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s-Convex Functions in the Fourth Sense and Some of Their Properties

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Abstract

In this paper, s-convex functions in the fourth sense is introduced. Its main characterizations, algebraic and functional properties are presented. Also, some relations between these functions and the other types of s-convex functions are given.

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1. Introduction

Convex functions have been an attraction center for many researchers since the very beginning of the last century after Jensen's systematic studies of these functions. It has been indispensible feature of the optimization problems after L. V. Kantorovich, Dantzig and Leontiev's solution methods in 1940s [23]. Since then, researchers have set forth the generalizations and extensions of the notion of convexity such as quasiconvex functions, Schur convex function, (K,S)-convex functions, B and B^{-1}-convex functions, s-convex functions, relative strongly exponentially convex functions, co-ordinated s-convex functions etc. which are employed in equilibrium theory, signal processing, stocastic analysis, microeconomics, and fractal theory [1–5, 9–12, 14, 16, 20, 22, 24–26]. One of them is given very recently by Micherda in [17]. That study presents a generalization of convexity, namely, (k,h)- convexity, in which two functions on $(0,1)$ are used, one defines the convexity of set and the other function determines the convexity type of the function. Its definition is given as follows:
Let $k:(0,1)\to\mathbb{R}$ and D be subset of X. If $k(\lambda)x+k(1-\lambda)y\in D$ for all $x,y\in D$ and $\lambda\in(0,1)$, then D is called k-convex set.
Let $D\subseteq X$ be a k-convex set and let $k,h:(0,1)\to\mathbb{R}$ and $f:D\to\mathbb{R}$. If for all $x,y\in D$ and $\lambda\in(0,1)$,

$$f(k(\lambda)x+k(1-\lambda)y)\leq h(\lambda)f(x)+h(1-\lambda)f(y) \tag{1.1}$$

is satisfied, then f is said to be (k,h)-convex function. In case of $k(\lambda)=h(\lambda)=\lambda$, definitions of classical convex set and function are obtained.
In the case $k(\lambda)=\lambda^{\frac{1}{p}}$ and $h(\lambda)=\lambda^{\frac{1}{p}}$ for $0<p\leq 1$ in (1.1), p-convexity concepts, which have been already introduced by [6, 7, 21], are given as follows:

Definition 1.1. *[7] Let $U\subseteq\mathbb{R}^n$ and $0<p\leq 1$. If for each $x,y\in U$, $\lambda,\mu\geq 0$ such that $\lambda^p+\mu^p=1$, $\lambda x+\mu y\in U$, then U is called a p-convex set in $\mathbb{R}^n$.*

Definition 1.2. *[21] Let $U\subseteq\mathbb{R}^n$ and let $f:U\to\mathbb{R}$ be a function. If the set*

$$epif=\left\{(x,\alpha)\in\mathbb{R}^{n+1}:x\in U,\ \alpha\in\mathbb{R},\ f(x)\leq\alpha\right\}$$

is p-convex set, then f is called a p-convex function.

The following theorem gives us a characterization of p-convex functions:

Theorem 1.3. *[21] Let $U\subseteq\mathbb{R}^n$ and let $f:U\to\mathbb{R}$ be a function. Then, f is a p-convex function if and only if U is a p-convex set, for all $\lambda,\mu\geq 0$ such that $\lambda^p+\mu^p=1$ and for each $x,y\in U$*

$$f(\lambda x+\mu y)\leq\lambda f(x)+\mu f(y) \tag{1.2}$$

is satisfied.

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The case $k(\lambda)=\lambda^{\frac{1}{s}}$ and $h(\lambda)=\lambda$ for $0<s\leq 1$ in (1.1) corresponds to the following type of s-convexity and was used in the theory of Orlicz spaces [18]:

Definition 1.4. *Let $U\subseteq\mathbb{R}^n$ be a s-convex set such that $s\in(0,1]$. A function $f:U\to\mathbb{R}$ is said to be s-convex in the first sense if*

$$f(\lambda x+\mu y)\leq\lambda^s f(x)+\mu^s f(y)$$

for all $x,y\in U$ and $\lambda,\mu\geq 0$ with $\lambda^s+\mu^s=1$.

