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Slant helices in dual Lorentzian Space D_1^3

Derya Saglam[1*], Serhat Ozkan[1], Duygu Ozdamar[1]

Abstract

In this paper, we consider a unit speed dual Lorentzian curve $\tilde{\alpha}$ in dual Lorentzian space D_1^3 and denote by $\left\{\widetilde{T},\widetilde{N},\widetilde{B}\right\}$ the dual Frenet frame of $\tilde{\alpha}$. We say that α is a slant helix if there exists a non-zero dual constant vector field $\widetilde{U}$ in D_1^3 such that the dual function $\left\langle \widetilde{N},\widetilde{U} \right\rangle$ is a dual constant. Moreover, we give some characterizations of slant helices in terms of their dual curvatures. Finally, we show that dual tangent indicatrices and dual binormal indicatrices of slant helices are dual helices.

Keywords: Dual Lorentzian space; Dual Frenet equations; Dual slant helices; Dual curves

Introduction

Dual numbers were introduced by W. K. Clifford (1849-1879) as a tool for his geometrical investigations. After him E. Study used dual members and dual vectors in his research on the geometry of lines and kinematics. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the points of dual unit sphere S^2 and directed lines in R^3 [2,8].

If we take the Minkowski 3-space R_1^3 instead of R^3 of the E. Study mapping can be stated as follows: The dual timelike and spacelike unit vectors of dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2 in the dual Lorentzian space D_1^3 are in one-to-one correspondence with the directed timelike and spacelike lines in R_1^3, respectively. Then a differentiable curve on H_0^2 corresponds to a timelike ruled surface in R_1^3. Similarly, the timelike (resp. spacelike) curve on S_1^2, corresponds to any spacelike(resp. timelike) ruled surface in R_1^3 [11, 8].

We will survey briefly the fundamental concepts and properties in the Lorentzian space. We refer mainly to O'Neill [5, 8].

Let R_1^3 be the 3-dimensional Lorentzian space with Lorentzian metric $\left\langle , \right\rangle = dx_1^2 + x_2^2 - x_3^2$. It is known that in R_1^3 there are three categories of curves and vectors, namely, spacelike, timelike and null, depending on their causal characters. Let $\vec{x}$ be tangent vector of Lorentzian space. Then $\vec{x}$ is said to be spacelike $\left\langle \vec{x},\vec{x} \right\rangle \rangle 0$ or $\vec{x}=\vec{0}$, timelike if $\left\langle \vec{x},\vec{x} \right\rangle \langle 0$, null (lightlike) if $\left\langle \vec{x},\vec{x} \right\rangle = 0$ and $\vec{x}\neq\vec{0}$. Let $\alpha: I \subset R \to R_1^3$ be a regular curve in R_1^3. Then, the curve α is spacelike if all velocity vectors are spacelike. Similarly, it is called timelike and null curve if all its velocity vectors are timelike and null vectors, respectively [8].

A dual number $\overline{x}$ has the form $x + \varepsilon x^*$ with properties

$$\varepsilon \neq 0, 0\varepsilon = \varepsilon 0 = 0, 1\varepsilon = \varepsilon 1 = \varepsilon, \varepsilon^2 = 0$$

where x and x^* are real numbers and ε is the dual unit (for the properties of dual vectors, see [7,12]). An ordered triple of dual numbers ($\overline{x}_1, \overline{x}_2, \overline{x}_3$) is called a dual vector and the set of dual vectors is denoted by

$$D^3 = D \times D \times D = \{\tilde{x} \big| \tilde{x} \quad = \left(\overline{x}_1 + \varepsilon \overline{x}_1^*, \overline{x}_2 + \varepsilon \overline{x}_2^*, \overline{x}_3 + \varepsilon \overline{x}_3^*\right)$$
$$= \left(\overline{x}_1, \overline{x}_2, \overline{x}_3\right) + \varepsilon \left(x_1^*, x_2^*, x_3^*\right)$$
$$= \vec{x} + \varepsilon \vec{x}^*, \quad \vec{x}, \vec{x}^* \in R^3\}.$$

