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ON THE BOUNDS FOR THE NORMS OF FTOEPLITZ AND ALMOST FHANKEL MATRICES

Necati TAŞKARA, Aynur YALÇINER, Hasan KÖSE

Selçuk Üniversitesi, Fen Edebiyat Fakültesi, Matematik Bölümü, Konya

ABSTRACT

In this study, we obtain defined Ftoeplitz and Almost Fhankel matrices. Then, we obtained upper bounds for the Euclidean norm of these matrices.

Key Words: Norm, FToeplitz, FHankel, Euclidean Norm

FTOEPLITZ VE HEMEN HEMEN FHANKEL MATRİSLERİNİN NORMLARI İÇİN SINIRLAR

ÖZET

Bu çalışmada, FToeplitz ve Hemen Hemen FHankel matrislerini tanımlayarak bu matrislerin Euclides normları için üst sınırlar elde edildi.

Anahtar Kelimeler: Norm, FToeplitz, FHankel, Euclidean Norm

1. INTRODUCTION

Toeplitz matrices defined by O. Toeplitz at 1910. Let us consider

$$a_0 + 2\sum_{n=1}^{\infty} r^n (a_n \cos nx + b_n \sin nx) \tag{1}$$

where $a_k, b_k \in R \quad (k = 1, \ldots, n)$. Let $t_n = a_n - ib_n$ and

$t_{-n} = a_n + ib_n$ such that $[t_{i-j}]$ is a hermitian matrix for $i, j = 0,...,n$ and $n = 0,1,...,b_0 = 0$. The hermitian form

$$T_n(x) = \sum t_{i-j} x_i \overline{x}_j, \qquad i, j = 0,...,n \tag{2}$$

is called Toeplitz form associated with the harmonic function in (1) and also the matrix associated with the form in (2) is called Toeplitz matrix and this matrix is represented $T_n = [t_{ij}]_{i,j=0}^{n}$ such that $t_{ij} = t_{i-j}$. Let $f(x)$ be a real valued function and let the Fourier series of $f(x)$ function be

$$f(x) = \sum_{-\infty}^{\infty} t_n e^{inx}.$$

Then

$$t_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-imx} f(x)dx, \ (m = \pm 1, \pm 2, ...). \tag{3}$$

Thus, the formulea (1.3) allow to compute the entries t_{ij} of T_n with the proper $f(x)$ function.

Let $\gcd(i, j)$ be the greatest common divisior of i and j integers. A GCD matrix $A = [a_{ij}]$ be defined as $a_{ij} = \gcd(i, j)$ where $i, j = 1,...,n$ [1]. The GCD matrices are defined by S. Beslin and S. Ligh [1]. Beslin and Ligh have given the formulas for the determinants of these matrices. E. Altınışık has obtained the formulas for the inverses of these matrices [4].

The ℓ_p norm $\|A\|_p$ of an $m \times n$ matrix A is defined by

$$\|A\|_p = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^p \right)^{1/p} \tag{4}$$

where $p \geq 2$. The ℓ_p norm is called as the Euclidean (or Frobenius) norm in the case $p = 2$ and denoted by $\|A\|_E$ or $\|A\|_F$.

The spectral norm $\|A\|_2$ of an m×n matrix A with complex entries defined by

$$\|A\|_2 = \sqrt{\max_{1\le i\le n}\left|\lambda_i(A^*A)\right|}$$

where A^* denotes the conjugate transpose of the matrix A.

We know that

$$\frac{1}{\sqrt{n}}\|A_n\|_F \le \|A_n\|_2$$

from equivalent norms [3].

In Section 2 of this paper, we have studied on the properties of Fibonacci numbers. In Section 3, we have established upper bounds for the euclidean norms of the FToeplitz and FHankel matrices.

2. PROPERTIES OF FIBONACCI NUMBERS

Definition 2. 1. It is called Fibonacci numbers with $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 3$.