In this definition, the concept of s-convex set is the same concept as p-convex set in Definition 1.1.
In case $k(\lambda)=\lambda$ and $h(\lambda)=\lambda^s$ for $0<s\leq 1$ in (1.1), the following type of s-convexity is obtained as follows:

Definition 1.5. *[8] Let $U\subseteq\mathbb{R}^n$ be a convex set and $s\in(0,1]$. A function $f:U\to\mathbb{R}$ is said to be s-convex in the second sense if the inequality*

$$f(\lambda x+\mu y)\leq\lambda^s f(x)+\mu^s f(y) \tag{1.3}$$

holds for all $x,y\in U$ and all $\lambda,\mu\geq 0$ with $\lambda+\mu=1$.

In the case $k(\lambda)=\lambda$ and $h(\lambda)=\lambda^s$ for $0<s\leq 1$ in (1.1), the following type of s-convexity is given as follows:

Definition 1.6. *[15] Let $U\subseteq\mathbb{R}^n$ be a convex set and $s\in(0,1]$. A function $f:U\to\mathbb{R}$ is said to be s-convex in the third sense if the inequality*

$$f(\lambda x+\mu y)\leq\lambda^{\frac{1}{s}} f(x)+\mu^{\frac{1}{s}} f(y)$$

holds for all $x,y\in U$ and all $\lambda,\mu\geq 0$ with $\lambda^s+\mu^s=1$.

The classes of s-convex functions in first, second and third senses are denoted by K_s^1,K_s^2,K_s^3 respectively. It can be easily seen that in the case $s=1$, each type of s-convexity is reduced to the ordinary convexity of functions.
In this paper, the s-convex function in the fourth sense is introduced, examples and some characterizations are given. The conditions under which this type of s-convexity is preserved are given. Some relations to other kinds of s-convexity are investigated.

2. s-Convex Functions in the Fourth Sense

Definition 2.1. *Let U be a convex subset of a vector space X and let $s\in(0,1]$. A function $f:U\to\mathbb{R}$ is said to be s-convex in the fourth sense if the inequality*

$$f(\lambda x+\mu y)\leq\lambda^{\frac{1}{s}} f(x)+\mu^{\frac{1}{s}} f(y) \tag{2.1}$$

is satisfied for each $x,y\in U$ and for all $\lambda,\mu\geq 0$ such that $\lambda+\mu=1$. The inequality (2.1) is equivalent to the following inequalities:

$$f(\lambda^s x+\mu^s y)\leq\lambda f(x)+\mu f(y),$$

where $\lambda,\mu\geq 0$ such that $\lambda^s+\mu^s=1$ and

$$f(\lambda x+(1-\lambda)y)\leq\lambda^{\frac{1}{s}} f(x)+(1-\lambda)^{\frac{1}{s}} f(y),$$

where $\lambda\in[0,1]$.
The class of these functions is denoted by K_s^4.
On the other hand, the function $f:U\to\mathbb{R}$ is said to be s-concave in the fourth sense if the inequality

$$f(\lambda x+\mu y)\geq\lambda^{\frac{1}{s}} f(x)+\mu^{\frac{1}{s}} f(y) \tag{2.2}$$

is satisfied for each $x,y\in U$ and for all $\lambda,\mu\geq 0$ such that $\lambda+\mu=1$.

Throughout the paper, $U\subseteq X$ is taken as a convex set and $\mathbb{R}_+=[0,\infty)$, $\mathbb{R}_-=(-\infty,0]$.