D^3 is a module on the ring D. For any $\tilde{x} = \vec{x} + \varepsilon \vec{x}^*, \tilde{y} = \vec{y} + \varepsilon \vec{y}^* \in D^3$, if the Lorentzian inner product of dual vectors $\tilde{x}$ and $\tilde{y}$ defined by

$$\langle \tilde{x}, \tilde{y} \rangle = \langle \vec{x}, \vec{y} \rangle + \varepsilon \left(\langle \vec{x}, \vec{y}^* \rangle + \langle \vec{x}^*, \vec{y} \rangle \right)$$

then the dual space D^3 together with this Lorentzian inner product is called dual Lorentzian space and it is shown by D_1^3 [8]. A dual vector $\tilde{x}$ in D_1^3 is said to be spacelike, timelike and lightlike (null) if the vector $\vec{x}$ is spacelike, timelike and lightlike (null), respectively. Dual Lorentzian cross product of dual vectors $\tilde{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3)$ and $\tilde{y} = (\overline{y}_1, \overline{y}_2, \overline{y}_3)$ in D_1^3 is defined by

$$\tilde{x} \times \tilde{y} = \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \overline{x}_1 & \overline{x}_2 & \overline{x}_3 \\ \overline{y}_1 & \overline{y}_2 & \overline{y}_3 \end{vmatrix}$$

$$= (\overline{x}_3 \overline{y}_2 - \overline{x}_2 \overline{y}_3, \overline{x}_3 \overline{y}_1 - \overline{x}_1 \overline{y}_3, \overline{x}_1 \overline{y}_2 - \overline{x}_2 \overline{y}_1)$$

[8]. If $\vec{x} \neq 0$, the norm $\|\tilde{x}\|$ of $\tilde{x} = \vec{x} + \varepsilon \vec{x}^*$ is defined by

$$\|\tilde{x}\| = \sqrt{|\langle \tilde{x}, \tilde{x} \rangle|} .$$

A dual vector $\tilde{x}$ with norm 1 is called a dual unit vector. Let $\tilde{x} = \vec{x} + \varepsilon \vec{x}^* \in D_1^3$. Then,

i) The set
$S_1^2 = \{ \tilde{x} = \vec{x} + \varepsilon \vec{x}^* \big| \|\tilde{x}\| = (1,0); \vec{x}, \vec{x}^* \in R^3$ and the vector $\vec{x}$ is spacelike$\}$ is called the pseudo dual sphere in D_1^3.

ii) The set
$H_0^2 = \{ \tilde{x} = \vec{x} + \varepsilon \vec{x}^* \big| \|\tilde{x}\| = (1,0); \vec{x}, \vec{x}^* \in R^3$ and the vector $\vec{x}$ is timelike$\}$ is called the pseudo dual hyperbolic space in D_1^3 [7, 11, 8].

If every real valued functions $\alpha_i(t)$ and $\alpha_i^*(t)$ $1 \leq i \leq 3$, are differentiable, dual Lorentzian curve

$$\tilde{\alpha} : I \subset R \rightarrow D_1^3$$
$$t \rightarrow D_1^3 = (\alpha_1(t) + \varepsilon \alpha_1^*(t), \alpha_2(t) + \varepsilon \alpha_2^*(t), \alpha_3(t) + \varepsilon \alpha_3^*(t))$$
$$= \vec{\alpha}(t) + \varepsilon \vec{\alpha}^*(t)$$

is differentiable in D_1^3. The real part $\vec{\alpha}(t)$ of the dual Lorentzian curve $\tilde{\alpha} = \tilde{\alpha}(t)$ is called indicatrix. The dual arc length of the curve $\tilde{\alpha}(t)$ from t_1 to t is defined as

(1.1) $$\overline{s} = \int_{t_1}^{t} \|\tilde{\alpha}'(t)\| dt = \int_{t_1}^{t} \|\tilde{\alpha}'(t)\| dt + \varepsilon \int_{t_1}^{t} \left\langle \vec{T}, \vec{\alpha}^{*'}(t) \right\rangle dt = s + \varepsilon s^*,$$

where $\vec{T}$ is a unit tangent vector of the indicatrix $\vec{\alpha}(t)$. From now on we will take the arc length s of $\vec{\alpha}(t)$ as the parameter instead of t [7, 8].

Now we can give dual Frenet equations for unit speed dual timelike and spacelike curves in D_1^3 similar to that in R_1^3 [1, 3, 10, 7].

