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An inertial parallel CQ subgradient extragradient method for variational inequalities application to signal-image recovery

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Abstract

In this paper, we introduce an inertial parallel CQ subgradient extragradient method for finding a common solutions of variational inequality problems. The novelty of this paper is using linesearch methods to find unknown L constant of L-Lipschitz continuous mappings. Strong convergence theorem has been proved under some suitable conditions in Hilbert spaces. Finally, we show applications to signal and image recovery, and show the good efficiency of our proposed algorithm when the number of subproblems is increasing.

Keywords: CQ algorithm Subgradient extragradient method Parallel algorithm Signal recovery Image restoration.
2010 MSC: 65K15; 68W10.

1. Introduction and Preliminaries

Let H be a real Hilbert space endowed with an inner product $\langle .,. \rangle$ and the induced norm $\|.\|$. A mapping A: $H \to H$ is said to be

(i) monotone if $\langle Ax - Ay, x - y \rangle \geq 0$ for all $x, y \in H$;

(ii) maximal monotone if it is monotone and its graph

$$G(A) := \{(x, Ax) : x \in H\}$$

is not a proper subset of one of any other monotone mapping;

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(*iii*) L-Lipchitz continuous if there exists a positive constant L such that

$$\|Ax - Ay\| \leq L\|Ax - Ay\| \text{ for all } x, y \in H.$$

It is well-known that a monotone mapping $A : H \to H$ is maximal if and only if for each $(x, y) \in H \times H$ such that $\langle x - u, y - v\rangle \geq 0$ for all $(u, v) \in G(A)$, it follows that $y = Ax$. Let C be nonempty closed convex subset of H and $A : H \to H$ is a nonlinear operator. The variational inequality problem (VIP) can be formulated as the problem of finding a point $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0, \forall x \in C. \tag{1}$$

The set of solutions of VIP (1) is denoted by $VI(A, C)$. However, the convergence of this method requires slightly strong assumptions that operators are strongly monotone or inverse strongly monotone. Many algorithms have been proposed and studied for solving VIP(1) of these algorithms involve projection methods [5, 6, 10, 11, 39, 40, 43, 46, 47, 51]. The VIP(1) serves as a powerful mathematical tool and generalizes many mathematical methods, in the sense that, it includes many special problems [29] such as convex feasibility problems, linear programming problem, minimizer problem, saddle - point problems, Hierarchical variational inequality problems. It is well known that VI(C,A) is equivalent to the following fixed point equation (see [2, 3, 4, 16, 17, 44, 19, 21, 23, 26, 29, 31, 32, 33]), $x = P_C(x - \lambda Ax), \lambda > 0$ and $r_\lambda(x) := x - P_C(x - \lambda Ax) = 0$. By using the idea of the projection method, Korelevich [24] proposed the extragradient method for solving the VIP(1) under the assumptions of Lipschitz continuous and pseudo-monotone of the operator. In this method, if a closed convex set has a simple structure, then the projections onto it can be discovered easily, the extragradient method is computable and very useful. However, we have to use the projection onto C into two times in the extragradient method to obtain the next approximation x_{n+1} over each iteration.

Later on, Censor et al. [8] proposed the subgradient extragradient method for sloving VIP (1). The second projection onto the closed convex set of the extragradient method was replaced by the projection onto a half Space. Censor et al. [7] used the hybrid method with subgradient extragradient method for obtaining the strong convergence result. This algorithm is defined as follows:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda Ax_n), \\ z_n = \alpha_n x_n + (1 - \alpha_n)P_{T_n}x_n, \\ C_n = \{z \in H : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0. \end{cases} \tag{2}$$

Recently, Gibali [15] suggested a self-adaptive subgradient extragradient method by adopting Armijo-like searches [52] and obtained convergence result for VI(A,C) in $\mathbb{R}^n$ when the pseudo-monotonicity and continuity of the operator are required.