Example 2.2. *Let $a,b\in\mathbb{R}$,*

$$L^+_{\frac{1}{s}}[a,b]=\left\{x\in L_{\frac{1}{s}}[a,b]\;\middle|\; x:[a,b]\to\mathbb{R}_+\right\}$$

and $f:L^+_{\frac{1}{s}}[a,b]\to\mathbb{R}$ defined by $f(x)=c\int_a^b|x(t)|^{\frac{1}{s}}dt$, where $c<0$. Then $f\in K_s^4$.
Let $x,y\in L^+_{\frac{1}{s}}[a,b]$ and $0<\lambda<1$. Then, the following relation holds:

$$\begin{aligned}
f(\lambda x+(1-\lambda)y) &= c\int_a^b|\lambda x(t)+(1-\lambda)y(t)|^{\frac{1}{s}}dt\\
&\leq c\int_a^b\left(\lambda^{\frac{1}{s}}|x(t)|^{\frac{1}{s}}+(1-\lambda)^{\frac{1}{s}}|y(t)|^{\frac{1}{s}}\right)dt\\
&=\lambda^{\frac{1}{s}}c\int_a^b|x(t)|^{\frac{1}{s}}dt+(1-\lambda)^{\frac{1}{s}}c\int_a^b\left|y(t)^{\frac{1}{s}}\right|dt\\
&=\lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y).
\end{aligned}$$

So, $f\in K_s^4$.

Example 2.3. *Let $U \subseteq \mathbb{R}^n$ and $k \in \mathbb{R}_+$. If we define $f : U \to \mathbb{R}$ such that $f(x) = -k$ then $f \in K_s^4$. Thus, for $\lambda \in [0,1]$, it can be written*

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= -k \\ &= -(\lambda + (1-\lambda))k \\ &= -\lambda k - (1-\lambda)k \\ &\leq -\lambda^{\frac{1}{s}} k - (1-\lambda)^{\frac{1}{s}} k \\ &= \lambda^{\frac{1}{s}} f(x) + (1-\lambda)^{\frac{1}{s}} f(y). \end{aligned}$$

So, $f \in K_s^4$.

Example 2.4. *Let*

$$U = \{x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n | \ x_1 + x_2 + \cdots + x_n \geq 0\}$$

and $k \in \mathbb{R}_+$. If we define $f : U \to \mathbb{R}$ such that $f(x_1, x_2, \ldots, x_n) = -k(x_1 + x_2 + \cdots + x_n)$, then $f \in K_s^4$. Because, we have $\lambda \in [0,1]$, it can be written

$$\begin{aligned} f(\lambda x + (1-\lambda)y) &= f(\lambda x_1 + (1-\lambda)y_1, \lambda x_2 + (1-\lambda)y_2, \ldots, \lambda x_n + (1-\lambda)y_n) \\ &= -k(\lambda x_1 + (1-\lambda)y_1 + \lambda x_2 + (1-\lambda)y_2 \cdots + \lambda x_n + (1-\lambda)y_n) \\ &= \lambda(-k)(x_1 + x_2 + \cdots + x_n) + (1-\lambda)(-k)(y_1 + y_2 + \cdots + y_n) \\ &\leq \lambda^{\frac{1}{s}}(-k)(x_1 + x_2 + \cdots + x_n) + (1-\lambda)^{\frac{1}{s}}(-k)(y_1 + y_2 + \cdots + y_n) \\ &= \lambda^{\frac{1}{s}} f(x) + (1-\lambda)^{\frac{1}{s}} f(y). \end{aligned}$$

So, it is obtained that $f \in K_s^4$.

Theorem 2.5. *If $f : U \to \mathbb{R}$ be a s-convex function in the fourth sense, then the following inequality is valid for all $x, y \in U$:*

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2^{\frac{1}{s}}}. \tag{2.3}$$

Proof. It is clear by taking $\lambda = \mu = \frac{1}{2}$. □

Corollary 2.6. *If $f : U \to \mathbb{R}$ is s-convex function in the fourth sense, then $f \leq 0$.*

Indeed, accepting $y = x$ in (2.3), we have $f(x) \leq 2^{1-\frac{1}{s}} f(x)$, so $(1 - 2^{1-\frac{1}{s}})f(x) \leq 0$. Thus, $f(x) \leq 0$.
Similary, it is deduced that if f is s-concave function in the fourth sense, then $f \geq 0$.