Timelike Case ($\langle \tilde{T}, \tilde{T} \rangle = -1$) :

Let $\tilde{\alpha}$ be a unit speed dual timelike curve in D_1^3. The dual Frenet frame $\{ \tilde{T}, \tilde{N}, \tilde{B} \}$ of $\tilde{\alpha}$ is given by

$$\tilde{T} = \tilde{\alpha}', \ \tilde{N} = \frac{\tilde{T}'}{\kappa} = \frac{\tilde{\alpha}''}{\|\tilde{\alpha}''\|}, \ \tilde{B} = \tilde{T} \times \tilde{N}$$

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where $\times$ is dual Lorentzian cross product. The dual Frenet equations are

(1.2)
$$\frac{d}{ds}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix} = \begin{pmatrix}0 & \overline{\kappa} & 0\\\overline{\kappa} & 0 & \overline{\tau}\\0 & -\overline{\tau} & 0\end{pmatrix}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix}$$

where $\overline{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\overline{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion.

Spacelike Case ($\left\langle \widetilde{T}, \widetilde{T}\right\rangle = 1$) :

We separate three cases depends on causal character of $\widetilde{T}'$.

i) Let $\widetilde{T}'$ be a dual spacelike vector field. Since $\overline{\kappa} = \left\|\widetilde{T}'\right\| = \sqrt{\left\langle\widetilde{T}',\widetilde{T}'\right\rangle}$, $\overline{\tau} = -\left\langle\widetilde{N}',\widetilde{B}\right\rangle$, $\widetilde{N} = \widetilde{T}'/\overline{\kappa}$ and $\widetilde{B} = \widetilde{T}\times\widetilde{N}$ then the dual Frenet equations:

(1.3)
$$\frac{d}{ds}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix} = \begin{pmatrix}0 & \overline{\kappa} & 0\\-\overline{\kappa} & 0 & \overline{\tau}\\0 & \overline{\tau} & 0\end{pmatrix}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix}$$

where $\overline{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\overline{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion.

ii) Let $\widetilde{T}'$ be a dual timelike vector field. Since $\overline{\kappa} = \left\|\widetilde{T}'\right\| = \sqrt{-\left\langle\widetilde{T}',\widetilde{T}'\right\rangle}$, $\overline{\tau} = \left\langle\widetilde{N}',\widetilde{B}\right\rangle$ then the dual Frenet equations:

(1.4)
$$\frac{d}{ds}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix} = \begin{pmatrix}0 & \overline{\kappa} & 0\\\overline{\kappa} & 0 & \overline{\tau}\\0 & \overline{\tau} & 0\end{pmatrix}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix}$$

where $\overline{\kappa} = \kappa + \varepsilon\kappa^*$ is nowhere pure dual curvature and $\overline{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion.

iii) Let $\widetilde{T}'$ be a dual lightlike vector field. The dual Frenet frame is $\widetilde{T} = \tilde{\alpha}'$, $\widetilde{N} = \widetilde{T}'$ and $\widetilde{B}$ is the unique dual lightlike vector field orthogonal to $\widetilde{T}$ such that $\left\langle\widetilde{N},\widetilde{B}\right\rangle = 1$. Then the dual Frenet equations:

(1.5)
$$\frac{d}{ds}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix} = \begin{pmatrix}0 & 1 & 0\\0 & \overline{\tau} & 0\\-1 & 0 & -\overline{\tau}\end{pmatrix}\begin{pmatrix}\widetilde{T}\\\widetilde{N}\\\widetilde{B}\end{pmatrix}$$

where $\overline{\tau} = \tau + \varepsilon\tau^*$ is nowhere pure dual torsion.

Slant helices ever been studied by many researchers [1, 4, 6, 9, 13, 14]. Recently, slant helices have been studied in Minkowski space E_1^3 and the authors have given characterizations of the curves [1]. In this paper we study dual slant helix in dual Lorentzian space and investigate dual version of some well-known results for dual slant helices in dual Lorentzian space. Also we take a dual Lorentzian curve $\tilde{\alpha} = \tilde{\alpha}(\overline{s})$ in dual Lorentzian space D_1^3 and denote by $\left\{\widetilde{T}, \widetilde{N}, \widetilde{B}\right\}$ the dual Frenet frame of $\tilde{\alpha}$. We say that $\tilde{\alpha}$ is a dual slant helix if there exists a fixed direction $\widetilde{U}$ of D_1^3 such that the function $\left\langle\widetilde{N},\widetilde{U}\right\rangle$ is dual constant [13, 14]. In this work we give characterizations of dual slant helices in terms of the dual curvature and dual torsion of $\tilde{\alpha}$. Finally, we discuss the dual tangent and dual binormal indicatrices of dual slant curves, proving that they are dual helices in D_1^3.

