HSJG
Hagia Sophia Journal of Geometry
e-ISSN 2687-4261
Vol. 3, No. 2, 24-30, 2021
Research Article

# On $f-$biharmonic Curves in the Three-dimensional Lorentzian Sasakian Manifolds

Murat ALTUNBAŞ

**Abstract**
The necessary and sufficient conditions for a proper $f-$biharmonic curve in the three-dimensional Lorentzian Sasakian manifolds are obtained. Moreover, we give some results for $f-$biharmonic Legendre curves.

**Keywords and 2010 Mathematics Subject Classification**
Keywords: Lorentzian metric —$f-$biharmonic curves—Lorentzian Sasakian manifolds
MSC: 53B25, 53C25

*Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, Erzincan, 24100, Turkey*
✉maltunbas@erzincan.edu.tr
**Corresponding author**: Murat ALTUNBAŞ
**Article History:** Received 30 June 2021; Accepted 17 December 2021

## 1. Introduction

Harmonic maps between (pseudo-) Riemannian manifolds have been studied extensively since the eminent work of Eells and Sampson [1]. In addition to this, biharmonic maps, which are generalizations of harmonic maps, constitute one of the dynamical topic of differential geometry (for a survey of biharmonic maps see [2]). Non-harmonic biharmonic maps are said to be proper biharmonic maps. Chen and Ishikawa showed that there does not exist proper biharmonic curves in Euclidean 3-space [3]. Moreover, they investigated proper biharmonic curves in Minkowski 3-space (see [4]). Caddeo, Montaldo and Piu studied biharmonic curves on a surface [5]. Caddeo, Oniciuc and Piu demonstrated that all non-geodesic biharmonic curves are helices in three-dimensional Heisenberg space [6]. Ou and Wang characterized non-geodesic biharmonic curves in Sol space and proved that there exists no non-geodesic biharmonic helix in Sol space [7]. Caddeo, Montaldo, Oniciuc and Piu found explicit formulae for biharmonic curves in Cartan-Vranceanu three-dimensional spaces [8].

In [9], Lu gave a generalization of biharmonic maps and introduced $f-$biharmonic maps. He derived the first variation formula and calculated the $f-$biharmonic map equation. Ou considered $f-$biharmonic curves on a generic manifold and gave a characterization for them in $n-$dimensional space forms [10]. Guvenc and Ozgur studied $f-$biharmonic Legendre curves in Sasakian space forms [11]. Karaca and Ozgur investigated $f-$biharmonic curves in Sol spaces, Cartan Vranceanu three-dimensional spaces and homogenous contact three-manifolds [12]. Du and Zhang examined $f-$biharmonic curves in Lorentz–Minkowski space [13].

In this paper, we investigate the curves of the three-dimensional Lorentzian Sasakian manifolds in order to specify $f-$bihamonicity properties of them. We consider the Lorentzian Bianchi-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian manifolds. Throughout the paper, all geometric objects (curves, manifolds, vector fields, functions etc.) are assumed to be smooth.

# 2. Preliminaries

## 2.1 Contact Lorentzian manifolds

A $(2n+1)-$dimensional differentiable manifold $M$ is said to be an almost contact manifold if it admits a global form $\eta$ such that $\eta \wedge (d\eta)^n \neq 0$ everywhere. When a contact form $\eta$ is given, we have a unique vector field $\xi$ satisfying

$$\eta(\xi) = 1 \text{ and } d\eta(\xi, X) = 0,$$

where $X$ is a vector field on $M$. The vector field $\xi$ is called characteristic vector field. It is known that there exists a Lorentzian metric $g$ and a $(1,1)-$tensor field such that

$$\eta(X) = -g(X,\xi),\ d\eta(X,Y) = g(X,\phi Y),\ \phi^2(X) = -X + \eta(X)\xi, \tag{1}$$

where $X$ and $Y$ are vector fields on $M$. From (1), it follows that

$$\phi\xi = 0,\ \eta \circ \phi = 0,\ g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y).$$

A Lorentzian manifold $M$ equipped with the tensors $(g, \phi, \xi, \eta)$ satisfying (1) is called Lorentzian contact metric manifold.

A Lorentzian contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)Y = g(X,Y)\xi + \eta(Y)X,$$

for any vector fields $X, Y$ on $M$, where $\nabla$ is Levi-Civita connection of $g$ [14].

