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Mathematics

Dual Darboux Frame of a Spacelike Ruled Surface and Darboux Approach to Mannheim Offsets of Spacelike Ruled Surfaces

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Abstract

In this paper, we define dual Darboux frame of a spacelike ruled surface. Then, we study Mannheim offsets of spacelike ruled surfaces in dual Lorentzian space by considering the E. Study Mapping. We represent spacelike ruled surfaces by dual Lorentzian unit spherical curves and define Mannheim offsets of the spacelike ruled surfaces by means of dual Darboux frame. We obtain relationships between the invariants of Mannheim spacelike offset surfaces and offset angle, offset distance. Moreover, we obtain some conditions for Mannheim offsets of spacelike ruled surfaces to be developable.

AMS: 53A25, 53C40, 53C50.

Keywords: Spacelike ruled surface; Mannheim offset; dual Darboux frame.

1. Introduction

In differential geometry, a surface is said to be “ruled” if through every point of the surface there is a straight line that lies on the surface. Then, a ruled surface can locally be described as the set of points swept by a moving straight line. For example, a cone is formed by keeping one point of a line fixed whilst moving another point along a circle. Because of they are one of the simplest objects in geometric modeling, these surfaces are largely used in surface theory and also used in many areas of science such as Computer Aided Geometric Design (CAGD), mathematical physics, moving geometry, kinematics for modeling the problems and model-based manufacturing of mechanical products. For example, a wood used as a building material is straight and can be considered as a straight line. So, the engineers can use ruled surfaces if they plan to construct a material with curvature [9].

An offset surface is a surface obtained by using a reference surface’s normal. Offsetting of curves and surfaces is one of the most important geometric operations in CAD/CAM due to its immediate applications in geometric modeling, NC machining, and robot navigation [5]. Especially, the offsets of the ruled surfaces have an important role in (CAGD) [15,16]. The well-known offset of the ruled surfaces is Bertrand offsets which were defined by Ravani and Ku by considering a generalization of the theory of Bertrand curve for trajectory ruled surfaces in line geometry [17]. Moreover, there exists a one-to-one correspondence between the lines of line space and the points of dual unit sphere. This famous correspondence is known as E. Study Mapping [2]. Hence, the geometry of ruled surfaces can be studied by considering dual curves lying on dual unit sphere. An example of this notion was given by Küçük and Gürsoy for the integral invariants of closed Bertrand trajectory ruled surfaces [7]. They have studied Bertrand offsets of closed ruled surfaces in dual space and introduced some relationships for these surface offsets.

Recently, a new definition of special curve couple was given by Wang and Liu [22]. They have called these curves as Mannheim partner curves. Then, Orbay and et al have given a generalization of the theory of Mannheim partner curves to the ruled surface and called Mannheim offset [9]. They have obtained the conditions for Mannheim offset surfaces to be developable. The corresponding characterizations of Mannheim offsets of ruled surfaces in Minkowski 3-space have been given in ref. [10,14]. Furthermore, in [11] Önder and Uğurlu have studied Mannheim offsets of ruled surfaces in dual space with Blaschke approach and obtained the relations between the integral invariants of closed ruled surfaces. Moreover, they have shown that the striction lines of developable Mannheim offset surfaces are Mannheim partner curves. They have also studied the Mannheim offsets of ruled surfaces in dual Lorentzian space by considering the Blaschke frame [12]. Also, they have given the dual Darboux frame of the timelike ruled surfaces with timelike rulings and studied the Mannheim offsets of these surfaces [13].

In this paper, we define the dual Darboux frame of a spacelike ruled surface and give the real and dual curvatures of this surface. Then, we introduce the Mannheim offsets of the spacelike ruled surfaces in

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view of dual Darboux frame. Using the dual representations of spacelike ruled surfaces, we give some theorems and new results which characterize the developable Mannheim spacelike surface offsets and we give a new relationship between the developable Mannheim offsets and their striction lines

2. Preliminaries

3-dimensional Minkowski space IR_1^3 is the real vector space IR^3 provided with the standard flat metric given by

$$\langle \vec{a}, \vec{a} \rangle = -a_1 b_1 + a_2 b_2 + a_3 b_3,$$

where $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3) \in IR^3$. The Lorentzian character of a vector $\vec{a} = (a_1, a_2, a_3)$ is defined as follows:

i) $\vec{a}$ is called timelike if $\langle \vec{a}, \vec{a} \rangle < 0$,

ii) $\vec{a}$ is called spacelike if $\langle \vec{a}, \vec{a} \rangle > 0$ or $\vec{a} = 0$,

iii) $\vec{a}$ is called lightlike (null) if $\langle \vec{a}, \vec{a} \rangle = 0$ and $\vec{a} \neq 0$.