Some Characterizations of Slant Helices in D_1^3

Theorem 2.1. *Let $\tilde{\alpha}$ be a unit speed dual timelike curve with not pure dual curvature $\overline{\kappa}$ and not pure dual torsion $\overline{\tau}$ in D_1^3. Then $\tilde{\alpha}$ is a dual slant helix if and only if one the next two dual functions:*

(2.1)
$$\frac{\overline{\kappa}^2}{\left(\overline{\tau}^2 - \overline{\kappa}^2\right)^{3/2}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)' \quad \text{or} \quad \frac{\overline{\kappa}^2}{\left(\overline{\kappa}^2 - \overline{\tau}^2\right)^{3/2}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$$

is dual constant where $\overline{\tau}^2 - \overline{\kappa}^2$ is not pure dual.

Proof. Let $\tilde{\alpha}$ be a unit speed dual timelike curve in D_1^3. In order to prove the theorem, we first assume that $\tilde{\alpha}$ is a dual slant helix. Let $\widetilde{U}$ be the fixed dual vector field such that the function $\left\langle \widetilde{N}, \widetilde{U} \right\rangle = \overline{c}$ is dual constant. There exist smooth dual functions $\overline{a}_1$ and $\overline{a}_3$ such that

(2.2) $\widetilde{U} = \overline{a}_1 \widetilde{T} + \overline{c} \widetilde{N} + \overline{a}_3 \widetilde{B}$

Since $\widetilde{U}$ is dual constant, a differentiation in (2.2) with using (1.2) gives

(2.3) $\begin{cases} \overline{a}_1' + \overline{c}\,\overline{\kappa} = 0 \\ \overline{a}_1 \overline{\kappa} - \overline{a}_3 \overline{\tau} = 0 \\ \overline{c}\,\overline{\tau} + \overline{a}_3' = 0 \end{cases}$

From the second equation of (2.3), we obtain

(2.4) $\overline{a}_1 = \overline{a}_3 \left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)$

On the other hand

(2.5) $\left\langle \widetilde{U}, \widetilde{U} \right\rangle = -\overline{a}_1^{\,2} + \overline{c}^{\,2} + \overline{a}_3^{\,2} = \text{dual constant}$

Considering (2.4) and (2.5) together, let $\overline{b}$ be the not pure dual constant given by

$$\delta \overline{b}^2 := \overline{a}_3^{\,2} \left(\left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)^2 - 1 \right), \quad \delta \in \{-1, 0, 1\}.$$

If $\delta=0$ then $\overline{a}_3 = 0$ and from (2.3) $\overline{a}_3' = 0$, $\overline{a}_1 = \overline{c} = 0$. Then $\widetilde{U} = 0$ and it is contradiction. Thus $\delta=1$ or $\delta=-1$ and since $\overline{\tau}^2 - \overline{\kappa}^2$ is not pure dual, then

$$\overline{a}_3 = \pm \dfrac{\overline{b}}{\sqrt{\left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)^2 - 1}} \quad \text{or} \quad \overline{a}_3 = \pm \dfrac{\overline{b}}{\sqrt{1 - \left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)^2}}.$$

From the third equation of (2.3), we obtain

$$\dfrac{d}{ds} \left(\pm \dfrac{\overline{b}}{\sqrt{\left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)^2 - 1}} \right) = \overline{c}\,\overline{\tau} \quad \text{or} \quad \dfrac{d}{ds} \left(\pm \dfrac{\overline{b}}{\sqrt{1 - \left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)^2}} \right) = -\overline{c}\,\overline{\tau}.$$

According to this

$$\dfrac{\overline{\kappa}^2}{\left(\overline{\tau}^2 - \overline{\kappa}^2 \right)^{\frac{3}{2}}} \left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)' = \pm \dfrac{\overline{c}}{\overline{b}} \quad \text{or} \quad \dfrac{\overline{\kappa}^2}{\left(\overline{\kappa}^2 - \overline{\tau}^2 \right)^{\frac{3}{2}}} \left(\dfrac{\overline{\tau}}{\overline{\kappa}} \right)' = \pm \dfrac{\overline{c}}{\overline{b}}.$$

Thus we obtain (2.1). Conversely, we suppose that the condition (2.1) is satisfied. For the sake of simplicity, we assume that the first dual function in (2.1) is a dual constant, namely $\overline{c}$ (the other case is analogous). We define