**Definition 1.** *A 1-dimensional integral submanifold of a contact manifold is called a Legendre curve [17].*

## 2.2 Frenet-Serret equations

Let $\gamma : I \to M$ be a unit speed curve in a three-dimensional Lorentzian manifold $M$ such that $\gamma'$ satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$. The constant $\varepsilon_1$ is said to be the causal character of $\gamma$. A unit speed curve is called spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve is called a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. A Frenet curve has an orthonormal frame field $\{T = \gamma', N, B\}$ along $\gamma$. Then the Frenet-Serret equations are given by

$$\begin{aligned}
\nabla_T T &= \varepsilon_2 \kappa N, \\
\nabla_T N &= -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \\
\nabla_T B &= \varepsilon_2 \tau N,
\end{aligned}$$

where $\kappa = \left\| \nabla_{\gamma'} \gamma' \right\|$ is the geodesic curvature and $\tau$ is the geodesic torsion of $\gamma$. The vector fields $T, N$ and $B$ are called tangent vector field, principal normal vector field and binormal vector field of $\gamma$, respectively.

The constants $\varepsilon_2$ and $\varepsilon_3$ are defined by $g(N,N) = \varepsilon_2$ and $g(B,B) = \varepsilon_3$, and called second causal character and third causal character of $\gamma$, respectively. The equation $\varepsilon_1 \varepsilon_2 = -\varepsilon_3$ holds.

A Frenet curve $\gamma$ is a geodesic if and only if $\kappa = 0$.

**Proposition 2.** *Let $\{T, N, B\}$ are orthonormal frame field in a Lorentzian 3-manifold. Then, [17],*

$$T \wedge_L N = \varepsilon_3 B,\ N \wedge_L B = \varepsilon_1 T,\ B \wedge_L T = \varepsilon_2 N.$$

**Proposition 3.** *The torsion of a Legendre curve is 1 in three-dimensional Sasakian Lorentzian manifolds [15].*

## 2.3 $f-$Biharmonic maps

A map $\varphi : (M_m, g) \to (N_n, h)$ between two pseudo-Riemannian manifolds is called harmonic if it is a critical point of the energy

$$E(\varphi) = \frac{1}{2} \int_{\Omega} \|d\varphi\|^2 dv_g,$$

where $\Omega$ is a compact domain of $M_m$. The tension field $\tau(\varphi)$ of $\varphi$ is defined by

$$\tau(\varphi) = tr(\nabla^{\varphi} d\varphi) = \sum_{i=1}^{m} \varepsilon_i (\nabla^{\varphi}_{e_i} d\varphi(e_i) - d\varphi(\nabla_{e_i} e_i)),$$

where $\nabla^\varphi$ and $\{e_i\}$ denote the induced connection by $\varphi$ on the bundle $\varphi^* TN_n$. A map $\varphi$ is called harmonic if its tension field vanishes. The bienergy $E_2(\varphi)$ of a map $\varphi$ is defined by

$$E_2(\varphi) = \frac{1}{2}\int_\Omega \|\tau(\varphi)\|^2 dv_g,$$

and $\varphi$ is called biharmonic if it is a critical point of the bienergy, where $\Omega$ is a compact domain of $M_m$. Clearly, all harmonic maps are biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps. The bitension field $\tau_2(\varphi)$ of $\varphi$ is defined by

$$\tau_2(\varphi) = \sum_{i=1}^{m} \varepsilon_i((\nabla^\varphi_{e_i}\nabla^\varphi_{e_i} - \nabla^\varphi_{\nabla_{e_i}e_i})\tau(\varphi) - R^N(\tau(\varphi), d\varphi(e_i))d\varphi(e_i)), \tag{2}$$

where $R^N$ denotes the curvature tensor of $N_n$. A map $\varphi$ is called biharmonic if its bitension field vanishes.

A map $\varphi$ is called $f-$harmonic with a function $f : M \to \mathbb{R}$, if it is a critical point of the energy

$$E_f(\varphi) = \frac{1}{2}\int_\Omega f\|d\varphi\|^2 dv_g,$$

where $\Omega$ is a compact domain of $M_m$. The $f-$tension field $\tau_f(\varphi)$ of $\varphi$ is given by

$$\tau_f(\varphi) = f\tau(\varphi) + d\varphi(grad f) \tag{3}$$

see [16]. The $f-$bitension field $\tau_{2,f}(\varphi)$ of $\varphi$ is defined by

$$\tau_{2,f}(\varphi) = f\tau_2(\varphi) + \Delta f\tau(\varphi) + 2\nabla^\varphi_{grad f}\tau(\varphi). \tag{4}$$

A map $\varphi$ is called $f-$biharmonic if its $f-$bitension field vanishes ( [9, 13]). Non-harmonic and non-biharmonic $f-$biharmonic curves are called proper $f-$biharmonic curves and if $f$ is a constant, then an $f-$biharmonic curve turns to be a biharmonic curve [9].

