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On Hyperbolic Jacobsthal-Lucas Sequence

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Abstract

In this study, we define the hyperbolic Jacobsthal-Lucas numbers and we obtain recurrence relations, Binet's formula, generating function and the summation formulas for these numbers.

1. Introduction and preliminaries

In this study, we introduce hyperbolic Jacobsthal-Lucas numbers and give some properties of them. Firstly, we present some background information about hyperbolic numbers and Jacobsthal-Lucas numbers. One can see [1]-[8] for details. Jacobsthal-Lucas sequence J_n is defined by the second-order recurence relation

$$J_{n+2} = J_{n+1} + 2J_n$$

with initial values $J_0 = 2, J_1 = 1$. The first few terms of this sequence are given as follows:

$$2, 1, 5, 7, 17, 31, 65, 127, 257, 511, 1025, 2047, \dots$$

Binet's formula and generating function of Jacobsthal-Lucas sequence are given by

$$J_n = 2^n + (-1)^n$$

and

$$\sum_{n=0}^{\infty} J_n x^n = \frac{2 - x^2}{1 - x - 2x^2}$$

respectively.
The set of hyperbolic numbers H can be described as

$$H = \{z = x + hy : h \notin R, h^2 = 1, x, y \in R\}.$$

Addition, substruction and multiplication of any two hyperbolic numbers z_1 and z_2 are defined by

$$z_1 \pm z_2 = (x_1 + hy_1) \pm (x_2 + hy_2) = (x_1 \pm x_2) + h(y_1 \pm y_2),$$
$$z_1 \times z_2 = (x_1 + hy_1) \times (x_2 + hy_2) = x_1x_2 + y_1y_2 + h(x_1y_2 + y_1x_2),$$

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and the division of two hyperbolic numbers are given by

$$\frac{z_1}{z_2}=\frac{x_1+hy_1}{x_2+hy_2}=\frac{(x_1+hy_1)(x_2-hy_2)}{(x_2+hy_2)(x_2-hy_2)}=\frac{x_1x_2+y_1y_2}{x_2^2-y_2^2}+h\frac{x_1y_2+y_1x_2}{x_2^2-y_2^2}.$$

The hyperbolic conjugation of $z=x+hy$ is defined by

$$\bar{z}=x-hy.$$

For more information and properties related hyperbolic numbers, see [9]-[18].

2. Hyperbolic Jacobsthal-Lucas sequence

In [14], author presented hyperbolic Fibonacci sequence and examined its properties. In this study, we define hyperbolic Jacobsthal-Lucas sequence and examined some of its properties.
The hyperbolic Jacobsthal-Lucas numbers are defined by

$$HJ_n=J_n+hJ_{n+1}$$

with initial conditions $HJ_0=2+h$, $HJ_1=1+5h$ where $h^2=1$. Then the first few terms of hyperbolic Jacobsthal-Lucas numbers are

$$2+h, 1+5h, 5+7h, 7+17h, 17+31h, 31+65h, 65+127h, ...,$$

It can be easily shown that

$$HJ_n=HJ_{n-1}+2HJ_{n-2}.$$

In fact, by using the definition of the hyperbolic Jacobsthal-Lucas numbers, we have

$$\begin{aligned}HJ_n&=J_n+hJ_{n+1}=J_{n-1}+2J_{n-2}+h(J_n+2J_{n-1})\\&=2J_{n-2}+h2J_{n-1}+J_{n-1}+hJ_n\\&=HJ_{n-1}+2HJ_{n-2}.\end{aligned}$$

Theorem 2.1. *Let HJ_n be $n-th$ hyperbolic Jacobsthal-Lucas number, then we obtain*

$$\lim_{x\to\infty}\frac{HJ_{n+1}}{HJ_n}=2.$$

Proof. We have

$$\lim_{x\to\infty}\frac{J_{n+1}}{J_n}=2.$$

for the Jacobsthal-Lucas sequence J_n. Then using this value for the hyperbolic Jacobsthal-Lucas HJ_n, we get

$$\begin{aligned}\lim_{x\to\infty}\tfrac{HJ_{n+1}}{HJ_n}&=\lim_{x\to\infty}\tfrac{J_{n+1}+hJ_{n+2}}{J_n+hJ_{n+1}}\\&=\lim_{x\to\infty}\tfrac{J_{n+1}+h(J_{n+1}+2HJ_n)}{J_n+hJ_{n+1}}\\&=\lim_{x\to\infty}\tfrac{(\frac{J_{n+1}}{J_n})+h((\frac{J_{n+1}}{J_n})+2)}{1+(h\frac{J_{n+1}}{J_n})}\\&=\tfrac{2+4h}{1+2h}=2.\end{aligned}$$