(2.6) $\widetilde{U} = \dfrac{\overline{\tau}}{\sqrt{\overline{\tau}^2 - \overline{\kappa}^2}} \widetilde{T} + \overline{c} \widetilde{N} + \dfrac{\overline{\kappa}}{\sqrt{\overline{\tau}^2 - \overline{\kappa}^2}} \widetilde{B}$

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A differentiation of (2.6) together the dual Frenet equations gives $\widetilde{U}' = 0$, that is, $\widetilde{U}$ is a dual constant vector. Moreover $\langle \widetilde{N}, \widetilde{U} \rangle = \overline{c}$ and so $\tilde{\alpha}$ is a dual slant helix. Hence we prove this theorem.

Theorem 2.2. *Let $\tilde{\alpha}$ be a unit speed dual spacelike curve in D_1^3.*

(a) If the dual principle normal vector of the dual curve $\tilde{\alpha}$ whose dual curvature $\overline{\kappa}$ and dual torsion $\overline{\tau}$ are not pure dual, is spacelike, then $\tilde{\alpha}$ is a dual slant helix if and only if one the next two dual functions

$$(2.7) \qquad \frac{\overline{\kappa}^2}{\left(\overline{\tau}^2 - \overline{\kappa}^2\right)^{\frac{3}{2}}} \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)' \quad \text{or} \quad \frac{\overline{\kappa}^2}{\left(\overline{\kappa}^2 - \overline{\tau}^2\right)^{\frac{3}{2}}} \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$$

is dual constant where $\overline{\tau}^2 - \overline{\kappa}^2$ is not pure dual.

(b) If the dual principle normal vector of the dual curve $\tilde{\alpha}$ whose dual curvature $\overline{\kappa}$ and dual torsion $\overline{\tau}$ are not pure dual, is timelike, then $\tilde{\alpha}$ is a dual slant helix if and only if the dual function

$$\frac{\overline{\kappa}^2}{\left(\overline{\tau}^2 + \overline{\kappa}^2\right)^{\frac{3}{2}}} \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$$

is dual constant where $\overline{\tau}^2 + \overline{\kappa}^2$ is not pure dual.

(c) Any dual spacelike curve, whose dual curvature $\overline{\kappa}$ and dual torsion $\overline{\tau}$ are not pure dual, with dual lightlike principle normal vector is a dual slant helix.

Proof. Let $\tilde{\alpha}$ be a unit speed dual spacelike curve in D_1^3. In the case that the dual principle normal vector $\widetilde{N}$ of $\tilde{\alpha}$ is spacelike or timelike, the proof of Theorem 2.2 is similar to the given for Theorem 2.1.

Now we consider that the dual principle normal vector $\widetilde{N}$ of the dual curve is a lightlike vector. We show that any such dual curve is a dual slant helix. Let $\overline{a}_2$ be any non-trivial solution of the O.D.E.

$$\overline{y}' + \overline{\tau}\,\overline{y} = \overline{0}$$

and define $\widetilde{U} = \overline{a}_2 \widetilde{N}$. By using (1.5) and $\overline{a}_2' + \overline{a}_2 \overline{\tau} = \overline{0}$ so $\widetilde{U}' = 0$, that is, $\widetilde{U}$ is a non-zero dual constant vector field of D_1^3 and then $\langle \widetilde{N}, \widetilde{U} \rangle = \overline{0}$. It proves that $\tilde{\alpha}$ is a dual slant helix. □

Indicatries and Involutes of a Dual Slant Helices

In this section we study the dual tangent indicatrix, dual binormal indicatrix of a dual slant helix and its involutes. We restrict to non-null dual curves whose dual principle normal vector $\widetilde{N}$ is spacelike or timelike. Thus we are considering dual timelike curves or dual spacelike curves with dual spacelike or dual timelike principle normal vector. Given a unit speed dual curve $\tilde{\alpha} : I \to D_1^3$, the dual tangent indicatrix (resp. dual binormal indicatrix) is the dual curve $\widetilde{\mathbf{T}} : I \to D_1^3, \widetilde{\mathbf{T}}(t) = \widetilde{T}(t)$, (resp. the dual curve $\widetilde{\mathbf{B}} : I \to D_1^3, \widetilde{\mathbf{B}}(t) = \widetilde{B}(t)$), where $\widetilde{T}$ (resp. $\widetilde{B}$) is the dual tangent vector (resp. dual binormal vector) to $\tilde{\alpha}$. If the image of a dual curve lies in H_0^2 or S_1^2 we say that the curve is dual spherical. In particular, the dual tangent indicatrix and the dual binormal indicatrix are dual spherical.