## 3. $f-$Biharmonic curves in Lorentzian Sasakian manifolds

We recall fundamental concepts about the Lorentzian Bianchi-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian manifolds from [17]. Let us consider the set

$$D = \{(x,y,z) \in \mathbb{R}^3 : 1 + \frac{c}{2}(x^2+y^2) > 0\},$$

where $c$ is a real number. On the region $D$, the contact form $\eta$ is taken

$$\eta = dz + \frac{ydx - xdy}{1+\frac{c}{2}(x^2+y^2)}.$$

Then, the characteristic vector field of $\eta$ is $\xi = \frac{\partial}{\partial z}$.

Next, the Lorentzian metric is equipped as

$$g_c = \frac{dx^2+dy^2}{\{1+\frac{c}{2}(x^2+y^2)\}^2} - (dz + \frac{ydx - xdy}{1+\frac{c}{2}(x^2+y^2)})^2.$$

The Lorentzian orthonormal frame field $(e_1, e_2, e_3)$ on $(D, g_c)$ is given by

$$e_1 = \{1+\frac{c}{2}(x^2+y^2)\}\frac{\partial}{\partial x} - y\frac{\partial}{\partial z},\ e_2 = \{1+\frac{c}{2}(x^2+y^2)\}\frac{\partial}{\partial y} + x\frac{\partial}{\partial z},\ e_3 = \frac{\partial}{\partial z}.$$

Then the endomorphism field $\phi$ is given by

$$\phi(e_1) = e_2,\ \phi(e_2) = -e_1,\ \phi(e_3) = 0.$$

The Levi-Civita connection $\nabla$ of $(D, g_c)$ is described as

$$\begin{gathered}\nabla_{e_1} e_1 = cye_2,\ \nabla_{e_1} e_2 = -cye_1 + e_3,\ \nabla_{e_1} e_3 = e_2,\\ \nabla_{e_2} e_1 = -cxe_2 - e_3,\ \nabla_{e_2} e_2 = cxe_1,\ \nabla_{e_2} e_3 = -e_1,\\ \nabla_{e_3} e_1 = e_2, \nabla_{e_3} e_2 = -e_1, \nabla_{e_3} e_3 = 0.\end{gathered}$$

The contact form $\eta$ on $D$ fulfills

$$d\eta(X,Y) = g_c(X, \phi Y),\ X,Y \in \chi(D).$$

Moreover the structure $(g_c, \phi, \xi, \eta)$ is Sasakian. The non-vanishing components of the curvature tensor $R$ of $(D, g_c)$ are given by

$$\begin{aligned} R(e_1,e_2)e_1 &= -(2c+3)e_1,\ R(e_1,e_3)e_3 = -e_1,\\ R(e_2,e_1)e_1 &= -(2c+3)e_2, R(e_2,e_3)e_3 = -e_2,\\ R(e_3,e_1)e_1 &= e_3,\ R(e_3,e_2)e_2 = e_3.\end{aligned}$$

For the sectional curvature $K$ of $(D, g_c)$, we have

$$K(\xi, e_i) = -R(\xi, e_i, \xi, e_i) = -1,\ i = 1,2,$$

and

$$K(e_1,e_2) = R(e_1,e_2,e_1,e_2) = 2c+3.$$

So, $(D, g_c)$ is of constant holomorphic sectional curvature $H = 2c+3$.

For the case $H = -1$ (i.e. $c = -2$), the Lorentzian Sasakian manifold $D$ turns to be anti-de Sitter 3-space.

Now, suppose that $\gamma : I \to (D, g_c)$ is a curve parametrized by arc-length and $\{T, N, B\}$ is an orthonormal frame field tangent to $D$ along $\gamma$, where $T = T_1e_1 + T_2e_2 + T_3e_3$, $N = N_1e_1 + N_2e_2 + N_3e_3$ and $B = B_1e_1 + B_2e_2 + B_3e_3$.