Similarly, the Lorentzian character of a curve $\alpha(s)$ in IR_1^3 is given by considering its velocity vectors $\alpha'(s)$. Then a curve $\alpha(s)$ has the same character with its velocity vector $\alpha'(s)$ [8]. The norm of a vector $\vec{a}$ is defined by $\|\vec{a}\| = \sqrt{|\langle \vec{a}, \vec{a} \rangle|}$. The Lorentzian cross product of two vectors $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ is defined by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & -\vec{e}_2 & -\vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2 b_3 - a_3 b_2,\ a_1 b_3 - a_3 b_1,\ a_2 b_1 - a_1 b_2)$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \qquad \vec{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \text{ and } \vec{e}_1 \times \vec{e}_2 = -\vec{e}_3,\ \vec{e}_2 \times \vec{e}_3 = \vec{e}_1,\ \vec{e}_3 \times \vec{e}_1 = -\vec{e}_2.$$

By using this definition it can be easily shown that $\langle \vec{a} \times \vec{b}, \vec{c} \rangle = -\det(\vec{a}, \vec{b}, \vec{c})$ [20].

The sets of the unit timelike and spacelike vectors are called hyperbolic unit sphere and Lorentzian unit sphere, respectively, and denoted by

$$H_0^2 = \left\{ \vec{a} = (a_1, a_2, a_3) \in IR_1^3 : \langle \vec{a}, \vec{a} \rangle = -1 \right\}$$

and

$$S_1^2 = \left\{ \vec{a} = (a_1, a_2, a_3) \in IR_1^3 : \langle \vec{a}, \vec{a} \rangle = 1 \right\}$$

respectively (See [18]).

Lorentzian character of a surface is defined using its normal vector. Then a surface in IR_1^3 is called a timelike (spacelike) surface if the normal vector of the surface is a spacelike (timelike) vector [1].

3. Dual Numbers and Dual Lorentzian Vectors

Let the set of the pairs (a,a^*) be $D = IR \times IR = \{\bar{a} = (a,a^*):\ a,a^* \in IR\}$. For $\bar{a} = (a,a^*)$, $\bar{b} = (b,b^*) \in D$ the following operations are defined on D:

Equality : $\bar{a} = \bar{b} \Leftrightarrow a = b,\ a^* = b^*$

Addition : $\bar{a} + \bar{b} = (a+b,\ a^* + b^*)$

Multiplication : $\bar{a}\bar{b} = (ab,\ ab^* + a^*b)$

The element $\varepsilon = (0,1) \in D$ is called dual unit which has the following properties

$$\varepsilon \neq 0,\quad \varepsilon^2 = 0,\quad \varepsilon 1 = 1\varepsilon = \varepsilon. \tag{1}$$

Let consider the element $\bar{a} \in D$ of the form $\bar{a} = (a,0)$. Then the mapping $f : D \to IR,\ \ f(a,0) = a$ is an isomorphism. So, we can write $a = (a,0)$. Then, by the multiplication rule we have that $\bar{a} = a + \varepsilon a^*$ and $\bar{a} = a + \varepsilon a^*$ is called dual number. The set of dual numbers is given by

$$D = \{\bar{a} = a + \varepsilon a^* : a, a^* \in IR,\ \varepsilon^2 = 0\}. \tag{2}$$

which forms a commutative group under addition [2,4].

A Dual function is a function with a dual variable $\bar{x}$. Then the general expression of a dual function is

$$f(\bar{x}) = f(x + \varepsilon x^*) = f(x) + \varepsilon x^* f'(x), \tag{3}$$

where $f'(x)$ is derivative of $f(x)$ with respect to x and $x, x^* \in IR$ [3]. Using (3), some well-known functions can be given as follows

$$\begin{cases} \cosh(\bar{x}) = \cosh(x + \varepsilon x^*) = \cosh(x) + \varepsilon x^* \sinh(x), \\ \sinh(\bar{x}) = \sinh(x + \varepsilon x^*) = \sinh(x) + \varepsilon x^* \cosh(x), \\ \sqrt{\bar{x}} = \sqrt{x + \varepsilon x^*} = \sqrt{x} + \varepsilon \dfrac{x^*}{2\sqrt{x}},\ \ (x > 0). \end{cases} \tag{4}$$

Let $D^3 = D \times D \times D$ be the set of all triples of dual numbers, i.e.,

$$D^3 = \{(\tilde{[illegible]}\ \bar{a}_3) : \bar{a}_i \in D,\ i = 1,2,3\}. \tag{5}$$

Then the set D^3 is called dual space. The elements of D^3 are called dual vectors. Analogue to the dual numbers, a dual vector [illegible] may be expressed in the form [illegible] $= (\vec{a}, \vec{a}$ [illegible]$)$, where $\vec{a}$ and $\vec{a}$ are the vectors of IR^3. Then for any vectors [illegible] and [illegible] of D^3, the scalar product and the vector product are defined by

$$\langle [illegible] \rangle + \varepsilon\big(\langle \vec{a}, [illegible] \rangle + \langle \vec{a}\ , [illegible] \rangle\big), \tag{6}$$

and

$$[illegible]\ \varepsilon\big(\vec{a} \times [illegible] + \vec{a}\ \times [illegible]\big), \tag{7}$$

respectively, where $\langle \vec{a}, b \rangle$ and $\vec{a} \times b$ are the inner product and the vector product of the vectors $\vec{a}$ and $\vec{a}$ in IR^3, respectively.