We say that $\tilde{\alpha}$ is a dual helix in dual Lorentzian space D_1^3 if there exits a fixed direction $\widetilde{U}$ of D_1^3 such that the function $\langle \widetilde{T}, \widetilde{U} \rangle$ is a dual constant. Dual helices are characterized by the fact that the ratio $\frac{\overline{\tau}}{\overline{\kappa}}$ is a dual constant along the dual curve, where $\overline{\tau}$ and $\overline{\kappa}$ is not pure dual torsion and dual curvature of $\tilde{\alpha}$ respectively.

Theorem 3.1. *Let $\tilde{\alpha}$ be a unit speed dual timelike curve or a spacelike (with dual spacelike or timelike principle normal vector) curve. Let be dual curvature $\overline{\kappa}$, dual torsion $\overline{\tau}$, the dual functions $\overline{\tau}^2 - \overline{\kappa}^2$ and $\overline{\tau}^2 + \overline{\kappa}^2$ are not pure dual. If $\tilde{\alpha}$ is a dual slant helix in D_1^3, then the dual tangent indicatrix $\widetilde{\mathbf{T}}$ of $\tilde{\alpha}$ is a dual (spherical) helix.*

Proof. Denote the dual curvature and the dual torsion of $\widetilde{\mathbf{T}}$ by $\overline{\kappa}_{\widetilde{\mathbf{T}}}$ and $\overline{\tau}_{\widetilde{\mathbf{T}}}$ respectively. We will prove that the ratio $\overline{\tau}_{\widetilde{\mathbf{T}}}/\overline{\kappa}_{\widetilde{\mathbf{T}}}$ is dual constant. Let the dual tangent indicatrix $\widetilde{\mathbf{T}}$ be not dual arclength parametrized. In general and if $\tilde{\beta}(t)$ is non-parametrized by the dual arclength dual curve, the corresponding formulae of the dual curvature and the dual torsion are:

$$\overline{\kappa}_{\tilde{\beta}}^2(t) = \varepsilon \frac{\left\|\tilde{\beta}'(t)\right\|^2 \left\|\tilde{\beta}''(t)\right\|^2 - \left\langle \tilde{\beta}'(t), \tilde{\beta}''(t)\right\rangle^2}{\left\|\tilde{\beta}'(t)\right\|^6}, \quad \overline{\tau}_{\tilde{\beta}}(t) = -\varepsilon \frac{\det(\tilde{\beta}'(t), \tilde{\beta}''(t), \tilde{\beta}'''(t))}{\overline{\kappa}_{\tilde{\beta}}^2(t) \left\|\tilde{\beta}'(t)\right\|^6}$$

where ε is 1 or -1 depending on $\tilde{\beta}''(t)$ is a dual spacelike or dual timelike vector, respectively.

Consider that $\tilde{\alpha}$ is a dual spacelike curve with dual principle normal vector $\widetilde{N}$ spacelike or timelike. Denote $\varepsilon = 1$ or -1 depending on $\widetilde{N}$ is spacelike or timelike, respectively. Then the dual tangent indicatrix $\widetilde{\mathbf{T}}$ is a dual spacelike curve or a dual timelike curve. For both cases,

$$\overline{\kappa}_{\widetilde{\mathbf{T}}}^2 = \frac{1}{\overline{\kappa}^2}\left(\overline{\kappa}^2 - \varepsilon\overline{\tau}^2\right), \quad \det(\widetilde{\mathbf{T}}', \widetilde{\mathbf{T}}'', \widetilde{\mathbf{T}}''') = \varepsilon\overline{\kappa}^5\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)', \quad \overline{\tau}_{\widetilde{\mathbf{T}}} = \varepsilon\frac{\overline{\kappa}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'}{\overline{\kappa}^2 - \overline{\tau}^2}.$$