Theorem 2.7. *Let $f : U \to \mathbb{R}$ be a s-convex function in the fourth sense. Then the inequality (2.1) holds for all $x, y \in U$ and $\lambda, \mu \geq 0$ such that $\lambda + \mu \leq 1$.*

Proof. Assume that $x, y \in U$, $\lambda, \mu \in \mathbb{R}_+$ and $0 < \lambda + \mu < 1$. Put $\gamma = \lambda + \mu$, $\alpha = \frac{\lambda}{\gamma}$ and $\beta = \frac{\mu}{\gamma}$. Then, $\alpha + \beta = \frac{\lambda}{\gamma} + \frac{\mu}{\gamma} = 1$ and we have

$$\begin{aligned} f(\lambda x + \mu y) &= f(\alpha\gamma x + \beta\gamma y) \\ &\leq \alpha^{\frac{1}{s}} f(\gamma x) + \beta^{\frac{1}{s}} f(\gamma y) \\ &= \alpha^{\frac{1}{s}} f(\gamma x + (1-\gamma).0) + \beta^{\frac{1}{s}} f(\gamma y + (1-\gamma).0) \\ &\leq \alpha^{\frac{1}{s}} \left[\gamma^{\frac{1}{s}} f(x) + (1-\gamma)^{\frac{1}{s}}.f(0)\right] + \beta^{\frac{1}{s}} \left[\gamma^{\frac{1}{s}} f(y) + (1-\gamma)^{\frac{1}{s}}.f(0)\right] \\ &= \alpha^{\frac{1}{s}} \gamma^{\frac{1}{s}} f(x) + \beta^{\frac{1}{s}} \gamma^{\frac{1}{s}} f(y) + (\alpha^{\frac{1}{s}} + \beta^{\frac{1}{s}})(1-\gamma)^{\frac{1}{s}}.f(0) \\ &\leq \alpha^{\frac{1}{s}} \gamma^{\frac{1}{s}} f(x) + \beta^{\frac{1}{s}} \gamma^{\frac{1}{s}} f(y) \\ &= \lambda^{\frac{1}{s}} f(x) + \mu^{\frac{1}{s}} f(y). \end{aligned}$$

□

Jensen inequality [13] is very important inequality in convex function theory. The following theorem shows the Jensen inequality for s-convex function in the fourth sense.

Theorem 2.8. *Let $f : U \to \mathbb{R}$ be a s-convex function in the fourth sense and $x_1, x_2 \ldots, x_m \in U$, $\lambda_1, \lambda_2 \ldots, \lambda_m \in \mathbb{R}_+$ with $\lambda_1 + \lambda_2 + \cdots + \lambda_m = 1$. Then*

$$f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m) \leq \lambda_1^{\frac{1}{s}} f(x_1) + \lambda_2^{\frac{1}{s}} f(x_2) + \cdots + \lambda_m^{\frac{1}{s}} f(x_m).$$

Proof. We use induction on m. The inequality is trivially true when $m=2$. Assume that it is true when $m=k$, where $k>2$. Now we show the validity when $m=k+1$. Let a real number x be defined by the equation $x=\lambda_1 x_1+x_2+\cdots+\lambda_{k+1}x_{k+1}$ where $x_1,\ldots,x_{k+1}\in U$, $\lambda_1,\ldots,\lambda_{k+1}\geq 0$ with $\lambda_1+\cdots+\lambda_{k+1}=1$. At least one of $\lambda_1,\ldots,\lambda_{k+1}$ must be less than 1. Let us say $\lambda_{k+1}<1$ and write $\lambda_1+\cdots+\lambda_k=1-\lambda_{k+1}$.One can find $\lambda_*<1$ such that $\lambda_1+\cdots+\lambda_k=\lambda_*$. Since $\left(\frac{\lambda_1}{\lambda_*}\right)+\cdots+\left(\frac{\lambda_k}{\lambda_*}\right)=1$ and the assumption of hypothesis, we get

$$f\left(\frac{\lambda_1}{\lambda_*}x_1+\cdots+\frac{\lambda_k}{\lambda_*}x_k\right)\leq\left(\frac{\lambda_1}{\lambda_*}\right)^{\frac{1}{s}}f(x_1)+\cdots+\left(\frac{\lambda_k}{\lambda_*}\right)^{\frac{1}{s}}f(x_k).$$

