INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 15 NO. 1 PAGE 75–78 (2022)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.957190

An Analogue of Thébault's Theorem Linking the Radical Center of Four Spheres with the Insphere and the Monge Point of a Tetrahedron

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(Communicated by Lyudmila N. Romakina)

ABSTRACT

In 1953, Victor Thébault conjectured a link between the altitudes of a tetrahedron and the radical center of the four spheres with the centers at the vertices of this tetrahedron and the corresponding tetrahedron altitudes as radii. This conjecture was proved in 2015. In this paper, we propose an analogue of Thébault's theorem. We establish a link between the radical center of the four spheres, the insphere, and the Monge point of a tetrahedron.

Keywords: *Solid geometry, Thébault's theorem, tetrahedron, radical center of spheres, Monge point;, insphere.*
AMS Subject Classification (2020): *Primary: 51M04; Secondary: 51M05; 51M20; 51N20.*

1. Introduction

The famous French problemist Victor Thébault (1882 – 1960) conjectured the following geometric fact linking the radical center of four spheres with other elements of a tetrahedron. Let AA', BB', CC', DD' be the altitudes of a tetrahedron $ABCD$ with feet A', B', C', and D', respectively. Let P be the radical center of the spheres with centers A, B, C, D and radii AA', BB', CC', and DD', respectively. Then each plane passing through the midpoint of the segment $B'C'$, $C'A'$, $A'B'$, $D'A'$, $D'B'$ or $D'C'$ perpendicular to the segment BC, CA, AB, DA, DB or DC, respectively, contains the point P [5, 6]. This hypothesis was proved in 2015 in [3]. We found a fact that is similar to the result of Thébaultt, but now linking the radical center of four spheres with the insphere and the Monge point of a tetrahedron. To avoid ambiguities, we give here the following definitions.

Let T be a tetrahedron in the Euclidean space $\mathbb{E}^3$. A sphere that touches four faces of the tetrahedron T is called the *insphere* of T. There are six planes, each of which passes through the midpoint of the edge of the tetrahedron T perpendicular to its opposite edge. These six planes have a common point, which is called the *Monge point* of T [4, 7, 8].

The *power* of a point P with respect to a sphere ω with a center O and radius R is the number

$$Pow(\omega, P) = |OP|^2 - R^2.$$

Let ω_1, ω_2, ω_3, and ω_4 be spheres with noncomplanar centers in $\mathbb{E}^3$. There exists an unique point that has the same power with respect to each of these spheres. This point is called the *radical center* of the spheres ω_1, ω_2, ω_3, and ω_4 [2, 1].

Received : *24-06-2021,* ***Accepted :*** *25-03-2022*
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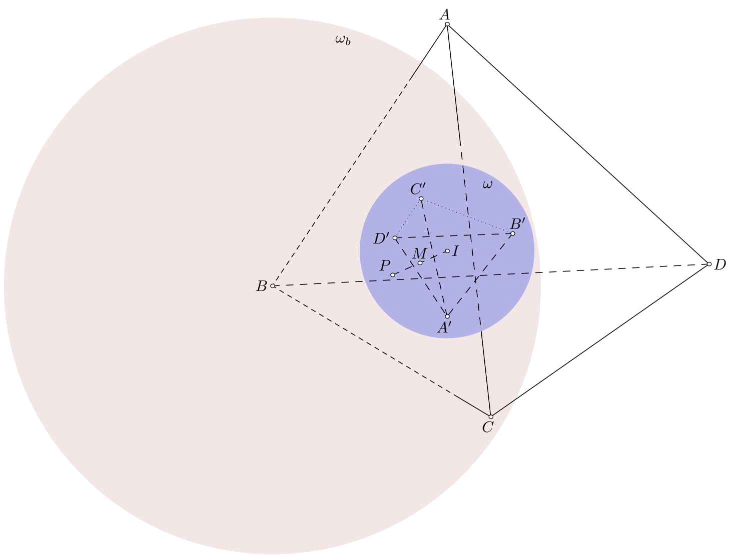

Figure 1. The contact points A', B', C', and D' of the insphere ω with the tetrahedron $ABCD$, the sphere ω_b, and the points I, M, P.