In the case that $\tilde{\alpha}$ is a dual timelike curve, then $\widetilde{\mathbf{T}}$ is a dual spacelike curve and

$$\overline{\kappa}_{\widetilde{\mathbf{T}}}^2 = -\frac{1}{\overline{\kappa}^2}\left(\overline{\kappa}^2 - \overline{\tau}^2\right), \quad \det(\widetilde{\mathbf{T}}', \widetilde{\mathbf{T}}'', \widetilde{\mathbf{T}}''') = -\overline{\kappa}^5\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)', \quad \overline{\tau}_{\widetilde{\mathbf{T}}} = \frac{\overline{\kappa}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'}{\overline{\kappa}^2 - \overline{\tau}^2}$$

Thus we obtain

$$\frac{\overline{\tau}_{\widetilde{\mathbf{T}}}}{\overline{\kappa}_{\widetilde{\mathbf{T}}}} = \varepsilon\frac{\overline{\kappa}^2\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'}{\left(\overline{\kappa}^2 - \varepsilon\overline{\tau}^2\right)^{\frac{3}{2}}} \quad \text{or} \quad \frac{\overline{\tau}_{\widetilde{\mathbf{T}}}}{\overline{\kappa}_{\widetilde{\mathbf{T}}}} = \frac{\overline{\kappa}^2\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'}{\left(\overline{\tau}^2 - \overline{\kappa}^2\right)^{\frac{3}{2}}}.$$

By considering Theorems 2.1 and 2.2, the ratio $\overline{\tau}_{\widetilde{\mathbf{T}}}/\overline{\kappa}_{\widetilde{\mathbf{T}}}$ is dual constant. Hence $\widetilde{\mathbf{T}}$ is a dual helix.

Theorem 3.2. *Let $\tilde{\alpha}$ be a unit speed dual timelike curve or a dual spacelike (with spacelike or timelike dual principle normal vector) curve. Let be dual curvature $\overline{\kappa}$, dual torsion $\overline{\tau}$, the dual functions $\overline{\tau}^2 - \overline{\kappa}^2$ and $\overline{\tau}^2 + \overline{\kappa}^2$ are not pure dual. If $\tilde{\alpha}$ is a dual slant helix in D_1^3, then the dual binormal indicatrix $\widetilde{\mathbf{B}}$ of $\tilde{\alpha}$ is a dual (spherical) helix.*

Proof. Denote $\overline{\kappa}_{\widetilde{\mathbf{B}}}$ and $\overline{\tau}_{\widetilde{\mathbf{B}}}$ the dual curvature and dual torsion of the dual curve $\widetilde{\mathbf{B}}$ respectively. Consider $\tilde{\alpha}$ a dual spacelike curve. Then the dual binormal indicatrix $\widetilde{\mathbf{B}}$ is a dual timelike or a dual spacelike curve, depending on $\widetilde{N}$ is timelike or spacelike, respectively.

$$\overline{\kappa}_{\widetilde{\mathbf{B}}}^2 = \frac{\overline{\kappa}^2 - \varepsilon\overline{\tau}^2}{\overline{\tau}^2}, \quad \det(\widetilde{\mathbf{B}}', \widetilde{\mathbf{B}}'', \widetilde{\mathbf{B}}''') = \varepsilon\overline{\kappa}^2\overline{\tau}^3\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)', \quad \overline{\tau}_{\widetilde{\mathbf{B}}} = \frac{\overline{\kappa}^2}{\overline{\tau}\left(\overline{\kappa}^2 - \varepsilon\overline{\tau}^2\right)}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'.$$

where ε is 1 or -1 depending on $\widetilde{N}$ is a spacelike or timelike, respectively.

If $\tilde{\alpha}$ is timelike, then $\widetilde{B}$ is a dual spacelike curve. We have

$$\overline{\kappa}_{\widetilde{B}}^2 = \frac{\overline{\tau}^2 - \overline{\kappa}^2}{\overline{\tau}^2}, \quad \det(\widetilde{\mathbf{B}}',\widetilde{\mathbf{B}}'',\widetilde{\mathbf{B}}''') = \overline{\kappa}^2 \overline{\tau}^3 \left(\frac{\overline{\tau}}{\overline{\kappa}}\right)', \quad \overline{\tau}_{\widetilde{B}} = \frac{\overline{\kappa}^2}{\overline{\tau}\left(\overline{\tau}^2 - \overline{\kappa}^2\right)}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'.$$