By using s-convexity of f in the fourth sense,

$$\begin{aligned} f(x) &= f\left(\lambda_*\left(\tfrac{\lambda_1}{\lambda_*}x_1+\cdots+\tfrac{\lambda_k}{\lambda_*}x_k\right)+\lambda_{k+1}x_{k+1}\right) \\ &\leq \lambda_*^{\frac{1}{s}}f\left(\tfrac{\lambda_1}{\lambda_*}x_1+\cdots+\tfrac{\lambda_k}{\lambda_*}x_k\right)+\lambda_{k+1}^{\frac{1}{s}}f(x_{k+1}) \\ &\leq \lambda_1^{\frac{1}{s}}f(x_1)+\cdots+\lambda_{k+1}^{\frac{1}{s}}f(x_{k+1}) \end{aligned}$$

is obtained. This completes the proof by induction. □

3. Some Properties of s-Convex Functions in the Fourth Sense

Theorem 3.1. *Let $s_1\leq s_2$. If $f:U\to\mathbb{R}$ is a s_2-concave function in the fourth sense, then f is a s_1-concave function in the fourth sense.*

Proof. Let $x,y\in U$ and $\lambda\in[0,1]$. Then, according to Theorem 2.7, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\geq \lambda^{\frac{1}{s_2}}f(x)+(1-\lambda)^{\frac{1}{s_2}}f(y) \\ &\geq \lambda^{\frac{1}{s_1}}f(x)+(1-\lambda)^{\frac{1}{s_1}}f(y), \end{aligned}$$

which means that $f\in K_{s_1}^4$. □

Theorem 3.2. *Let $s_1\leq s_2$ and $f:U\to\mathbb{R}$. If $f\in K_{s_2}^4$, then $f\in K_{s_1}^4$.*

Proof. Let $x,y\in U$ and $\lambda\in[0,1]$. Then, according to Theorem 2.7, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\leq \lambda^{\frac{1}{s_2}}f(x)+(1-\lambda)^{\frac{1}{s_2}}f(y) \\ &\leq \lambda^{\frac{1}{s_1}}f(x)+(1-\lambda)^{\frac{1}{s_1}}f(y), \end{aligned}$$

which means that $f\in K_{s_1}^4$. □

Theorem 3.3. *If $f:U\to\mathbb{R}_-$ is a convex function, then f is a s-convex function in the fourth sense.*

Proof. Let $x,y\in U$ and $\lambda\in[0,1]$. Then, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\leq \lambda f(x)+(1-\lambda)f(y) \\ &\leq \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

□

Theorem 3.4. *If $f:U\to\mathbb{R}$ is a concave function, then f is a s-concave function in the fourth sense.*

Proof. Let $x,y\in U$ and $\lambda\in[0,1]$. Then, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\geq \lambda f(x)+(1-\lambda)f(y) \\ &\geq \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

□

Theorem 3.5. *If $f:U\to\mathbb{R}_+$ be a s-concave function in the second sense, then f is a s-concave function in the fourth sense.*

Proof. Let $x,y\in U$ and $\lambda\in[0,1]$. Then, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &\geq \lambda^s f(x)+(1-\lambda)^s f(y) \\ &\geq \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

□

Theorem 3.6. *Let $f:U\to\mathbb{R}$ and $x,y\in U$. If the function $g:[0,1]\to\mathbb{R}$ defined by $g(\lambda)=f(\lambda x+(1-\lambda)y)$ is a s-convex function in the fourth sense, then f is also a s-convex function in the fourth sense.*

Proof. Let $x,y\in U$ and $\lambda\in[0,1]$. Then

$$\begin{aligned} f(\lambda x+(1-\lambda)y)=g(\lambda) &= g(\lambda\cdot 1+(1-\lambda)\cdot 0)\\ &\leq \lambda^{\frac{1}{s}}g(1)+(1-\lambda)^{\frac{1}{s}}g(0)\\ &= \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