Very recently, Shehu and Iyiola [34] proposed the modified viscosity algorithm with adoption of Armijo-line step size rule which is called viscosity type subgradient extragradient line method for a Lipschitz continuous monotone mapping that the Lipschitz constant is unknown in an infinite dimensional Hilbert space. This method is defined as follow:

$$\begin{cases} x_0 \in H, \\ y_n = P_C(x_n - \lambda_n Ax_n), \lambda_n = \rho^{l_n} \\ (l_n \text{ is the smallest nonnegative integer } l \\ \ \text{such that } \lambda_n\|Ax_n - Ay_n\| \leq \mu\|r_{\rho^l}(x_n)\|), \\ z_n = P_{T_n}(x_n - \lambda_n Ay_n), \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)z_n, n \geq 1, \end{cases} \tag{3}$$

where $T_n := \{z \in H : \langle x_n - \lambda_n A x_n - y_n, z - y_n\rangle \le 0\}, \rho, \mu \in (0,1)$ and $\{\alpha_n\} \subseteq (0,1)$.

Our interest in this paper is to study the common variational inequality problems (CVIP). The CVIP is to find $x^* \in C$ such that

$$\langle A_i x^*, x - x^*\rangle \ge 0, \forall x \in C,\ i = 1, ..., N, \tag{4}$$

where $A_i : H \to H$ is a nonlinear operator for all $i = 1, 2, ..., N$.

In 2012, Censor et al. [9] presented the algorithm for solving the CVIP (4) here, finite elements are computed in parallel of each iterations. The closed convex subset $C_n^1, C_n^2, ..., C_n^N$ are constructed getting x_{n+1} which is projected onto the intersection of these closed convex subset. This algorithm is generated by $x_1 \in H$ and compute

$$\begin{cases} y_n^i = P_{K_i}(x_n - \lambda_n^i A_i x_n), \\ z_n^i = P_{K_i}(x_n - \lambda_n^i A_i y_n^i), \\ C_n^i = \{z \in H : \langle x_n - z_n^i, z - x_n - \gamma_n^i (z_n^i - x_n)\rangle \le 0\}, \\ C_n = \bigcap_{i=1}^{N} C_n^i, \\ W_n = \{z \in H : \langle x_1 - x_n, z - x_n\rangle \le 0\}, \\ x_{n+1} = P_{C_n \cap W_n} x_1. \end{cases} \tag{5}$$

This method has been extensively used due to its simplicity many authors improved it in various ways (see [14, 18, 20, 25, 31, 35, 36, 37, 48, 49, 50]).

Inspired by the previous results, we introduce the new algorithm by modifying the hybrid subgradient extragradient method combining inertial technique with adoption of Armijo-line step size rule and projection onto the set of intersection sets of half-spaces to find common solution of variational inequality problems (CVIP). We prove strong convergence theorem under some suitable conditions in Hilbert spaces. Moreover, we apply our main results in image and signal recovery problems.

2. Main Result

In this section, we introduce an inertial parallel CQ subgradient extragradient method for variational inequalities and prove the convergence theorem of the algorithms. Let $A_i : H \to H$ be a family of L_i -Lipschitz continuous for all $i = 1, 2, ..., N$ with $F = \cap_{i=1}^{N} VI(A_i, C_i) \neq \phi$. The algorithm is generated as follow:

Algorithm 2.1. *(Inertial parallel CQ subgradient extragradient method)*
Initialization: *Take $\rho > 0, \mu \in (0,1), \theta \in [0,1)$ and $\{\theta_n\} \subseteq [0, \theta]$. Select arbitrary points $x_0, x_1 \in H$. For i = 1, 2, ..., N set $n := 1$*
Step 1. *Compute s_n,*

$$s_n = x_n + \theta_n (x_n - x_{n-1}).$$

Step 2. *Compute y_n,*

$$y_n^i = P_C(s_n - \lambda_n^i A_i s_n),$$

where $\lambda_n^i = \rho^{l^i}$ and l^i is the smallest nonnegative integer such that