The norm of a dual vector $\tilde{a}$ is given by

$$\|\tilde{a}\| \quad \| \quad \| \quad ; \frac{\langle \vec{a}, \vec{a} \rangle}{\|\vec{a}\|} \qquad \neq 0) . \tag{8}$$

If $\|\tilde{a}\|$, then $\tilde{a}$ is called dual unit vector. The set of such vectors is defined by

$$\tilde{S} \quad \{\tilde{a} \qquad \tilde{\ } \tilde{\ } \qquad \}, \tag{9}$$

and called dual unit sphere [2,4].

The Lorentzian inner product of two dual vectors $\tilde{a}$, $\tilde{b} \in D^3$ is defined by

$$\langle \tilde{a}, \tilde{b} \rangle \quad + \varepsilon (\langle \vec{a}, b \rangle + \langle \vec{a}, b \rangle),$$

where $\langle \vec{a}, b \rangle$ is the Lorentzian inner product of the vectors $\vec{a}$ and $\vec{b}$ in IR_1^3. Then the Lorentzian character of a dual vector $\tilde{a}$ is defined as the Lorentzian character of real part $\vec{a}$, i.e., dual vector $\tilde{a}$ has the same character with real vector $\vec{a}$ [18].

Dual Lorentzian vectors form a space called dual Lorentzian space which is denoted as follows

$$D_1^3 = \{ \tilde{a} \qquad : \ \vec{a}, \vec{a} \in IR_1 \ \} .$$

Let $\tilde{a}$ $\tilde{\ }$ $_1^3$. Then the following product is called dual Lorentzian cross product

$$\tilde{a} \quad \tilde{\ } \qquad \varepsilon (\vec{a} \times b + \vec{a} \times b),$$

where $\vec{a} \times b$ denotes the Lorentzian cross product in IR_1^3 .

Let $\tilde{a}$ $\in D_1^3$. If $\|\tilde{a}\|$ 0 (resp. $\|\tilde{a}\|$), then $\tilde{a}$ is called dual unit timelike (resp. spacelike) vector for which followings hold

$$< \vec{a}, \vec{a} > = -1 \ (resp. < \vec{a}, \vec{a} > = 1), \ < \vec{a}, \vec{a} > = 0. \tag{10}$$

Then we called the unit sphere consists of all unit dual timelike vectors as dual hyperbolic unit sphere which is represented by $\tilde{H}_0$,

$$\tilde{H}_0 \quad \{\tilde{a} \qquad \tilde{\ } \tilde{\ } \qquad \}. \tag{11}$$

Similarly, we called the unit sphere consists of all unit dual spacelike vectors as dual Lorentzian unit sphere which is represented by $\tilde{S}_1$,

$$\tilde{S}_1 \quad \{\tilde{a} \qquad \tilde{\ } \tilde{\ } \qquad \}. \tag{12}$$

(See [18]).

Definition 3.1. ([18]) ***i) Dual Lorentzian timelike angle:*** The dual angle $\bar{\theta} = \theta + \varepsilon\theta^*$ between a dual spacelike vector $\tilde{\ }$ and a dual timelike vector $\tilde{\ }$ in D_1^3 is defined by $| < \tilde{\ } \tilde{\ } \quad \| \tilde{\ } \| \| \tilde{\ } \|$ and called dual *Lorentzian timelike angle*.

ii) Dual Central angle: The dual angle $\bar{\theta}=\theta+\varepsilon\theta^{*}$ between dual spacelike vectors $\tilde{a}$ and $\tilde{b}$ in D_1^3 that span a dual timelike vector subspace is defined by $|\langle \tilde{a},\tilde{b}\rangle| = \|\tilde{a}\|\|\tilde{b}\|\cosh\bar{\theta}$ and called dual *central angle*.

4. Dual Darboux Frame of a Spacelike Ruled Surface

From E. Study mapping, the lines of the line space IR^3 correspond to dual unit vectors [2,4]. Then, the dual spherical curve lying fully on $\tilde{S}^2$ represents a ruled surface in IR^3. In this section, we introduce this correspondence rule for spacelike ruled surfaces and give the dual Darboux frame for these surfaces.

In the Minkowski 3-space IR_1^3, to determine an oriented spacelike line L it is enough to know a point $p\in L$ and a unit spacelike vector $\vec{a}$. Then, the moment vector $\vec{a}^{*}=\vec{p}\times\vec{a}$ can be defined. The moment vector $\vec{a}^{*}$ does not depend on the chosen of point p. For another point q in L we can write $\vec{q}=\vec{p}+\lambda\vec{a}$ and then $\vec{a}^{*}=\vec{p}\times\vec{a}=\vec{q}\times\vec{a}$. Reciprocally, when such a pair $(\vec{a},\vec{a}^{*})$ is given, one recovers the spacelike line L as $L=\left\{(\vec{a}\times\vec{a}^{*})+\lambda\vec{a}:\ \vec{a},\vec{a}^{*}\in IR_1^3,\ \lambda\in IR\right\}$, written in parametric equations. The vectors $\vec{a}$ and $\vec{a}^{*}$ are not independent of one another and they satisfy the following relationships

$$\langle\vec{a},\vec{a}\rangle=1,\quad \langle\vec{a},\vec{a}^{*}\rangle=0. \tag{13}$$