Then, $f\in K_s^4$. □

Theorem 3.7. *If $f_i:U\to\mathbb{R}_-$ are s-convex functions in the fourth sense for $i=1,2,\cdots,m$, then $f=\sum_{i=1}^{m}a_if_i$ is a s-convex function in the fourth sense where $a_i\geq 0$.*

Proof. For $x,y\in U$ and $\lambda\in[0,1]$, we have

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &= \sum_{i=1}^{m}a_if_i(\lambda x+(1-\lambda)y)\\ &\leq \sum_{i=1}^{m}a_i\left(\lambda^{\frac{1}{s}}f_i(x)+(1-\lambda)^{\frac{1}{s}}f_i(y)\right)\\ &= \lambda^{\frac{1}{s}}\sum_{i=1}^{m}a_if_i(x)+(1-\lambda)^{\frac{1}{s}}\sum_{i=1}^{m}a_if_i(y)\\ &= \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

This shows that $f\in K_s^4$. □

Theorem 3.8. *If $f_i:U\to\mathbb{R}_-$ are s-convex functions in the fourth sense for $i=1,2,\cdots,m$, then $f:U\to\mathbb{R}_-$ defined by $f=\max_{1\leq i\leq m}\{f_i\}$ is a s-convex function in the fourth sense.*

Proof. For each $x,y\in U$ and $\lambda\in[0,1]$, we can write

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &= \max_{1\leq i\leq m}\{f_i(\lambda x+(1-\lambda)y)\}\\ &= f_t(\lambda x+(1-\lambda)y)\\ &\leq \lambda^{\frac{1}{s}}f_t(x)+(1-\lambda)^{\frac{1}{s}}f_t(y)\\ &\leq \lambda^{\frac{1}{s}}\max_{1\leq i\leq m}\{f_i(x)\}+(1-\lambda)^{\frac{1}{s}}\max_{1\leq i\leq m}\{f_i(y)\}\\ &= \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

Thus, $f=\max_{1\leq i\leq m}\{f_i\}$ is a s-convex function in the fourth sense. □

Theorem 3.9. *If $f_i:U\to\mathbb{R}$ are s-concave functions in the fourth sense for $i=1,2,\cdots,m$, then $f:U\to\mathbb{R}$ defined by $f=\min_{1\leq i\leq m}\{f_i\}$ is a s-concave function in the fourth sense.*

Proof. For each $x,y\in U$ and $\lambda\in[0,1]$, we can write

$$\begin{aligned} f(\lambda x+(1-\lambda)y) &= \min_{1\leq i\leq m}\{f_i(\lambda x+(1-\lambda)y)\}\\ &= f_t(\lambda x+(1-\lambda)y)\\ &\geq \lambda^{\frac{1}{s}}f_t(x)+(1-\lambda)^{\frac{1}{s}}f_t(y)\\ &\geq \lambda^{\frac{1}{s}}\min_{1\leq i\leq m}\{f_i(x)\}+(1-\lambda)^{\frac{1}{s}}\min_{1\leq i\leq m}\{f_i(y)\}\\ &= \lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y). \end{aligned}$$

Thus, $f=\min_{1\leq i\leq m}\{f_i\}$ is a s-concave function in the fourth sense. □

Next, it will be given some properties of composition of functions in different types of convexity.

Theorem 3.10. *If the function $f:U\to\mathbb{R}_-$ is a s-convex function in the fourth sense and $g:f(U)\to\mathbb{R}$ is an increasing linear function, then $g\circ f:U\to\mathbb{R}$ is a s-convex function in the fourth sense.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(g\circ f)(\lambda x+(1-\lambda)y) &= g(f(\lambda x+(1-\lambda)y))\\ &\leq g(\lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y))\\ &= \lambda^{\frac{1}{s}}g(f(x))+(1-\lambda)^{\frac{1}{s}}g(f(y))\\ &= \lambda^{\frac{1}{s}}(g\circ f)(x)+(1-\lambda)^{\frac{1}{s}}(g\circ f)(y).\end{aligned}$$