$$\rho^{l^i} \|A_i s_n - A_i y_n^i\| \le \mu \|s_n - y_n^i\|. \tag{6}$$

Step 3. *Compute z_n^i,*

$$z_n^i = P_{T_n^i}(s_n - \lambda_n^i A_i y_n^i),\ i = 1, ..., N,$$

where $T_n^i = \{v \in H : \langle s_n - \lambda_n^i A_i s_n - y_n^i, v - y_n^i\rangle \le 0\}$.
Step 4. *Compute $\bar{z}_n$, i.e.,*

$$\bar{z}_n = argmax\{\|z_n^i - s_n\| : i = 1, ..., N\}.$$

Step 5. *Compute* $x_{n+1} = P_{C_n \cap Q_n} x_1$, *where*

$$C_n = \{\upsilon \in H : \|\bar{z}_n - \upsilon\| \leq \|s_n - \upsilon\|\},$$

and

$$Q_n = \{\upsilon \in H : \langle \upsilon - x_n, x_n - x_0 \rangle \geq 0\}.$$

Step 6. *Set* $n := n + 1$ *and back to* ***Step 1.***

Lemma 2.2. *For all* $i = 1, 2, ..., N$, *there exists a nonnegative integer* l^i *satisfying (6).*

Proof. Suppose $\|s_n - y^i_{n_0}\| = 0$ for some $n_0 \geq 1$. Take $l^i = n_0$, which satisfies (6). Suppose that $\|s_n - y^i_{n_1}\| \neq 0$ for some $n_1 \geq 1$ and assume the contrary that $\rho^{n_1}\|A_i s_n - A_i y^i_{n_1}\| > \mu\|s_n - y^i_{n_1}\|$. Then, by Lemma 6.3 of [12] and the fact that $\rho \in (0, 1)$, we obtain

$$\begin{aligned} \|A_i s_n - A_i y^i_{n_1}\| &> \frac{\mu}{\rho^{n_1}}\|s_n - y^i_{n_1}\| \\ &\geq \frac{\mu}{\rho^{n_1}} \min\{1, \rho^{n_1}\}\|s_n - y^i_1\| \\ &= \mu\|s_n - y^i_1\|. \end{aligned} \tag{7}$$

Using the fact that P_C is continuous, we have that for all $i = 1, 2, ..., N$,

$$y^i_{n_1} = P_C(s_n - \rho^{n_1} A_i s_n) \to P_C(s_n), n_1 \to \infty.$$

We consider two cases: $s_n \in C$ and $s_n \notin C$.

(i) If $s_n \in C$, then $s_n = P_C(s_n)$. Now, since $\|s_n - y^i_{n_1}\| \neq 0$ and $\rho^{n_1} \leq 1$, it follows from Lemma 6.3 of [12] that

$$\begin{aligned} 0 &< \|s_n - y^i_{n_1}\| \\ &\leq \max\{1, \rho^{n_1}\}\|s_n - y^i_1\| \\ &= \|s_n - y^i_1\|. \end{aligned}$$

Letting $n_1 \to \infty$ in (7), we have that

$$0 = \|A_i s_n - A_i s_n\| \geq \mu\|s_n - y^i_1\| > 0.$$

This is a contradiction and hence (6) is valid.

(ii) If $s_n \notin C$, then

$$\rho^i_{n_1}\|A_i s_n - A_i y^i_{n_1}\| \to 0, n_1 \to \infty.$$

while

$$\begin{aligned} \lim_{n_1 \to \infty} \mu\|s_n - P_C(s_n - \rho^{n_1} A_i s_n)\| &= \mu \lim_{n_1 \to \infty} \|s_n - P_C(s_n - \rho^{n_1} A_i s_n)\| \\ &= \mu\|s_n - P_C(s_n)\| > 0. \end{aligned}$$

This is a contradiction. Therefore, Algorithm 2.1 is well defined and implementable.