Some Fibonacci numbers are given in the following:

n	1	2	3	4	5	6	7	8	9	10
F_n	1	1	2	3	5	8	13	21	34	55

We can give some properties of Fibonacci numbers as in the following:

1. $n \in \mathbb{Z}^+, \sum_{i=1}^{n} F_i = F_{n\text{-}2}\ \text{-}\ 1,$

2. $n \in \mathbb{Z}^+, \sum_{i=1}^{n} F_i^2 = F_n F_{n+1},$

3. $m, n \in \mathbb{Z}^+, F_n F_m + F_{n+1} F_{m+1} = F_{m+n+1},$

4. $m,n,r\in\mathbb{Z}^+, F_{m+n-2}F_{m+r-1} - F_{m+n-1}F_{m+r-2} = (-1)^{m+r-2}F_{n-r}$,

5. $n\in\mathbb{Z}^+$ then $F_n = \dfrac{\alpha^n - \beta^n}{\alpha - \beta}$ such that $\alpha = \frac{1}{2}(1+\sqrt{5})$ and $\beta = \frac{1}{2}(1-\sqrt{5})$.

Definition 2. 2. Let $a_1, a_2, ..., a_{2n-1}\in$ and $H_n = [h_{ij}]$ be an $n\times n$ matrix such that $h_{ij} = a_{i+j-1}$. Then the matrix H_n is called Hankel matrix.

Definition 2. 3. Let (a_k) be an integer squence such that $a_k \neq 0$ for $k\geq 1$ and let $A=[a_{ij}]$ be an $n\times n$ matrix such that $a_{ij} = \dfrac{1}{a_{i+j-1}}$. Then the matrix A is called reciprocal Hankel matrix associate with (a_k) squence and is indicate $H_n(a_k)$.

Definition 2. 4. Let F_k be kth Fibonacci number. The matrix T is called almost gcd FToeplitz matrix such that the entries of T are $t_{ij} = 1/F[\gcd(i,j)]$.

Definition 2. 5. Let F_k be kth Fibonacci number. The matrix T_n is called FToeplitz matrix such that the entries of T_n are

$$t_{ij} = \begin{cases} a, & i=j\ (a\in IR) \\ \dfrac{1}{F(i-j)} & i\neq j \end{cases}$$

Definition 2. 6. Let F_k be kth Fibonacci number and H_n be an nxn matrix such that $h_{ij} = 1/F_i + F_j$. Then the matrix H_n is called almost FHankel matrix.

3. NORMS OF FTOEPLITZ AND ALMOST FHANKEL MATRICES

Theorem 3. 1. Let T be $n\times n$ almost gcd FTeoplitz matrix. Then the inequalities are $\|T\|_F \leq n$ valid where $\|\cdot\|_F$ is euclidean norm.

Proof. The main diagonal elements of the matrix T are $1/F[\gcd(i,j)]$. Since greatest of the elements that it is to prevent Toeplitz matrix of the matrix T is equal to 1/2 and the number of these elements is most $2n$ and by the definition of euclidean norm, we have

$$\|T\|_F^2 \leq (n^2 - \frac{3n}{2}).1 + 2n\left(\frac{1}{2}\right)^2 + \sum_{k=1}^{n} \frac{1}{(F_k)^2}$$

For all k, since $\sum_{k=1}^{n} 1/(F_k)^2 \leq 2.4$, we obtain $\|T\|_F^2 \leq n^2 - n + 2.4 \leq n^2$. This is completed the proof.

Theorem 3. 2. Let T_n be $n \times n$ FTeoplitz matrix. Then the inequalities are

$$\frac{1}{\sqrt{n}}\|T_n\|_F \leq \sqrt{a^2 + 2\pi}$$

valid where $\|.\|_F$ is euclidean norm.

Proof. By the definition FTeoplitz matrix and Frobenius norm

$$\|T_n\|_F^2 = na^2 + 2\sum_{k=1}^{n-1} \frac{n-k}{F_k^2}. \tag{5}$$

If we divide by n both sides of the equality (3.1), then

$$\frac{1}{n}\|T_n\|_F^2 = a^2 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\frac{1}{F_k^2}. \tag{6}$$

If we take the limit of the right hand side of the in equality (3.2) as $n \to \infty$, we have

$$\lim_{n\to\infty}\left(a^2 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right)\frac{1}{F_k^2}\right) = a^2 + 2\pi.$$

Hence

$$\frac{1}{n}\|T_n\|_F \leq a^2 + 2\pi. \tag{7}$$

If we take the square root of the in equality (3.3), then we obtain

$$\frac{1}{\sqrt{n}}\|\mathrm{T}_n\|_F \leq \sqrt{a^2 + 2\pi}\,.$$

Theorem 3. 3. Let H_n be $n \times n$ almost FHankel matrix. Then the inequality is $\|H_n\|_F \leq 2.6$ valid where $\|.\|_F$ is euclidean norm.