The components $a_i,\ a_i^{*}$ $(1\le i\le 3)$ of the vectors $\vec{a}$ and $\vec{a}^{*}$ are called the normalized Plucker coordinates of the spacelike line L. From (10), (11) and (13) we see that the dual spacelike unit vector $\tilde{a}=\vec{a}+\varepsilon\vec{a}^{*}$ corresponds to spacelike line L. This correspondence is known as E. Study Mapping: There exists a one-to-one correspondence between the spacelike vectors of dual Lorentzian unit sphere $\tilde{S}_1^2$ and the directed spacelike lines of the Minkowski space IR_1^3 [18]. Using this mapping, the study of spatial motion of a spacelike line corresponds to the study of dual Lorentzian spherical curve lying on $\tilde{S}_1^2$.

The relations between a ruled surface and dual spherical curves have been introduced by Veldkamp in detailed [21]. Now, we use the similar procedure to introduce the dual Darboux frame of a spacelike ruled surface.

Let $(\tilde{k})$ be a dual Lorentzian curve on $\tilde{S}_1^2$ and let the dual position vector of $(\tilde{k})$ be unit spacelike vector $\tilde{e}(u)=\vec{e}(u)+\varepsilon\vec{e}^{*}(u)$. The real part $\vec{e}$ draws a curve on S_1^2 and is called the (real) indicatrix of $(\tilde{k})$ which will be accepted as not a single line in this study. Let consider the parameter u as the arc-length parameter s of the real indicatrix and denote the differentiation with respect to s by primes. Then we have $\langle\vec{e}',\vec{e}'\rangle=-1$. The vector $\vec{e}'=\vec{t}$ is the unit tangent vector of the indicatrix and it is also the unit normal of the surface. The equation $\vec{e}^{*}(s)=\vec{p}(s)\times\vec{e}(s)$ has infinity of solutions for the function $\vec{p}(s)$. If $\vec{p}_o(s)$ is a solution, then the set of all solutions is given by $\vec{p}(s)=\vec{p}_o(s)+\lambda(s)\vec{e}(s)$, where λ is a real scalar function of s. Therefore we have $\langle\vec{p}',\vec{e}'\rangle=\langle\vec{p}_o',\vec{e}'\rangle+\lambda$. By taking $\lambda=\lambda_o=-\langle\vec{p}_o',\vec{e}'\rangle$ we have that $\vec{p}_o(s)+\lambda_o(s)\vec{e}(s)=\vec{c}(s)$ is the unique solution for $\vec{p}(s)$ with $\langle\vec{c}',\vec{e}'\rangle=0$. Then, the given dual curve $(\tilde{k})$ corresponding to the spacelike ruled surface

$$\varphi_e=\vec{c}(s)+v\vec{e}(s), \tag{14}$$

may be represented by

$$\tilde{e}(s)=\vec{e}+\varepsilon\vec{c}\times\vec{e}, \tag{15}$$

where

$$\langle\vec{e},\vec{e}\rangle=1,\ \langle\vec{e}',\vec{e}'\rangle=-1,\ \langle\vec{c}',\vec{e}'\rangle=0,$$

and $\vec{c}$ is position vector of the striction curve. Then we have

$$\|\tilde{\ }\| \quad \det(\vec{c},\vec{e},\iota) = 1+\varepsilon\Delta, \tag{16}$$

where $\Delta = \det(\vec{c},\vec{e},\iota)$ which characterizes the developable spacelike surface, i.e, the spacelike surface is developable if and only if $\Delta = 0$. Then, the dual arc-length $\bar{s}$ of the dual curve $(\tilde{\ })$ is given by

$$\bar{s} = \int_0^s \|\tilde{\ }\| \quad \int_0^s (1+\varepsilon\Delta)du = s + \varepsilon\int_0^s \Delta du\,. \tag{17}$$

From (17) we have $\bar{s}' = 1+\varepsilon\Delta$. Therefore, the dual unit tangent to the dual curve $\tilde{\ }$ is given by

$$\frac{d\tilde{\ }}{d\bar{s}} = \frac{\tilde{\ }}{\bar{s}'} = \frac{\tilde{\ }}{1+\varepsilon\Delta} = \tilde{\ } \quad \times\iota)\,. \tag{18}$$

Introducing the dual unit vector $\tilde{\ } \quad \tilde{\ } \quad \times\vec{g}$ we have the dual frame $\{\tilde{\ } \quad \tilde{\ }\}$ which is known as dual geodesic trihedron or dual Darboux frame of φ_e (or $(\tilde{\ })$). The real parts $\{\vec{e},\iota,\vec{g}\}$ of dual Darboux frame vectors form an orthonormal frame which is called geodesic trihedron of the indicatrix $\vec{e}(s)$ with the derivations

$$\vec{e}\,' = \iota, \quad \iota\,' = \vec{e}+\gamma\vec{g}, \quad \vec{g}\,' = \gamma\iota\,, \tag{19}$$

where γ is called the conical curvature [6,19].