□

Lemma 2.3. *Suppose that $x^* \in F$ and the sequences $\{y_n^i\}, \{z_n^i\}$ generated by Step 1 and Step 2 of Algorithm 2.1. Then*

$$\begin{aligned}\|z_n^i - x^*\|^2 \le \|x_n - x^*\|^2 + (1+c)2\theta_n\langle x_n - x_{n-1}, y_n^i - x^*\rangle \\ - c\Big(\|x_n - y_n^i\|^2 + \|z_n^i - y_n^i\|^2\Big),\end{aligned} \tag{8}$$

where $c = 1 - \mu > 0$.

Proof. Let $x^* \in F$. For each i = 1,2,...N, let $u_n^i = s_n - \lambda_n^i A_i y_n^i, \forall n \ge 1$, we have

$$\begin{aligned}\|z_n^i - x^*\|^2 &= \|P_{T_n^i}(s_n - \lambda_n^i A_i y_n^i) - x^*\|^2 \\ &= \|P_{T_n^i}(u_n^i) - x^*\|^2 \\ &= \|(P_{T_n^i}(u_n^i) - u_n^i) + (u_n^i - x^*)\|^2 \\ &= \|u_n^i - x^*\|^2 + \|u_n^i - P_{T_n^i}(u_n^i)\|^2 + 2\langle P_{T_n^i}(u_n^i) - u_n^i, u_n^i - x^*\rangle\end{aligned} \tag{9}$$

since $x^* \in F \subseteq C \subseteq T_n^i$ by the property of the metric projection $P_{T_n^i}$, we derive

$$\begin{aligned}&2\|u_n^i - P_{T_n^i}(u_n^i)\|^2 + 2\langle P_{T_n^i}(u_n^i) - u_n^i, u_n^i - x^*\rangle \\ &\quad = 2\langle u_n^i - P_{T_n^i}(u_n^i), x^* - P_{T_n^i}(u_n^i)\rangle \le 0\end{aligned} \tag{10}$$

and

$$\|u_n^i - P_{T_n^i}(u_n^i)\|^2 + 2\langle P_{T_n^i}(u_n^i) - u_n^i, u_n^i - x^*\rangle \le -\|u_n^i - P_{T_n^i}(u_n^i)\|^2. \tag{11}$$

We then obtain from Algorithm 2.1 and Lemma 2.3 (ii) of [42] that

$$\begin{aligned}\|z_n^i - x^*\|^2 &\le \|u_n^i - x^*\|^2 - \|u_n^i - P_{T_n^i}(u_n^i)\|^2 \\ &= \|(s_n - \lambda_n^i A_i y_n^i) - x^*\|^2 - \|(s_n - \lambda_n^i A_i y_n^i) - z_n^i\|^2 \\ &= \|s_n - x^*\|^2 - \|s_n - z_n^i\|^2 + 2\lambda_n^i\langle x^* - z_n^i, A_i y_n^i\rangle.\end{aligned} \tag{12}$$

Since A_i is the monotone operator for all i = 1,2,...,N, we have

$$\begin{aligned}0 &\le \langle A_i y_n^i - A_i x^*, y_n^i - x^*\rangle \\ &= \langle A_i y_n^i, y_n^i - x^*\rangle - \langle A_i x^*, y_n^i - x^*\rangle \\ &\le \langle A_i y_n^i, y_n^i - x^*\rangle \\ &= \langle A_i y_n^i, y_n^i - z_n^i + z_n^i - x^*\rangle \\ &= \langle A_i y_n^i, y_n^i - z_n^i\rangle + \langle A_i y_n^i, z_n^i - x^*\rangle.\end{aligned}$$

Thus,

$$\langle x^* - z_n^i, A_i y_n^i, \rangle \le \langle A_i y_n^i, y_n^i - z_n^i\rangle. \tag{13}$$