□

Theorem 2.2. *The Binet formula for the hyperbolic Jacobsthal-Lucas numbers is given by*

$$HJ_n=(1+2h)2^n+(1-h)(-1)^n. \tag{2.1}$$

Proof. By using the Binet formula of the Jacobsthal-Lucas numbers

$$J_n=2^n+(-1)^n,$$

we get

$$\begin{aligned}HJ_n&=J_n+hJ_{n+1}\\&=2^n+(-1)^n+h(2^{n+1}+(-1)^{n+1})\\&=(1+2h)2^n+(1-h)(-1)^n.\end{aligned}$$

□

Theorem 2.3. *The generating function for the hyperbolic Jacobsthal-Lucas sequence is given by*

$$\sum_{n=0}^{\infty} HJ_n x^n = \frac{2+h+(1-4h)x}{1-x-2x^2}.$$

Proof. Let

$$g(x) = \sum_{n=0}^{\infty} HJ_n x^n$$

be generating function of hyperbolic Jacobsthal-Lucas numbers. Then we have the following equations:

$$\begin{aligned} g(x) &= HJ_0 + HJ_1 x + HJ_2 x^2 + HJ_3 x^3 + HJ_4 x^4 + \dots \\ -xg(x) &= -HJ_0 x - HJ_1 x^2 - HJ_2 x^3 - HJ_3 x^4 - HJ_4 x^5 - \dots \\ -2x^2 g(x) &= -2HJ_0 x^2 - 2HJ_1 x^3 - 2HJ_2 x^4 - 2HJ_3 x^5 - 2HJ_4 x^6 - \dots \\ (1-x-2x^2)g(x) &= HJ_0 + (HJ_1 - HJ_0)x. \end{aligned}$$

By rewriting the last equation, we get

$$g(x) = \frac{2+4h+(1-4h)x}{1-x-2x^2}$$

with $HJ_0 = 2+h$, $HJ_1 = 1+5h$. □

Theorem 2.4. *(Catalan's identity) The following identitiy holds for all natural numbers n and m:*

$$HJ_{n+m}HJ_{n-m} - HJ_n^2 = (-1+h)[(-2)^{n+m} + (-2)^{n-m} + (-2)^{n+1}].$$

Proof. By using the formula (2.1), we obtain

$$\begin{aligned} HJ_{n+m}HJ_{n-m} - HJ_n^2 &= \left((1+2h)\,2^{n+m} + (1-h)\,(-1)^{n+m}\right)\left((1+2h)\,2^{n-m} + (1-h)\,(-1)^{n-m}\right) \\ &\quad -((1+2h)2^n + (1-h)(-1)^n)^2 \\ &= ((5+4h)2^{2n} + (2-2h)(-1)^{2n} + (-1+h)2^n(-1)^n[2^m(-1)^{-m} + 2^{-m}(-1)^m]) \\ &\quad -((5+4h)2^{2n} + (2-2h)(-1)^{2n} + 2(-1+h)2^n(-1)^n) \\ &= (-1+h)[(-2)^{n+m} + (-2)^{n-m} + (-2)^{n+1}]. \end{aligned}$$

□

Theorem 2.5. *(d'Ocagne's identity) The following identitiy holds for any integers n and m:*

$$HJ_{m+1}HJ_n - HJ_m HJ_{n+1} = 3(-1+h)[(-2)^m(-1)^n - (-2)^n(-1)^m].$$

Proof. By the Binet formula (2.1), we get

$$\begin{aligned} HJ_{m+1}HJ_n - HJ_m HJ_{n+1} &= ((1+2h)2^{m+1} + (1-h)(-1)^{m+1})((1+2h)2^n + (1-h)(-1)^n) \\ &\quad -((1+2h)2^m + (1-h)(-1)^m)((1+2h)2^{n+1} + (1-h)(-1)^{n+1}) \\ &= 3(-1+h)[(-2)^m(-1)^n - (-2)^n(-1)^m]. \end{aligned}$$

