the addition,
(ii) \preceq^{l_1} is compatible with the positive scalar multiplication.

Proof.

- (i) Let $A, B, C \in \mathcal{P}(Y)$ and $A \preceq^{l_1} B$ be given. Since $A \preceq^{l_1} B$, we have $(B \ominus_\ell A) \cap K \neq \emptyset$. That means there exists $\bar{x} \in K$ such that $\bar{x} \in B \ominus_\ell A$. Then, we get $\bar{x} + A \subset B + K$. So, $\bar{x} + A + C \subset B + C + K$, that is $\bar{x} \in (B + C) \ominus_\ell (A + C)$. Therefore,

$$[(B + C) \ominus_\ell (A + C)] \cap K \neq \emptyset.$$

Thus, we obtain $A + C \preceq^{l_1} B + C$ that implies \preceq^{l_1} is compatible with the addition.

- (ii) Let $A \preceq^{l_1} B$. We show that $\lambda A \preceq^{l_1} \lambda B$ for all scalars $\lambda > 0$. Since $A \preceq^{l_1} B$, there exists an $\bar{x} \in K$ such $\bar{x} \in B \ominus_\ell A$, i.e., $\bar{x} + A \subset B + K$. So, we have $\lambda \bar{x} + \lambda A \subset \lambda B + \lambda K = \lambda B + K$ and $\lambda \bar{x} \in K$ because K is cone. Then, we obtain $(\lambda B \ominus_\ell \lambda A) \cap K \neq \emptyset$, i.e., $\lambda A \preceq^{l_1} \lambda B$. Hence, \preceq^{l_1} is compatible with the positive scalar multiplication. \square

Proposition 3.3. The order relation \preceq^{l_1} has the following properties;

- (i) \preceq^{l_1} is reflexive,
(ii) \preceq^{l_1} is transitive.

Proof.

- (i) Let $A \in \mathcal{P}(Y)$. Because $0_Y \in A \ominus_\ell A$ and $0_Y \in K$, we have $(A \ominus_\ell A) \cap K \neq \emptyset$. Hence, $A \preceq^{l_1} A$.
(ii) Assume that $A \preceq^{l_1} B$ and $B \preceq^{l_1} C$ for any $A, B, C \in \mathcal{P}(Y)$. We have $(B \ominus_\ell A) \cap K \neq \emptyset$ since $A \preceq^{l_1} B$. Then, there exists $x_1 \in K$ such

$$x_1 + A \subset B + K. \tag{3.1}$$

Since $B \preceq^{l_1} C$, $(C \ominus_\ell B) \cap K \neq \emptyset$ yields. Then, there exists $x_2 \in K$ such

$$x_2 + B \subset C + K. \tag{3.2}$$

From (3.1) and (3.2) we get $x_1 + x_2 + A \subset x_2 + B + K \subset C + K + K = C + K$. As $x_1 + x_2 + A \subset C + K$ and $x_1 + x_2 \in K$, we obtain $(C \ominus_\ell A) \cap K \neq \emptyset$, i.e., $A \preceq^{l_1} C$.

□

Remark 3.4. The order relation \preceq^{ℓ_1} isn't antisymmetric. For example, let $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$, $A = \{(1, 1)\}$ and $B = \{(2, 2)\}$ be given. Then, we get $B \ominus_{\ell} A = \{x \in \mathbb{R}^2 \mid x + A \subset B + K\} = [1, \infty) \times [1, \infty)$ and $A \ominus_{\ell} B = \{x \in \mathbb{R}^2 \mid x + B \subset A + K\} = [-1, \infty) \times [-1, \infty)$. So, we have $(B \ominus_{\ell} A) \cap K \neq \emptyset$ and $(A \ominus_{\ell} B) \cap K \neq \emptyset$, i.e., $A \preceq^{\ell_1} B$ and $B \preceq^{\ell_1} A$. But $A \neq B$. Hence, \preceq^{ℓ_1} isn't antisymmetric.

Corollary 3.5. The order relation \preceq^{ℓ_1} is a pre-order relation on $\mathcal{P}(Y)$.

Now, a relation between the order relation \preceq^{ℓ_1} and order relation \preceq^{m_1} are given.

Proposition 3.6. Let $A, B \in \mathcal{P}(Y)$. If $A \preceq^{m_1} B$, then $A \preceq^{\ell_1} B$.

Proof. Let $A \preceq^{m_1} B$, i.e., $(B \dot{-} A) \cap K \neq \emptyset$ be given. There exists an $x \in K$ such $x + A \subset B$. Because K is pointed, we get $x + A \subset B + K$, i.e., $x \in B \ominus_{\ell} A$. As $x \in K$ and $(B \ominus_{\ell} A) \cap K \neq \emptyset$, we obtain $A \preceq^{\ell_1} B$ □

Note that \preceq^{ℓ_1} doesn't imply \preceq^{m_1} . This is presented in the following example.