Hence, $g\circ f \in K_s^4$. □

Theorem 3.11. *Let $g : U \to \mathbb{R}_+$, $f : g(U) \to \mathbb{R}$ and f be decreasing linear function. If g is a s-concave function in the fourth sense, then $f\circ g \in K_s^4$.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y))\\ &\leq f(\lambda^{\frac{1}{s}}g(x)+(1-\lambda)^{\frac{1}{s}}g(y))\\ &= \lambda^{\frac{1}{s}}f(g(x))+(1-\lambda)^{\frac{1}{s}}f(g(y))\\ &= \lambda^{\frac{1}{s}}(f\circ g)(x)+(1-\lambda)^{\frac{1}{s}}(f\circ g)(y).\end{aligned}$$

□

Theorem 3.12. *Let $f : U \to \mathbb{R}_-$ and $g : \mathbb{R}_- \to \mathbb{R}$ be an increasing function. If $f \in K_s^4$ and $g \in K_s^3$, then $g\circ f \in K_{s^2}^4$.*

Proof. For each $x,y \in U$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}(g\circ f)(\lambda x+(1-\lambda)y) &= g(f(\lambda x+(1-\lambda)y))\\ &\leq g(\lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y))\\ &\leq \lambda^{\frac{1}{s^2}}g(f(x))+(1-\lambda)^{\frac{1}{s^2}}g(f(y))\\ &= \lambda^{\frac{1}{s^2}}(g\circ f)(x)+(1-\lambda)^{\frac{1}{s^2}}(g\circ f)(y).\end{aligned}$$

Hence, $g\circ f \in K_{s^2}^4$. □

Theorem 3.13. *If $f : U \to \mathbb{R}_+$ is a s-concave function in the fourth sense and $g : f(U) \to \mathbb{R}$ is a decreasing s-convex function in the third sense, then $g\circ f$ is a s^2-convex function in the fourth sense.*

Proof. For each $x,y \in U$ and $\lambda \in [0,1]$, we have

$$\begin{aligned}(g\circ f)(\lambda x+(1-\lambda)y) &= g(f(\lambda x+(1-\lambda)y))\\ &\leq g(\lambda^{\frac{1}{s}}f(x)+(1-\lambda)^{\frac{1}{s}}f(y))\\ &\leq \lambda^{\frac{1}{s^2}}g(f(x))+(1-\lambda)^{\frac{1}{s^2}}g(f(y))\\ &= \lambda^{\frac{1}{s^2}}(g\circ f)(x)+(1-\lambda)^{\frac{1}{s^2}}(g\circ f)(y).\end{aligned}$$

□

Theorem 3.14. *Let $g : U \to V$ be a linear transformation and $f : V \to \mathbb{R}$ be a function. If $f \in K_s^4$, then $f\circ g \in K_s^4$.*

Proof. Let $\lambda \in [0,1]$. Thus, we get

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y))\\ &= f(\lambda g(x)+(1-\lambda)g(y))\\ &\leq \lambda^{\frac{1}{s}}f(g(x))+(1-\lambda)^{\frac{1}{s}}f(g(y))\\ &= \lambda^{\frac{1}{s}}(f\circ g)(x)+(1-\lambda)^{\frac{1}{s}}(f\circ g)(y)\end{aligned}$$

for all $x,y \in U$. Hence, $f\circ g \in K_s^4$. □

Theorem 3.15. *Let $g : U \to \mathbb{R}_-$ and $f : \mathbb{R}_- \to \mathbb{R}$ be an increasing function. If $f \in K_s^1$ and $g \in K_s^4$, then $f\circ g : U \to \mathbb{R}$ is a convex function.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y))\\ &\leq f(\lambda^{\frac{1}{s}}g(x)+(1-\lambda)^{\frac{1}{s}}g(y))\\ &\leq \lambda f(g(x))+(1-\lambda)f(g(y)).\end{aligned}$$