The $f-$biharmonicity condition for curves on $(D, g_c)$ is obtained in the following theorem.

**Theorem 4.** *Let $\gamma : I \to (D, g_c)$ be a curve parametrized by arc-length. Then $\gamma$ is $f-$biharmonic if and only if the following relations are satisfied:*

$$\begin{gathered} 3\kappa\kappa' f + 2\kappa^2 f' = 0,\\ \kappa f'' + 2\kappa' f' + f[\kappa'' + \varepsilon_3\kappa^3 + \varepsilon_1\kappa\tau^2 + \kappa\varepsilon_2(\varepsilon_3 + 2(c+2)\eta(B)^2)] = 0,\\ -2\kappa\tau f' - f(2\kappa'\tau + \kappa\tau') + 2\varepsilon_1(c+2)\kappa f\eta(N)\eta(B) = 0. \end{gathered} \tag{5}$$

*Proof.* Let $\gamma = \gamma(s)$ be a curve parametrized by arc-length. We use formula (4). From [17], we have

$$\tau(\gamma) = \varepsilon_1\nabla_T T = -\varepsilon_3\kappa N, \tag{6}$$

$$\begin{aligned} R(T,N,T,N) = \varepsilon_3 + 2(c+2)B_3^2,\\ R(T,N,T,B) = 2\varepsilon_1(c+2)N_3B_3, \end{aligned} \tag{7}$$

$$\tau_2(\gamma) = 3\varepsilon_3\kappa\kappa' T + \varepsilon_2(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N + \varepsilon_1(2\kappa'\tau + \kappa\tau')B + \varepsilon_2\kappa R(T,N)T. \tag{8}$$

Moreover, from [13], we have

$$\begin{aligned} \nabla^{\gamma}_{grad f}\tau(\gamma) = f'\nabla_T(\nabla_T T) = \varepsilon_2 f'[\kappa' N + \kappa(-\varepsilon_1\kappa T - \varepsilon_3\tau B)],\\ \Delta f\tau(\gamma) = f''\nabla_T T = f''\varepsilon_2\kappa N. \end{aligned} \tag{9}$$

Therefore, combining equations (6), (8) and (9), we obtain

$$\begin{aligned} \tau_{2,f}(\gamma) &= 3\varepsilon_3\kappa\kappa' fT + \varepsilon_2 f(\kappa'' - \varepsilon_2\kappa(\varepsilon_1\kappa^2 + \varepsilon_3\tau^2))N + \varepsilon_1 f(2\kappa'\tau + \kappa\tau')B\\ &\quad + \varepsilon_2 f\kappa R(T,N)T + \varepsilon_2\kappa f'' N + 2\varepsilon_2 f'[\kappa' N + \kappa(-\varepsilon_1\kappa T - \varepsilon_3\tau B)]. \end{aligned} \tag{10}$$

If we take inner product of equation (10) with $T, N$ and $B$, respectively and use the equations (7), we get (5). ■

**Corollary 5.** *Let $\gamma : I \to (D, g_c)$ be a Legendre curve parametrized by arc-length. Then $\gamma$ is $f-$biharmonic if and only if the following relations are satisfied:*

$$\begin{gathered}3\kappa\kappa' f + 2\kappa^2 f' = 0,\\ \kappa f'' + 2\kappa' f' + f[\kappa'' + \varepsilon_3\kappa^3 + \varepsilon_1\kappa + \kappa\varepsilon_2(\varepsilon_3 + 2(c+2)\eta(B)^2)] = 0,\\ -\kappa f' + f(-\kappa' + \varepsilon_1(c+2)\kappa\eta(N)\eta(B)) = 0.\end{gathered}$$

Now, we express the following results for $c \neq -2$.