2. The main theorem

Theorem 2.1. *Let ω be the insphere of a tetrahedron $ABCD$. Assume that ω has the center at a point I and touches the faces (BCD), (CDA), (DAB), and (ABC) at points A', B', C', and D', respectively. Denote a sphere with the center A, B, C or D and radius AA', BB', CC' or, respectively, DD' by ω_a, ω_b, ω_c or ω_d, respectively. Let M be the Monge point of the tetrahedron $A'B'C'D'$, and let P be the reflection of I with respect to M. Then point P is the radical center of the spheres ω_a, ω_b, ω_c, and ω_d.*

Proof. We provide an analytical proof of the theorem by depicting the considered elements of the tetrahedron $ABCD$ in Figure 1.

Let us introduce a Cartesian coordinate system in the space $\mathbb{E}^3$ putting the origin to the point I and selecting the radius IA' of the insphere ω as a unit segment of the first coordinate axis. Assume that the coordinate plane (xy) contains the point B'. Using the spherical coordinates for the insphere ω (see, for instance, formulae (8), (9), (10) in [9]), we find the following coordinates of the considered points

$$I\,(0,0,0),\ A'\,(1,0,0)\,,\ B'\,(a,a',0)\,,\ C'\,(bc,cb',c')\,,\ D'\,(de,ed',e')\,, \tag{2.1}$$

where for the real numbers a, b, c, d, e, a', b', c', d', and e' from the segment $[-1,1]$ the following conditions hold

$$a'=\sqrt{1-a^2},\ b'=\sqrt{1-b^2},\ c'=\sqrt{1-c^2},\ d'=\sqrt{1-d^2},\ e'=\sqrt{1-e^2}.$$

The plane, which is orthogonal to the segment $C'D'$ and passes through the midpoint of the segment $A'B'$, has the equation

$$(de-bc)\,x+(ed'-cb')\,y+(e'-c')\,z-\frac{a'\,(ed'-cb')+(de-bc)\,(a+1)}{2}=0. \tag{2.2}$$

The plane, which is orthogonal to the segment $B'D'$ and passes through the midpoint of the segment $A'C'$, is given by the equation

$$(de-a)\,x+(ed'-a')\,y+(e')\,z+\frac{(bc+1)\,(a-de)-c'e'-cb'\,(ed'-a')}{2}=0. \tag{2.3}$$

And finally, the plane, which is orthogonal to the segment $A'D'$ and contains the midpoint of the segment $B'C'$, has the equation

$$(de-1)\,x+(ed')\,y+(e')\,z-\frac{(de-1)\,(a+bc)+c'e'+e\,(a'+cb')\,d'}{2}=0. \tag{2.4}$$

Solving the system of Eqs. (2.2), (2.3), and (2.4), we find the coordinates of the Monge point M of the tetrahedron $ABCD$:

$$M\left(\frac{a+de+bc+1}{2},\ \frac{ed'+cb'+a'}{2},\ \frac{c'+e'}{2}\right).$$

The point P, which is the reflection of I with respect to M, has the coordinates

$$P\left(a+de+bc+1,\ ed'+a'+cb',\ c'+e'\right). \tag{2.5}$$

The planes containing faces of the tetrahedron $ABCD$ are defined by the following conditions

$$(BCD)\perp IA',\ A'\in(BCD),\quad (CDA)\perp IB',\ B'\in(CDA),$$

$$(DAB)\perp IC',\ C'\in(DAB),\quad (ABC)\perp ID',\ D'\in(BCD).$$

From these conditions we find the equations of the planes (BCD), (CDA), (DAB), and (ABC), respectively:

$$x-1=0, \tag{2.6}$$

$$ax+a'y-1=0, \tag{2.7}$$

$$bcx+cb'y+c'z-1=0, \tag{2.8}$$

$$dex+ed'y+e'z-1=0. \tag{2.9}$$

Point C is the intersection of the planes (BCD), (CDA), and (ABC). Thus, solving the system of Eqs. (2.6), (2.7), and (2.9), we obtain the coordinates of the point C:

$$C\left(1,\ -\frac{a-1}{a'},\ \frac{-dea'+ed'a-ed'+a'}{e'a'}\right). \tag{2.10}$$