Hence, $f\circ g \in K_s^4$. □

Theorem 3.16. *Let $g : U \to \mathbb{R}_-$ and $f : \mathbb{R}_- \to \mathbb{R}$ be an increasing f is s-convex function (i.e. p-convex function) and $g \in K_s^4$, then $f\circ g \in K_s^4$.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y)) \\ &\leq f(\lambda^{\frac{1}{s}}g(x)+(1-\lambda)^{\frac{1}{s}}g(y)) \\ &\leq \lambda^{\frac{1}{s}}f(g(x))+(1-\lambda)^{\frac{1}{s}}f(g(y)) \\ &= \lambda^{\frac{1}{s}}(f\circ g)(x)+(1-\lambda)^{\frac{1}{s}}(f\circ g)(y).\end{aligned}$$

Hence, $f\circ g \in K_s^4$. □

Theorem 3.17. *Let $g: U \to \mathbb{R}_+$ and $f: \mathbb{R}_+ \to \mathbb{R}$ be a decreasing function. If $f \in K_s^1$ and g is a s-concave function in the fourth sense, then $f\circ g: U \to \mathbb{R}_+$ is a convex function.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y)) \\ &\leq f(\lambda^{\frac{1}{s}}g(x)+(1-\lambda)^{\frac{1}{s}}g(y)) \\ &\leq \lambda f(g(x))+(1-\lambda)f(g(y)) \\ &= \lambda(f\circ g)(x)+(1-\lambda)(f\circ g)(y).\end{aligned}$$

□

Theorem 3.18. *If $g: U \to \mathbb{R}_+$ is a s-concave function in the fourth sense and $f: g(U) \to \mathbb{R}$ is a decreasing s-convex function (i.e. p-convex function), then $f\circ g \in K_s^4$.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y)) \\ &\leq f(\lambda^{\frac{1}{s}}g(x)+(1-\lambda)^{\frac{1}{s}}g(y)) \\ &\leq \lambda^{\frac{1}{s}}f(g(x))+(1-\lambda)^{\frac{1}{s}}f(g(y)) \\ &= \lambda^{\frac{1}{s}}(f\circ g)(x)+(1-\lambda)^{\frac{1}{s}}(f\circ g)(y).\end{aligned}$$

□

The following theorem can be considered as a generalization of Theorem 3.18.

Theorem 3.19. *Let $s_2 \leq s_1$. If $g: U \to \mathbb{R}_+$ is a s_1-concave function in the fourth sense and $f: g(U) \to \mathbb{R}$ is a decreasing s_2-convex function (i.e. p-convex function), then $f\circ g \in K_{s_2}^4$.*

Proof. Let $x,y \in U$ and $\lambda \in [0,1]$.

$$\begin{aligned}(f\circ g)(\lambda x+(1-\lambda)y) &= f(g(\lambda x+(1-\lambda)y)) \\ &\leq f(\lambda^{\frac{1}{s_1}}g(x)+(1-\lambda)^{\frac{1}{s_1}}g(y)) \\ &\leq f(\lambda^{\frac{1}{s_2}}g(x)+(1-\lambda)^{\frac{1}{s_2}}g(y)) \\ &\leq \lambda^{\frac{1}{s_2}}f(g(x))+(1-\lambda)^{\frac{1}{s_2}}f(g(y)) \\ &= \lambda^{\frac{1}{s_2}}(f\circ g)(x)+(1-\lambda)^{\frac{1}{s_2}}(f\circ g)(y).\end{aligned}$$

□

4. Conclusion

Convex functions play an important role in many areas as optimization, control theory, game theory, probability, statistics, biological system, economy, medicine, art, linear programming and convex programming. Therefore, convexity has a huge impact on our daily lives with its myriad applications and it is one of the areas of great interest to mathematicians. s-Convex functions in the fourth sense are introduced in this paper, which is the continuation of the studies in which s-convex functions in the first, second and third sense are given. Some characterizations, algebraic and functional properties of these functions are presented. The conditions under which this type of s-convexity is preserved are given. Some relations to other kinds of s-convexity are investigated. Also, some relations between these functions and the other types of s-convex functions are given. It is thought that this study, in which a new type of convexity is defined, will contribute to the literature in the field of convexity.

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