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Thus we obtain

$$\frac{\overline{\tau}_{\widetilde{B}}}{\overline{\kappa}_{\widetilde{B}}} = \delta \frac{\overline{\kappa}^2\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'}{\left(\overline{\kappa}^2 - \varepsilon\overline{\tau}^2\right)^{\frac{3}{2}}} \quad \text{or} \quad \frac{\overline{\tau}_{\widetilde{B}}}{\overline{\kappa}_{\widetilde{B}}} = \delta \frac{\overline{\kappa}^2\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'}{\left(\overline{\kappa}^2 - \overline{\tau}^2\right)^{\frac{3}{2}}},$$

where δ is 1 or -1 depending on $\overline{\tau}$ is positive or negative, respectively. By using (2.1), (2.7) and (2.8) in the above equations are dual constant, which proves that the dual binormal indicatrix $\widetilde{B}$ is a dual helix in D_1^3.

Now we give a characterization of a dual slant helix in terms of its involutes. Firstly, we define an involute of a dual curve. $\tilde{\alpha}: I \subset R \rightarrow D_1^3$ is a dual curve, an involute of $\tilde{\alpha}$ is a dual curve $\tilde{\beta}: I \rightarrow D_1^3$ such that for each $s \in I$ the point $\tilde{\beta}(s)$ lies on the dual tangent line to $\tilde{\alpha}$ at s and $\left\langle \tilde{\alpha}'(s), \tilde{\beta}'(s)\right\rangle = 0$. If $\tilde{\alpha}$ is a non-null dual curve, the equation of an involute is $\tilde{\beta}(s) = \tilde{\alpha}(s) + (\overline{c} - s)\widetilde{T}(s)$, where $\overline{c}$ is dual constant and $\widetilde{T}$ is the unit dual tangent vector of $\tilde{\alpha}$.

Theorem 3.3. *Let $\tilde{\alpha}$ be a unit speed dual timelike curve or a spacelike (with spacelike or timelike dual principle normal vector) curve. Let be dual curvature $\overline{\kappa}$, dual torsion $\overline{\tau}$, the dual functions $\overline{\tau}^2 - \overline{\kappa}^2$ and $\overline{\tau}^2 + \overline{\kappa}^2$ are not pure dual. Let $\tilde{\beta}$ be an involute of $\tilde{\alpha}$. Then $\tilde{\alpha}$ is a dual slant helix if and only if $\tilde{\beta}$ is a dual helix.*

Proof. We denote by $\overline{\kappa}_{\tilde{B}}$ and $\overline{\tau}_{\tilde{B}}$ the dual curvature and the dual torsion of $\tilde{\beta}$, respectively. If $\tilde{\alpha}$ is a dual timelike curve, then

(3.1) $$\overline{\kappa}_{\tilde{B}} = \frac{\overline{\tau}^2 - \overline{\kappa}^2}{\overline{\kappa}^2\left(\overline{c} - \overline{s}\right)^2}, \quad \overline{\tau}_{\tilde{B}} = -\frac{\overline{\kappa}}{\left(\overline{c} - \overline{s}\right)\left(\overline{\tau}^2 - \overline{\kappa}^2\right)}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$$

Thus we obtain

$$\frac{\overline{\tau}_{\tilde{B}}}{\overline{\kappa}_{\tilde{B}}} = -\frac{\overline{\kappa}^2}{\left(\overline{\tau}^2 - \overline{\kappa}^2\right)^{\frac{3}{2}}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'.$$

If $\tilde{\alpha}$ is a dual spacelike curve

(3.2) $$\overline{\kappa}_{\widetilde{B}} = \frac{\overline{\kappa}^2 - \varepsilon\overline{\tau}^2}{\overline{\kappa}^2\left(\overline{c} - \overline{s}\right)^2}, \quad \overline{\tau}_{\tilde{B}} = -\frac{\overline{\kappa}}{\left(\overline{c} - \overline{s}\right)\left(\overline{\kappa}^2 - \varepsilon\overline{\tau}^2\right)}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'$$

and

$$\frac{\overline{\tau}_{\tilde{B}}}{\overline{\kappa}_{\tilde{B}}} = \frac{\overline{\kappa}^2}{\left(\overline{\kappa}^2 - \varepsilon\overline{\tau}^2\right)^{\frac{3}{2}}}\left(\frac{\overline{\tau}}{\overline{\kappa}}\right)'.$$

Here ε is 1 or -1 depending if $\widetilde{N}$ is a spacelike or a timelike dual vector, respectively. The proof finishes using the equations (3.1), (3.2) and Theorem 2.1 and Theorem 2.2.

□

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