Let now obtain the dual Darboux formulae of a spacelike ruled surface.

From (18) we have $\langle\tilde{\ },\tilde{\ }\rangle \quad \varepsilon 0$. Since $\tilde{\ } \quad \tilde{\ }$, we obtain

$$\left\langle \tilde{\ }, \frac{d\tilde{\ }}{d\bar{s}} \right\rangle, \quad \frac{\tilde{\ }}{d\bar{s}} = -\tilde{\ } \quad \frac{\tilde{\ }}{d\bar{s}}\,. \tag{20}$$

For the derivative of $\tilde{\ }$ let write

$$\frac{d\tilde{\ }}{d\bar{s}} = \bar{\alpha}\tilde{\ } \quad \tilde{\ }\,, \tag{21}$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$ are the dual functions of dual arc-length $\bar{s}$. Then from (20) it follows

$$\frac{d\tilde{\ }}{d\bar{s}} = \bar{\gamma}\tilde{\ }\,. \tag{22}$$

Since $\tilde{\ } \quad \tilde{\ } \quad \tilde{\ }$, (22) gives that

$$\frac{d\tilde{\ }}{d\bar{s}} = \tilde{\ } \quad \tilde{\ }\,. \tag{23}$$

Then from (18), (22) and (23) we have the following theorem.

Theorem 4.1. *The dual Darboux formulae of a spacelike ruled surface are given by*

$$\frac{d\tilde{\ }}{d\bar{s}} = \tilde{\ } \quad \frac{\tilde{\ }}{d\bar{s}} \quad \tilde{\ } \quad \frac{\tilde{\ }}{d\bar{s}} \quad \tilde{\ }\,. \tag{24}$$

From (24), the dual Darboux vector of the trihedron is obtained as $\tilde{\ } \quad \tilde{\ } \quad \tilde{\ }$.

Let now give the invariants of the surface. Since $\bar{s}' = 1+\varepsilon\Delta$, (22) gives that

$$\tilde{\xi} \quad \ldots \quad \varepsilon\Delta)\tilde{\ } . \tag{25}$$

From (18) and equality $\tilde{\xi} \ldots \times \vec{g}$, we obtain

$$\begin{aligned} \tilde{\xi} \quad & \ldots (\vec{c} \times \vec{g} \gamma \vec{c} \times \vec{t}\,) \\ &= \gamma\vec{t} + \varepsilon\gamma(\vec{c} \times \vec{t}\,) + \varepsilon(\vec{c} \times \vec{g}) \\ &= \gamma\tilde{\ } \ldots \vec{g}). \end{aligned} \tag{26}$$

Then (25) and (26) gives us

$$\bar{\gamma}(1+\varepsilon\Delta)\tilde{\ } \ldots \tilde{\ } \ldots \vec{g}), \tag{27}$$

and from (27) we have

$$\bar{\gamma}(1+\varepsilon\Delta) = \gamma - \varepsilon\delta, \tag{28}$$

where $\delta = \langle \vec{c}, \vec{e} \rangle$ and from (28) it follows that

$$\bar{\gamma} = \gamma - \varepsilon(\delta + \gamma\Delta). \tag{29}$$

Moreover, since $\vec{c}$ as well as $\vec{e}$ is perpendicular to $\vec{t}$, for the real scalar μ we may write $\vec{c} \times \vec{e} = \mu\vec{t}$. Then

$$\Delta = \det(\vec{c}, \vec{e}, \vec{t}\,) = -\langle \vec{c} \times \vec{e}, \vec{t}\,\rangle = -\mu\langle \vec{t}, \vec{t}\,\rangle = \mu .$$

Hence $\vec{e} \times (\vec{c} \times \vec{e}) = \Delta\vec{e} \times \vec{t} = -\Delta\vec{g}$ and $\vec{c} = \delta\vec{e} - \Delta\vec{g}$.

The functions $\gamma(s)$, $\delta(s)$ and $\Delta(s)$ are the invariants of spacelike ruled surface φ_e. They determine the surface uniquely up to its position in the space.