□

Theorem 2.6. *(Gelin-Cesaro's identity) The following identitiy holds for any integers n and m:*

$$HJ_{n+2}HJ_{n+1}HJ_{n-1}HJ_{n-2} - HJ_n^4 = \frac{9}{8}(-1+h)(-2)^n[(2)^{2n+1} - 13(1-h)(-2)^n + 4(1-h)].$$

Proof. Using

$$\begin{aligned} HJ_n &= (1+2h)2^n + (1-h)(-1)^n, \\ HJ_n &= (1+2h)[2^n + (-1+h)(-1)^n], \end{aligned}$$

and by setting $a = 2^n, b = (-1+h)(-1)^n$ we obtain following values:

1.$HJ_{n+2} = (1+2h)[4a+b]$

2.$HJ_{n+1} = (1+2h)[2a-b]$

3.$HJ_{n-1} = (1+2h)[\frac{a}{2} - b]$

4.$HJ_{n-2} = (1+2h)[\frac{a}{4}+b]$

from the above values, we can easily calculate

$$\begin{aligned} HJ_{n+2}HJ_{n+1}HJ_{n--1}HJ_{n-2} - HJ_n^4 &= (1+2h)^4[(8a^2-2ab-b^2)(\tfrac{a^2}{8}+\tfrac{ab}{4}-b^2)-(a^4+b^4+4a^3b+6a^2b^2+4ab^3)] \\ &= \tfrac{9}{8}(-1+h)(-2)^n[(2)^{2n+1}-13(1-h)(-2)^n+4(1-h)]. \end{aligned}$$

□

Theorem 2.7. *(Melham's identity) The following identity holds for any integers n and m:*

$$HJ_{n+1}HJ_{n+2}HJ_{n+6} - HJ_{n+3}^3 = 9(1-h)(-2)^n[2^{n+3}+10(1-h)(-1)^n].$$

Proof. Using

$$\begin{aligned} HJ_n &= (1+2h)2^n + (1-h)(-1)^n, \\ HJ_n &= (1+2h)[2^n + (-1+h)(-1)^n], \end{aligned}$$

and by setting $a = 2^n, b = (-1+h)(-1)^n$ we obtain following values:

1.$HJ_{n+1} = (1+2h)[2a-b]$,

2.$HJ_{n+2} = (1+2h)[4a+b]$,

3.$HJ_{n+6} = (1+2h)[64a+b]$,

4.$HJ_{n+3} = (1+2h)[8a-b]$.

From the above values, we can easily calculate

$$\begin{aligned} HJ_{n+1}HJ_{n+2}HJ_{n+6} - HJ_{n+3}^3 &= (1+2h)^3[(8a^2-2ab-b^2)(64a+b)-(8a-b)^3] \\ &= (1+2h)^3 9ab[8a-10b] \\ &= 9(1-h)(-2)^n[2^{n+3}+10(1-h)(-1)^n]. \end{aligned}$$

□

Theorem 2.8. *For $n \geq 0$, we obtain*

$$\sum_{k=0}^{n} HJ_k = \frac{1}{2}(HJ_{n+2} - (1+5h)).$$

Proof. We use the mathematical induction on n. For $n = 0$, we have

$$HJ_0 = \frac{1}{2}[HJ_2 - (1+5h)] = \frac{1}{2}[5+7h-1-5h] = 2+h.$$

Now assume that it is true for $n = k$, namelyand by setting

$$\sum_{k=0}^{k} HJ_k = \frac{1}{2}(HJ_{k+2} - (1+5h)).$$

From the induction hypothesis, we obtain

$$\begin{aligned} \sum_{k=0}^{k+1} HJ_k &= \frac{1}{2}(HJ_{k+2} - (1+5h)) + HJ_{k+1} \\ &= \frac{1}{2}(HJ_{k+2} - (1+5h) + 2HJ_{k+1}) \\ &= \frac{1}{2}(HJ_{k+3} - (1+5h)). \end{aligned}$$

□

3. Conclusion

The hyperbolic Jacobsthal-Lucas numbers with initial conditions $HJ_0 = 2+h, HJ_1 = +5h$ are defined by

$$HJ_n = J_n + hJ_{n+1}$$

where $h^2 = 1$.
In this paper, we give the hyperbolic Jacobsthal Lucas numbers and present some recurrence relations, Binet's formula, generating function and some special idetities for these numbers.

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Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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