Proof. Since all entries of the matrix H_n is equal or less than 1/2, i.e. $h_{kj} \leq \sqrt{2k-1} / F_k$ for all *k,j*, we can replace the entries of the matrix H_n with $\sqrt{2k-1} / F_k$ for $k = 1,\ldots,n.$ Then by the definition euclidean norm, we have

$$\|H_n\|_F^2 \leq \sum_{k=1}^{n-1} \frac{2k-1}{F_k^2}.$$

Since $\sum_{k=1}^{n-1}(2k-1)/F_k^2 \leq 6.7$ for all n, we obtain $\|H_n\|_F \leq 2.6$.

Table 1. Verification of Theorem 3.3 with different n values

n	$\|H_n\|_F$	$\left(\sum_{k=1}^{n} \frac{2k-1}{F_k^2}\right)^{1/2}$
3	1.227576655	2.291287848
5	1.434538134	2.527405345
10	1.499052953	2.586808136
20	1.500230616	2.587685852
50	1.500230784	2.587685963
80	1.500230784	2.587685963
90	1.500230784	2.587685963
100	1.500230784	2.587685963
150	1.500230784	2.587685963

Table 2. Verification of Theorem 3.2 with different n values for a=0

n	$\frac{1}{\sqrt{n}}\|T_n\|_F$	$\sqrt{a^2+2\pi}$
3	1.414213563	2.506628274
4	1.620185174	2.506628274
15	2.060213395	2.506628274
50	2.161062866	2.506628274
60	2.168086910	2.506628274
80	2.176835090	2.506628274
100	2.182067164	2.506628274
200	2.176835090	2.506628274
300	2.195958418	2.506628274

Table 3. Verification of Theorem 3.2 with different n values for a=1

n	$\frac{1}{\sqrt{n}}\|T_n\|_F$	$\sqrt{a^2+2\pi}$
3	1.732050808	2.698737725
4	1.903943276	2.698737725
15	2.290082800	2.698737725
50	2.381216646	2.698737725
60	2.387593108	2.698737725
80	2.395539816	2.698737725
100	2.400295213	2.698737725
200	2.409777852	2.698737725
300	2.412930454	2.698737725

Table 4. Verification of Theorem 3.2 with different n values for a=1/2

N	$\frac{1}{\sqrt{n}}\|T_n\|_F$	$\sqrt{a^2+2\pi}$
3	1.500000000	2.556009646
4	1.695582496	2.556009646
15	2.120018686	2.556009646
50	2.218150741	2.556009646
60	2.224994572	2.556009646
80	2.233519870	2.556009646
100	2.238619465	2.556009646
200	2.248783960	2.556009646
300	2.252161934	2.556009646

Table 5. Verification of Theorem 3.1 with different n values

N	$\|T\|_F$	n
3	2.872281323	3
4	3.789605667	4
5	4.732981206	5
6	5.560282017	6
10	9.177599832	10
15	13.56009108	15
20	18.06761831	20
50	44.71394731	50
100	88.67822829	100

REFERENCES

1. Beslin S. and Ligh S., Greatest Common Divisior Matrices, Linear Algebra and Its Applications, 118, 69-76, (1989).

2. Szegö G. and Grenander U., Toeplitz Forms and Their Applications, University of California Press, (1958).

3. Bozkurt D., On the ℓ_{p} Norms of Almost Cauchy-Toeplitz Matrices, Turkish Journal of Mathematics, 20: (4), (1996).

4. Altınışık E., Hemen Hemen Hilbert Smith Matrislerinin Karakterizasyonu, PhD. Thesis, S. Ü. Fen Bilimleri Enstitüsü, Konya, (2001).