4.1. Elements of Curvature of a Dual Lorentzian Curve

The dual radius of curvature of spacelike ruled surface $\tilde{\ } \ldots$ is can be calculated analogous to common Lorentzian curve theory as follows

$$\bar{R} = \frac{\left\| \frac{d\tilde{\ }}{d\bar{s}} \right\|^{\cdot}}{\left\| \frac{d\tilde{\ }}{d\bar{s}} \times \frac{\tilde{\ }}{d\bar{s}^2} \right\|} = \frac{1}{\sqrt{\cdots\bar{\gamma}^2}} . \tag{30}$$

The unit Darboux vector $\tilde{c}_o$ is given by

$$\tilde{c}_o \quad \frac{\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}}\tilde{\ } \quad \frac{1}{\sqrt{1+\bar{\gamma}^2}}\tilde{\ } . \tag{31}$$

It is clear that $\tilde{c}_o$ is spacelike. Then, the dual angle between $\tilde{c}_o$ and $\tilde{\ }$ satisfies the followings

$$\sin\bar{\rho} = \frac{1}{\sqrt{1+\bar{\gamma}^2}} = \bar{R}, \quad \cos\bar{\rho} = \frac{-\bar{\gamma}}{\sqrt{1+\bar{\gamma}^2}},$$

where $\bar{\rho}$ is the dual spherical radius of curvature.

Furthermore, the corresponding equalities for a dual hyperbolic curve $\tilde{\ } \ldots$ (timelike ruled surface φ_{e_1}) are given as follows:

$$\begin{cases} \bar{s}_1 = \int_0^{s_1} \left\| [illegible] \right\| = \int_0^{s_1} (1-\varepsilon\Delta_1)du_1 = s_1 - \varepsilon \int_0^{s_1} \Delta_1 du_1, \quad \Delta_1 = \det(\vec{c}_1, \vec{e}_1, \iota_1), \\ \delta_1 = \langle \vec{c}_1, \vec{e}_1 \rangle, \quad \gamma_1 = -\langle \vec{g}_1, \iota_1 \rangle, \quad \gamma_1 = \gamma_1 + \varepsilon(\delta_1 + \gamma_1 \Delta_1) \end{cases} \tag{32}$$

$$\begin{cases} \cosh \bar{\rho}_1 = -\dfrac{\bar{\gamma}_1}{\sqrt{|1-\bar{\gamma}_1^2|}}, \quad \sinh \bar{\rho}_1 = -\dfrac{1}{\sqrt{|1-\bar{\gamma}_1^2|}}, \quad if \ |\bar{\gamma}_1| > 1. \\ \sinh \bar{\rho}_1 = -\dfrac{\bar{\gamma}_1}{\sqrt{|1-\bar{\gamma}_1^2|}}, \quad \cosh \bar{\rho}_1 = -\dfrac{1}{\sqrt{|1-\bar{\gamma}_1^2|}}, \quad if \ |\bar{\gamma}_1| < 1. \end{cases} \tag{33}$$

and

$$\bar{R}_1 = \begin{cases} -\sinh \bar{\rho}_1, & if \ |\bar{\gamma}_1| > 1, \\ -\cosh \bar{\rho}_1, & if \ |\bar{\gamma}_1| < 1. \end{cases} \quad \text{and} \quad \bar{\gamma} = \begin{cases} \coth \bar{\rho}, & if \ |\bar{\gamma}| > 1, \\ \tanh \bar{\rho}, & if \ |\bar{\gamma}| < 1. \end{cases} \tag{34}$$

(See [13]).

5. Darboux Approach to Mannheim Offsets of Spacelike Ruled Surfaces

Let φ_e be a spacelike ruled surface generated by dual spacelike unit vector [illegible] and φ_{e_1} be a ruled surface generated by dual unit vector [illegible] and let the dual Darboux frames of these surfaces be $\{$[illegible]$\}$ and $\{$[illegible]$\}$, respectively. Then φ_e and φ_{e_1} are called Mannheim surface offsets, if

$$[illegible] \tag{35}$$

holds, where $\bar{s}$ and $\bar{s}_1$ are the dual arc-lengths of φ_e and φ_{e_1}, respectively. This definition gives that φ_{e_1} is a timelike ruled surface, but the generator of this surface can be timelike or spacelike. In this study, we consider the surface φ_{e_1} as a timelike ruled surface with timelike ruling. If the ruling is chosen as spacelike, using Definition 3.1 (ii), similar results can be found. So, in this study we mean that φ_e and φ_{e_1} are spacelike and timelike ruled surfaces, respectively, and for short we don't write the Lorentzian characters of the surfaces hereinafter.

Let now the ruled surfaces φ_e and φ_{e_1} form a Mannheim offset. Then by considering (35), the relationship between the trihedrons of φ_e and φ_{e_1} can be given as follows

$$[illegible] \tag{36}$$

where $\bar{\theta} = \theta + \varepsilon\theta^*$, $(\theta, \theta^* \in IR)$ is the dual Lorentzian angle between the dual generators [illegible] and [illegible] of φ_e and φ_{e_1}. The real angle θ is called the offset angle which is the angle between the rulings $\vec{e}$ and $\vec{e}_1$, and θ^* is called the offset distance which is measured from the striction point $\vec{c}$ of φ_e to striction point $\vec{c}_1$ of φ_{e_1}. And from (36) we may write $\vec{c}_1 = \vec{c} + \upsilon\ \iota$. Then, $\bar{\theta} = \theta + \varepsilon\theta^*$ is called dual Lorentzian offset angle of the Mannheim ruled surfaces φ_e and φ_{e_1} . If $\theta = 0$, then the Mannheim surface offsets are said to be right offsets.