Example 3.7. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$, $A = \text{conv}\{(0, 0), (1, 2)\}$ and $B = \text{conv}\{(0, 0), (2, 1)\}$.

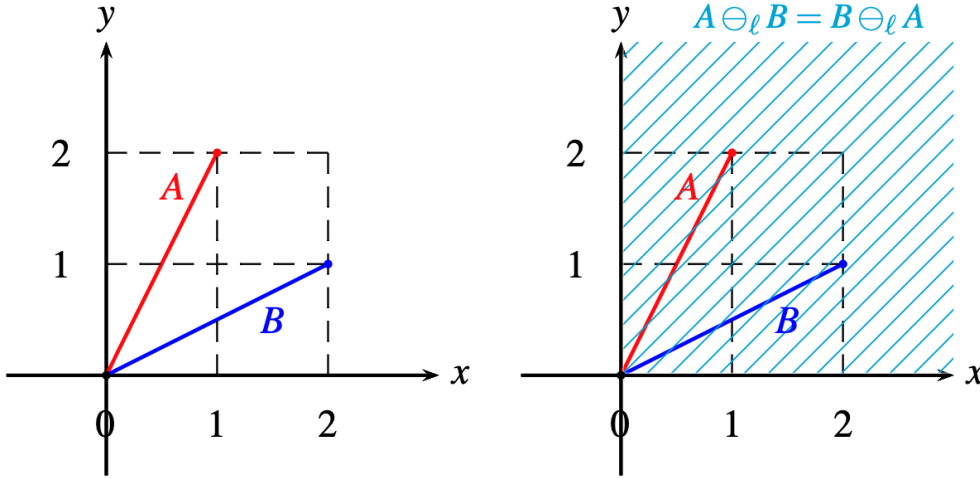


Figure 3.1. $A = \text{conv}\{(0, 0), (1, 2)\}$, $B = \text{conv}\{(0, 0), (2, 1)\}$, $A \ominus_{\ell} B$ and $B \ominus_{\ell} A$

As seen in Figure 3.1, since $(B \ominus_{\ell} A) = K$, we have $(B \ominus_{\ell} A) \cap K = K$, i.e., $A \preceq^{\ell_1} B$. On the other hand, as $B \dot{-} A = \emptyset$, we get $(B \dot{-} A) \cap K = \emptyset$, i.e., $A \not\preceq^{m_1} B$.

In the following definition, strict version of \preceq^{ℓ_1} is given.

Definition 3.8. Let $A, B \in \mathcal{P}(Y)$. The strict ℓ_1 order relation is defined by

$$A \prec^{\ell_1} B : \iff (B \ominus_{\ell} A) \cap \text{int}(K) \neq \emptyset$$

Note that \prec^{ℓ_1} implies \preceq^{ℓ_1} . Namely, if $A \prec^{\ell_1} B$, then $A \preceq^{\ell_1} B$ for all $A, B \in \mathcal{P}(Y)$.

Remark 3.9. The order relation \prec^{ℓ_1} is compatible with not only the addition but also the positive scalar multiplication. Moreover, it is reflexive and transitive. But it isn't antisymmetric.

One of the most important problems in the set order relations is that some sets can not be compared according to any order relation. Although two sets may not be compared wrt order relation \preceq^* , these sets can be compared wrt ℓ_1 order relation. This is illustrated in the accompanying example.

Example 3.10. Let $K = \mathbb{R}_+^2$, $A = \{(x, y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 \text{ and } 3 \leq y \leq 4\}$ and $B = \{(x, y) \in \mathbb{R}^2 \mid 3 \leq x \leq 4 \text{ and } 1 \leq y \leq 2\}$. As seen Figure 3.2, while $A \not\preceq^* B$ we obtain $(B \ominus_{\ell} A) \cap K \neq \emptyset$, i.e., $A \preceq^{\ell_1} B$.