From the system of Eqs. (2.6), (2.7), and (2.8), we obtain the coordinates of the point D:

$$D\left(1,\ -\frac{a-1}{a'},\ \frac{-bca'+cb'a-cb'+a'}{c'a'}\right). \tag{2.11}$$

Using the points coordinates from (2.1), (2.5), (2.10), and (2.11), we find the power of P with respect to spheres ω_d and ω_c, respectively:

$$\begin{aligned} Pow(\omega_d,P) &= d^2(P,D)-d^2(D',D)=(a+de+bc+1-1)^2 \\ &+\left(ed'+a'+cb'+\frac{a-1}{a'}\right)^2+\left(c'+e'-\frac{-bca'+cb'a-cb'+a'}{c'a'}\right)^2 \\ &-(de-1)^2-\left(ed'+\frac{a-1}{a'}\right)^2-\left(e'-\frac{-bca'+cb'a-cb'+a'}{c'a'}\right)^2 \\ &=2a+2de+2c'e'+2bc+2ea'd'+2ade+2ca'b'+2abc+2ceb'd'+2bcde-3. \end{aligned} \tag{2.12}$$

$$\begin{aligned} Pow(\omega_c,P) &= d^2(P,C)-d^2(C',C)=(a+de+bc+1-1)^2 \\ &+\left(ed'+a'+cb'+\frac{a-1}{a'}\right)^2+\left(c'+e'-\frac{-dea'+ed'a-ed'+a'}{e'a'}\right)^2 \\ &-(bc-1)^2-\left(cb'+\frac{a-1}{a'}\right)^2-\left(c'-\frac{-dea'+ed'a-ed'+a'}{e'a'}\right)^2 \\ &=2a+2de+2c'e'+2bc+2ea'd'+2ade+2ca'b'+2abc+2ceb'd'+2bcde-3. \end{aligned} \tag{2.13}$$

Expressions (2.12), (2.13) yield the equality $Pow(\omega_d,P)=Pow(\omega_c,P)$. Taking into account that the vertices of the tetrahedron $ABCD$ in the proof of this fact are chosen arbitrarily, we conclude that the power of the point P with respect to each of spheres ω_a, ω_b, ω_c, and ω_d has the same value.

This completes the theorem proof. □

3. Conclusion

The content of this paper deal with objects in the three-dimensional Euclidean space. Nevertheless, we believe that similar results for simplexes also exist in the Euclidean space $\mathbb{E}^n$ of any dimension n. The concept of Monge point in a simplex can be defined as follows.

Let F be a simplex in the space $\mathbb{E}^n$. All hyperplanes, each of which passes through the centroid of a $(n-2)$-face of F and perpendicular to the opposite edge of this face, have a common point. This point is called the *Monge point* of the simplex F [4].

We present the following conjecture.

Conjecture. *Let ω be the insphere of a simplex $A_0A_1\dots A_n$ in the space $\mathbb{E}^n$. Denote the centre of ω by I, and for $i=\overline{0,n}$, denote the tangency point of ω with the face $(A_0A_1\dots A_{i-1}A_{i+1}\dots A_n)$ by B_i. Let ω_i be the sphere with the center A_i and radius A_iB_i, and let M be the Monge point of simplex $B_0B_1\dots B_n$. Then the radical center of all spheres ω_i lies on the line IM.*

Acknowledgements

We express our sincere thanks to Professor Lyudmila N. Romakina for her enthusiasm in completing this manuscript. We also thank the two referees for their important comments.

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