Now, we give some theorems and results characterizing Mannheim offsets.

Theorem 5.1. *Let φ_e and φ_{e_1} form a Mannheim offset. The offset angle θ and the offset distance θ^* are given by*

$$\theta = -s + c,\ \ \theta^* = -\int_0^s \Delta du + c^*, \tag{37}$$

respectively, where c and c^* are real constants.

Proof. Since φ_e and φ_{e_1} form a Mannheim offset, we can write

$$\tilde{\ } \qquad \bar{\ }\tilde{\ } \qquad \bar{\ } \quad . \tag{38}$$

By differentiating (38) with respect to $\bar{s}$ we have

$$\frac{d\tilde{\ }}{d\bar{s}} = \text{cosn}\,\bar{\theta}\left(1 + \frac{d\bar{\theta}}{d\bar{s}}\right)\tilde{\ } \qquad \bar{\ }\left(\qquad \frac{d\bar{\theta}}{d\bar{s}} \right) \qquad \bar{\ }\tilde{\ } . \tag{39}$$

From (35) we have that $\frac{d\tilde{\ }}{d\bar{s}}$ and $\tilde{\ }$ are linearly dependent. Then, from (39) we get $\frac{d\bar{\theta}}{d\bar{s}} = -1$ and for the dual constant $\bar{c} = c + \varepsilon c^*$ we write

$$d\bar{\theta} = -d\bar{s}$$
$$\bar{\theta} = -\bar{s} + \bar{c}$$
$$\theta + \varepsilon\theta^* = -s - \varepsilon s^* + c + \varepsilon c^*.$$

Then from (17) we have

$$\theta = -s + c,\ \ \theta^* = -\int_0^s \Delta du + c^*,$$

where c and c^* are real constants.

From (37), the following corollary can be given.

Corollary 5.1. *Let φ_e and φ_{e_1} form a Mannheim offset. Then φ_e is developable if and only if offset distance is constant, i.e.* $\theta^* = c^* = \text{constant}$.

Theorem 5.2. *Let φ_e and φ_{e_1} form a Mannheim offset. Then the relationship between the dual arc-length parameters of φ_e and φ_{e_1} is*

$$\frac{d\bar{s}_1}{d\bar{s}} = \bar{\gamma}\cosh\bar{\theta}\ . \tag{40}$$

Proof. Since φ_e and φ_{e_1} form a Mannheim offset, Theorem 5.1 gives that

$$\frac{d\tilde{\ }}{d\bar{s}_1} = \tilde{\ }_1 \quad , \qquad \bar{\theta}\,\frac{d\bar{s}}{d\bar{s}_1}\,\tilde{\xi}\,. \tag{41}$$

Then using equality $\tilde{\ }_1 \quad \tilde{\xi}$, from (41) it follows

$$\bar{\gamma}\cosh\bar{\theta}\,\frac{d\bar{s}}{d\bar{s}_1} = 1 \tag{42}$$

and from (42) we get (40).

Corollary 5.2. *Let φ_e and φ_{e_1} form a Mannheim offset. Then the relationships between the real arc-length parameters of φ_e and φ_{e_1} are given as follows*

$$\frac{ds_1}{ds} = \gamma\cosh\theta, \quad \frac{ds\,ds_1^* - ds^*ds_1}{ds^2} = \theta^*\gamma\sinh\theta - (\delta + \gamma\Delta)\cosh\theta\,. \tag{43}$$

Proof. Since φ_e and φ_{e_1} form a Mannheim offset, (40) holds. By considering (4), the real and dual parts of (40) are

$$\frac{ds_1}{ds} = \gamma\cosh\theta, \quad \frac{ds\,ds_1^* - ds^*ds_1}{ds^2} = \theta^*\gamma\sinh\theta - (\delta + \gamma\Delta)\cosh\theta\,, \tag{44}$$

which are desired equalities.

In Corollary 5.1, we give the relationship between the offset distance θ^* and developable spacelike ruled surface φ_e. Now we give the condition for φ_{e_1} to be developable according to θ^*. From (17) and (32) we have

$$ds^* = \Delta ds, \quad ds_1^* = -\Delta_1 ds_1\,, \tag{45}$$

respectively. Then writing (45) in (44) and using (43) we get

$$\Delta_1 = -\theta^*\tanh\theta + \frac{\delta}{\gamma}\,,$$

and give the following corollaries:

Corollary 5.3. *Let φ_e and φ_{e_1} form a Mannheim offset. Then*

$$\Delta_1 = -\theta^*\tanh\theta + \frac{\delta}{\gamma}\,, \tag{46}$$

holds.