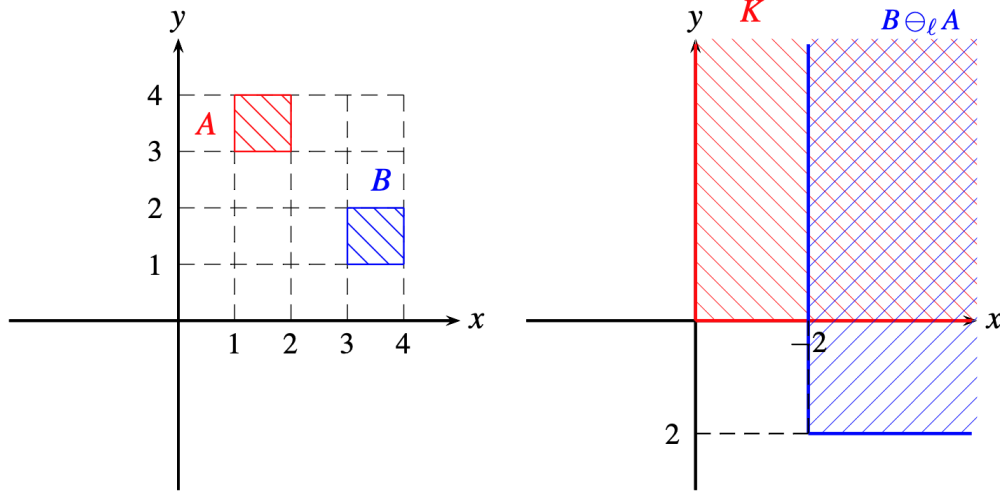


Figure 3.2. $A = \{(x,y) \in \mathbb{R}^2 \mid 1 \leq x \leq 2 \text{ and } 3 \leq y \leq 4\}$, $B = \{(x,y) \in \mathbb{R}^2 \mid 3 \leq x \leq 4 \text{ and } 1 \leq y \leq 2\}$ and $B \ominus_{\ell} A$

Proposition 3.11. Let $A, B \in \mathcal{P}(Y)$. Then, the following assertions are satisfied:

- (i) If there exist $a \in A$ and $b \in B$ such that $b \leq_K a$, then $A \preceq^{\ell_1} B$,
- (ii) if there exist $a \in A$ and $b \in B$ such that $b <_K a$, then $A \prec^{\ell_1} B$.

Proof. (i) Assume that there exist $a \in A$ and $b \in B$ such that $b \leq_K a$. By contradiction, suppose that $A \not\preceq^{\ell_1} B$. Then, $(B \ominus_{\ell} A) \cap K = \emptyset$, and we have $k + A \not\subseteq B + K$ for all $k \in K$. So, $k + a \notin B + K$ for all $k \in K$, $a \in A$. From here, we get $k + a \notin b + K$ for all $k \in K$, $a \in A$ and $b \in B$. Let $k = 0_Y$, then $a \notin b + K$ for all $a \in A$ and $b \in B$. Therefore, for all $a \in A$ and $b \in B$ we get $b \not\leq_K a$, which is a contradict.

- (ii) This can be proven by similarly to (i). □

4. Solution Concepts of SVOP According to Set and Vector Approaches

In this part of the study, we obtain relations between the solutions of $(v-SVOP)$ and (ℓ_1-SVOP) .

In the following definition, the efficient elements of a family are given wrt pre-order relation \preceq^{ℓ_1} .

Definition 4.1. Let $\mathcal{S} \subset \mathcal{P}(Y)$ and $A \in \mathcal{S}$ be given. We call that

- (i) A is an ℓ_1 -minimal (ℓ_1 -maximal) element of \mathcal{S} iff

$$B \preceq^{\ell_1} A \text{ for some } B \in \mathcal{S} \implies A \preceq^{\ell_1} B \quad (A \preceq^{\ell_1} B \text{ for some } B \in \mathcal{S} \implies B \preceq^{\ell_1} A),$$

- (ii) A is a weak ℓ_1 -minimal (weak ℓ_1 -maximal) element of \mathcal{S} iff

$$B \prec^{\ell_1} A \text{ for some } B \in \mathcal{S} \implies A \prec^{\ell_1} B \quad (A \prec^{\ell_1} B \text{ for some } B \in \mathcal{S} \implies B \prec^{\ell_1} A).$$

If we consider $SVOP$ wrt ℓ_1 order relation, then problem is denoted by

$$(\ell_1-SVOP) \begin{cases} \min(\max) F(x) \\ \text{s.t. } x \in S. \end{cases}$$

Let $x_0 \in S$ be given. x_0 is called a solution of (ℓ_1-SVOP) if $F(x_0) \in \mathcal{F}(S)$ is an ℓ_1 -minimal (ℓ_1 -maximal) set of $\mathcal{F}(S)$. Similarly, x_0 is called a weak solution of (ℓ_1-SVOP) if $F(x_0) \in \mathcal{F}(S)$ is a weak ℓ_1 -minimal (weak ℓ_1 -maximal) set of $\mathcal{F}(S)$.

The solution of (ℓ_1-SVOP) may not be the solution of $(v-SVOP)$. Now, we will give an example related to this situation.