**Proposition 6.** *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If $\kappa$ is a non-zero constant, then $\gamma$ is biharmonic.*

*Proof.* Under the assumption $\kappa$ is a non-zero constant, from the first equation in (5), obviously we get $f' = 0$. So, $\gamma$ is a biharmonic curve. ■

**Proposition 7.** *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If $\tau$ is a non-zero constant and $\eta(N)\eta(B) = 0$ (i.e., $N_3 B_3 = 0$), then $\gamma$ is biharmonic.*

*Proof.* Under the assumption $\tau$ is a non-zero constant and $\eta(N)\eta(B) = 0$, using the first and third equations in (5), we get

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{11}$$

and

$$\tau\left(\frac{\kappa'}{\kappa} + \frac{f'}{f}\right) = 0. \tag{12}$$

Putting equation (11) in (12) shows that $f$ is constant, therefore $\gamma$ is a biharmonic curve. ■

**Corollary 8.** *If $\gamma : I \to (D, g_c)$ is an $f-$biharmonic Legendre curve parametrized by arc-length and $\eta(N)\eta(B) = 0$, then $\gamma$ is biharmonic.*

**Proposition 9.** *Let $\gamma : I \to (D, g_c)$ be an $f-$biharmonic curve parametrized by arc-length. If $\tau$ is a non-zero constant, then $f = e^{\int \frac{3\varepsilon_1(c+2)\eta(N)\eta(B)}{\tau}}$.*

*Proof.* Under the assumption $\tau$ is a non-zero constant, if we use the first and third equations in (5), we obtain

$$\frac{\kappa'}{\kappa} = -\frac{2f'}{3f} \tag{13}$$

and

$$-2\kappa\tau f' - 2f\kappa'\tau + 2\varepsilon_1(c+2)\kappa f\eta(N)\eta(B) = 0. \tag{14}$$

Setting equation (13) in (14), we get the result. ■

**Corollary 10.** *If $\gamma : I \to (D, g_c)$ is an $f-$biharmonic Legendre curve parametrized by arc-length, then $f = e^{\int 3\varepsilon_1(c+2)\eta(N)\eta(B)}$.*

**Proposition 11.** *Let $\gamma : I \to (D, g_c)$ be a non-geodesic curve parametrized by arc-length and suppose that $\tau = 0$. In this case, $\gamma$ is $f-$biharmonic if and only if the following equations are valid:*

$$f^2\kappa^3 = c_1^2, \tag{15}$$

$$(f\kappa)'' = -f\kappa(\varepsilon_3\kappa^2 + \varepsilon_2(\varepsilon_3 + 2(c+2)\eta(B)^2)), \tag{16}$$

$$\eta(N)\eta(B) = 0, \tag{17}$$

*where $c_1 \in \mathbb{R}$.*

*Proof.* Under the assumption $\tau = 0$, if we use equations in (5) by integrating first equation, we deduce the results. ■

**Proposition 12.** *Let $\gamma : I \to (D, g_c)$ be a non-geodesic curve parametrized by arc-length and suppose that $\tau$ and $\kappa$ are non-constants. In this case, $\gamma$ is $f-$biharmonic if and only if the following equations are valid:*

$$f^2\kappa^3 = c_1^2, \tag{18}$$

$$(f\kappa)'' = -f\kappa(\varepsilon_3\kappa^2 + \varepsilon_1\tau^2 + \varepsilon_2(\varepsilon_3 + 2(c+2)\eta(B)^2), \tag{19}$$

$$\kappa^2 f^2 \tau = e^{\int \frac{2\varepsilon_1(c+2)\eta(N)\eta(B)}{\tau}}, \tag{20}$$

*where $c_1 \in \mathbb{R}$.*

*Proof.* Under the assumption $\tau$ and $\kappa$ are non-constants, if we use equations in (5) by integrating first and third equations, we obtain (18), (19) and (20). ■

From the last two propositions, we can give the following theorem.

**Theorem 13.** *An arc-length parametrized curve $\gamma : I \to (D, g_c)$ is proper $f-$biharmonic if and only if one of the following situations is true:*

*(i) $\tau = 0,\ f = c_1\kappa^{-3/2}$ and the curvature $\kappa$ solves the equation below:*

$$3(\kappa')^2 - 2\kappa\kappa'' = -4\kappa^2[\varepsilon_3\kappa^2 + \varepsilon_2(\varepsilon_3 + 2(c+2)\eta(B)^2)].$$

*(ii) $\tau \neq 0,\ \frac{\tau}{\kappa} = \frac{e^{\int \frac{2\varepsilon_1(c+2)\eta(N)\eta(B)}{\tau}}}{c_1^2},\ f = c_1\kappa^{-3/2}$ and the curvature $\kappa$ solves the equation below:*

$$3(\kappa')^2 - 2\kappa\kappa'' = -4\kappa^2[\varepsilon_3\kappa^2(1 - \varepsilon_2\frac{e^{\int \frac{4\varepsilon_1(c+2)\eta(N)\eta(B)}{\tau}}}{c_1^4}) + \varepsilon_2(\varepsilon_3 + 2(c+2)\eta(B)^2)].$$

*Proof.* (i) The first equation of (5) gives

$$f = c_1\kappa^{-3/2}. \tag{21}$$

By replacing the above equation into (16), we obtain the result.