Corollary 5.4. *Let φ_e and φ_{e_1} form a Mannheim offset. Then φ_{e_1} is developable if and only if $\theta^* = \frac{\delta}{\gamma}\coth\theta$ holds.*

Theorem 5.3. *Let φ_e and φ_{e_1} form a Mannheim offset. There exists the following relationship between the invariants of the surfaces and offset angle θ, offset distance θ^*,*

$$\delta_1 = \frac{\delta}{\gamma} \tanh\theta - \theta^* . \tag{47}$$

Proof. Let the striction lines of φ_e and φ_{e_1} be $c(s)$ and $c_1(s_1)$, respectively, and let φ_e and φ_{e_1} form a Mannheim offset. Then, from the Mannheim condition we can write

$$\vec{c}_1 = \vec{c} + \theta \; \vec{g} . \tag{48}$$

Differentiating (48) with respect to s_1 we have

$$\frac{d\vec{c}_1}{ds_1} = \left(\frac{d\vec{c}}{ds} + \theta^* \gamma \vec{t} + \frac{}{ds} \vec{g} \right) \frac{}{ds_1} . \tag{49}$$

From (32) we know that $\delta_1 = \langle d\vec{c}_1 / ds_1 , e_1 \rangle$. Then from (36) and (49) we obtain

$$\delta_1 = \left(\sinh\theta \langle d\vec{c} / ds, \vec{e} \rangle + \cosh\theta \langle d\vec{c} / ds, \vec{t} \rangle + \theta \; \gamma \cosh \langle \vec{t}, \vec{t} \rangle \right) \frac{}{ds_1} . \tag{50}$$

Since $\langle d\vec{c} / ds, e \rangle = \delta$, $\langle d\vec{c} / ds, \vec{t} \rangle = 0$, $\langle \vec{t}, \vec{t} \rangle = -1$, from (50) we write

$$\delta_1 = \left(\delta \sinh\theta - \theta^* \gamma \cosh\theta \right) \frac{ds}{ds_1} . \tag{51}$$

Furthermore, from (43) we have

$$\frac{ds}{ds_1} = \frac{1}{\gamma \cosh\theta} . \tag{52}$$

Then substituting (52) in (51) we obtain

$$\delta_1 = \frac{\delta}{\gamma} \tanh\theta - \theta^* .$$

Theorem 5.4. *If φ_e and φ_{e_1} form a Mannheim offset, then for conical curvature γ_1 of φ_{e_1} and offset angle θ*

$$\gamma_1 = -\tanh\theta , \tag{53}$$

holds.

Proof. From (32) and (36) we have

$$\begin{aligned} \gamma_1 &= -\langle \vec{g}_1 , t_1 \rangle \\ &= -\left\langle \frac{d}{ds_1} (\cosh\theta \vec{e} + \sinh\theta \vec{t}), \; \vec{g} \right\rangle \\ &= -\gamma \sinh\theta \frac{ds}{ds_1} . \end{aligned} \tag{54}$$

From the first equality of (43) and (54), we have $\gamma_1 = -\tanh\theta$.

Theorem 5.5. *If the surfaces φ_e and φ_{e_1} form a Mannheim offset, then the dual conical curvature $\bar{\gamma}_1$ of φ_{e_1} is obtained as*

$$\bar{\gamma}_1 = -\tanh\bar{\theta}\,. \tag{55}$$

Proof. From (32), (47), (53) and (54) by direct calculation we have (55).

From (55) we have the following corollary.

Theorem 5.6. *If the surfaces φ_e and φ_{e_1} form a Mannheim offset, then the dual curvature of φ_{e_1} is given by*

$$\bar{R}_1 = \cosh\bar{\theta}\,. \tag{56}$$

Proof. From (62) we have

$$\sqrt{\left|1-\bar{\gamma}_1^2\right|} = \mathrm{sech}\,\theta - \varepsilon\theta^* \tanh\theta\,\mathrm{sech}\,\theta = \mathrm{sech}\,\bar{\theta}\,.$$

Then from (34) we have

$$\bar{R}_1 = \frac{1}{\sqrt{\left|1-\bar{\gamma}_1^2\right|}} = \frac{\mathrm{sech}\,\theta + \varepsilon\theta^* \tanh\theta\,\mathrm{sech}\,\theta}{\mathrm{sech}^2\,\theta}$$
$$= \cosh\theta + \varepsilon\theta^* \sinh\theta$$
$$= \cosh\bar{\theta}.$$

Then we can give the following corollaries.

Corollary 5.5. *If φ_e and φ_{e_1} form a Mannheim offset and $\left|\bar{\gamma}_1\right| < 1$, then the dual spherical radius of curvature of φ_{e_1} is given by*

$$\cosh\bar{\rho}_1 = -\cosh\bar{\theta}\,. \tag{57}$$

If we assume that $\left|\bar{\gamma}_1\right| > 1$, then we have equalities for a timelike ruled surface whose Darboux vector is timelike and the obtained equalities will be analogue to given ones.

6. Conclusions

The dual Darboux frame of a spacelike ruled surface is introduced. Then the characterizations of Mannheim offsets of spacelike ruled surfaces are given in view of dual Darboux frame and new relations between the invariants of Mannheim offsets are obtained. The results of the paper are new characterizations of Mannheim offsets of a spacelike ruled surface and also give the relationships for these offsets to be developable according to offset angle and offset distance. Moreover, the relationship between the developable Mannheim offsets and their striction lines is given.

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