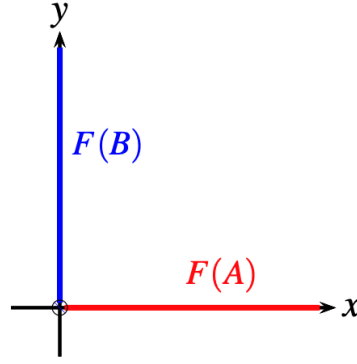


Figure 4.1. $F(A) = \{(x,0) \in \mathbb{R}^2 \mid x > 0\}$ and $F(B) = \{(0,y) \in \mathbb{R}^2 \mid y > 0\}$

Example 4.2. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$, set-valued map $F : \{A,B\} \rightrightarrows \mathbb{R}^2$ be defined as $F(A) = \{(x,0) \in \mathbb{R}^2 \mid x > 0\}$ and $F(B) = \{(0,y) \in \mathbb{R}^2 \mid y > 0\}$. Consider the following set-valued optimization problem

$$SVOP \begin{cases} \min F(x) \\ \text{s.t. } x \in \{A,B\}. \end{cases}$$

As seen in Figure 4.1, $F(A) \cap \min\{F(A) \cup F(B)\} = \emptyset$ and $F(B) \cap \min\{F(A) \cup F(B)\} = \emptyset$. So, A and B pairs of sets are not a solution of $(v-SVOP)$. On the other hand, A and B are solution of (ℓ_1-SVOP) because $F(A) \preceq^{\ell_1} F(B)$ implies $F(B) \preceq^{\ell_1} F(A)$, and $F(B) \preceq^{\ell_1} F(A)$ implies $F(A) \preceq^{\ell_1} F(B)$.

Conversely, the solution of $(v-SVOP)$ may not be the solution of (ℓ_1-SVOP) . The following example is related to this situation.

Example 4.3. Let $Y = \mathbb{R}^2$, $K = \mathbb{R}_+^2$, set-valued map $F : \{1,2\} \rightrightarrows \mathbb{R}^2$ be defined as $F(1) = \{(x,y) \in \mathbb{R}^2 \mid x = y \text{ and } x \geq 0\}$ and $F(2) = \{(x,y) \in \mathbb{R}^2 \mid y = -x \text{ and } x \geq 0\}$. Let's consider the following problem

$$SVOP \begin{cases} \min F(x) \\ \text{s.t. } x \in \{1,2\}. \end{cases}$$

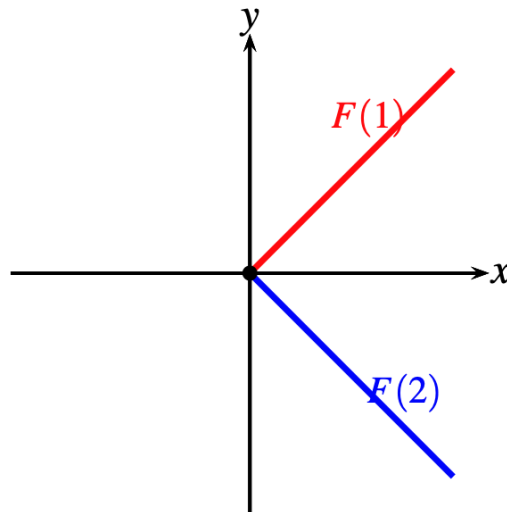


Figure 4.2. $F(1) = \{(x,y) \in \mathbb{R}^2 \mid x = y \text{ and } x \geq 0\}$ and $F(2) = \{(x,y) \in \mathbb{R}^2 \mid y = -x \text{ and } x \geq 0\}$

As seen in Figure 4.2, because $\min\{F(1) \cup F(2)\} \cap F(1) \neq \emptyset$ and $\min\{F(1) \cup F(2)\} \cap F(2) \neq \emptyset$, 1 and 2 are solution of $(v-SVOP)$.

Since $(F(2) \ominus_\ell F(1)) \cap K \neq \emptyset$, we have $F(1) \preceq^{\ell_1} F(2)$. As $F(1) \ominus_\ell F(2) = \emptyset$, we obtain $(F(1) \ominus_\ell F(2)) \cap K = \emptyset$. Hence, we get $F(2) \not\preceq^{\ell_1} F(1)$. Because $F(1) \preceq^{\ell_1} F(2)$ doesn't imply $F(2) \preceq^{\ell_1} F(1)$, 2 isn't a solution of $(\ell_1 - SVOP)$. Although 2 is a solution of $(v - SVOP)$, it isn't a solution of $(\ell_1 - SVOP)$.

5. Conclusion

In this study, a new pre-order relation on the family of nonempty sets is introduced, and set-valued optimization problems $(\ell_1 - SVOP)$ are derived. Some optimality conditions can be obtained by using different tools such as vectorization, directional derivative, scalarization, subdifferential etc. for $(\ell_1 - SVOP)$.

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