(ii) From the first equation of (5), we have

$$f = c_1\kappa^{-3/2}. \tag{22}$$

Setting the above equation in (20), we get

$$\frac{\tau}{\kappa} = \frac{e^{\int \frac{2\varepsilon_1(c+2)\eta(N)\eta(B)}{\tau}}}{c_1^2}. \tag{23}$$

And finally putting equations (22) and (23) in (19), we obtain the result. ■

Consequently, we can express the following corollary.

**Corollary 14.** *An arc-length parametrized $f-$biharmonic curve $\gamma : I \to (D, g_c)$ with constant geodesic curvature is biharmonic.*

## 4. Conclusions

In this paper, we obtain some characterizations for $f-$biharmonic curves in Lorentzian Bianchi-Cartan-Vranceanu model of 3-dimensional Lorentzian Sasakian manifolds.

## 5. Acknowledgements

The author would like to thank the referee for his/her valuable suggestions and comments.

## References

[1] Eells, J. Jr., & Sampson J. H., (1964). *Harmonic mappings of Riemannian manifolds*. Amer. J. Math. 86(1), 109-160.

[2] Montaldo, S., & Oniciuc, C., (2006). *A short survey on biharmonic maps between Riemannian manifolds*. Rev. Un. Mat. Argentina, 47(2), 1-22.

[3] Chen, B.Y., & Ishikawa S., (1991). *Biharmonic surfaces in pseudo-Euclidean spaces*. Mem. Fac. Sci. Kyushu Univ. Ser. A, 45(2), 323-347.

[4] Inoguchi, J. I., (2003). *Biharmonic curves in Minkowski 3-space*. Int. J. Math. Math. Sci., 21, 1365-1368.

[5] Caddeo, R., Montaldo, S., & Piu, P. (2001). *Biharmonic curves on a surface*. Rend. Mat. Appl., 21, 143-157.

[6] Caddeo, R., Oniciuc, C., & Piu, P. (2004). *Explicit formulas for non-geodesic biharmonic curves of the Heisenberg group*. Rend. Sem. Mat. Univ. Politec. Torino, 62, 265-277.

[7] Ou, Y. L., & Wang, Z. P., (2006). *Biharmonic maps into Sol and Nil space*. arXiv preprint math/0612329.

[8] Caddeo, R., Montaldo, S., Oniciuc, C., & Piu, P. (2006). *The classification of biharmonic curves of Cartan-Vranceanu 3-dimensional spaces*. The 7th Int. Workshop in Dif. Geo. App., Cluj University Press, 121-131.

[9] Lu, W. J., (2015). *On $f-$Biharmonic maps and bi-$f-$harmonic maps between Riemannian manifolds*. Sci. China Math. 58(7), 1483-1498.

[10] Ou, Y. L. (2014). *On $f-$biharmonic maps and $f-$biharmonic submanifolds*. Pacific J. Math., 271(2), 461-477.

[11] Güvenç, S., & Özgür, C., (2017). *On the characterizations of $f-$biharmonic Legendre curves in Sasakian space forms*. Filomat, 31(3), 639-648.

[12] Karaca, F., & Özgür, C., (2018). *On $f-$biharmonic curves*. International Electronic Journal of Geometry, 11(2), 18-27.

[13] Du, L., & Zhang, J., (2020). *Classification of f-biharmonic curves in Lorentz–Minkowski space*. Journal of Mathematics.

[14] Calvoruso, G., (2011). *Contact Lorentzian manifolds*. Dif. Geom. Appl., 29, 41-51.

[15] Lee, J. E., (2019). *Slant curves and contact Magnetic curves in Sasakian Lorentzian 3-Manifolds*. Symmetry, 11(6), 784.

[16] Oukkas, S., Nasri, R., & Djaa, M., (2010). *On the $f-$harmonic and $f-$biharmonic maps*. JP J. Geom. Topol., 10(1) 11-27.

[17] Lee, J. E., (2020). *Biharmonic curves in three-dimensional Lorentzian Sasakian space forms*. Comm. Korean Math. Soc., 35(3), 967-977.