Using (12) in (13), we obtain

$$\begin{aligned}\|z_n^i - x^*\|^2 &\le \|s_n - x^*\|^2 - \|s_n - z_n^i\|^2 + 2\lambda_n^i\langle A_i y_n^i, y_n^i - z_n^i\rangle \\ &= \|s_n - x^*\|^2 + 2\lambda_n^i\langle A_i y_n^i, y_n^i - z_n^i\rangle - \|s_n - y_n^i + y_n^i - z_n^i\|^2 \\ &= \|s_n - x^*\|^2 + 2\lambda_n^i\langle A_i y_n^i, y_n^i - z_n^i\rangle - 2\langle s_n - y_n^i, y_n^i - z_n^i\rangle \\ &\quad - \|s_n - y_n^i\|^2 - \|y_n^i - z_n^i\|^2 \\ &= \|s_n - x^*\|^2 + 2\langle s_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i\rangle - \|s_n - y_n^i\|^2 \\ &\quad - \|y_n^i - z_n^i\|^2.\end{aligned} \tag{14}$$

Observe that

$$\begin{aligned}\langle s_n - \lambda_n^i A_i y_n^i - y_n^i, z_n^i - y_n^i\rangle &= \langle s_n - \lambda_n^i A_i s_n - y_n^i, z_n - y_n^i\rangle \\ &\quad + \langle \lambda_n^i A_i s_n - \lambda_n^i A_i y_n^i, z_n^i - y_n^i\rangle \\ &\leq \langle \lambda_n^i A_i s_n - \lambda_n^i A_i y_n^i, z_n^i - y_n^i\rangle.\end{aligned}$$

Using the last inequality in (14), we have that

$$\begin{aligned}\|z_n^i - x^*\|^2 &\leq \|s_n - x^*\|^2 + 2\langle \lambda_n^i A_i s_n - \lambda_n^i A_i y_n^i, z_n^i - y_n^i\rangle - \|s_n - y_n^i\|^2 \\ &\quad - \|y_n^i - z_n^i\|^2 \\ &= \|s_n - x^*\|^2 + 2\lambda_n^i \langle A_i s_n - A_i y_n^i, z_n^i - y_n^i\rangle - \|s_n - y_n^i\|^2 \\ &\quad - \|y_n^i - z_n^i\|^2 \\ &\leq \|s_n - x^*\|^2 + 2\lambda_n^i \|A_i s_n - A_i y_n^i\|\|z_n^i - y_n^i\| - \|s_n - y_n^i\|^2 \\ &\quad - \|y_n^i - z_n^i\|^2 \\ &\leq \|s_n - x^*\|^2 + 2\mu \|s_n - y_n^i\|\|z_n^i - y_n^i\| - \|s_n - y_n^i\|^2 \\ &\quad - \|y_n^i - z_n^i\|^2 \\ &\leq \|s_n - x^*\|^2 + \mu\Big(\|s_n - y_n^i\|^2 + \|z_n^i - y_n^i\|^2\Big) - \|s_n - y_n^i\|^2 \\ &\quad - \|y_n^i - z_n^i\|^2 \\ &\leq \|s_n - x^*\|^2 + \Big(\mu\|s_n - y_n^i\|^2 - \|s_n - y_n^i\|^2\Big) \\ &\quad + \Big(\mu\|z_n^i - y_n^i\|^2 - \|y_n^i - z_n^i\|^2\Big) \\ &= \|s_n - x^*\|^2 - (1-\mu)\|s_n - y_n^i\|^2 - (1-\mu)\|y_n^i - z_n^i\|^2 \\ &\leq \|s_n - x^*\|^2 - (1-\mu)\Big(\|s_n - y_n^i\|^2 + \|y_n^i - z_n^i\|^2\Big) \\ &\leq \|s_n - x^*\|^2 - c\ \Big(\|s_n - y_n^i\|^2 + \|y_n^i - z_n^i\|^2\Big). \end{aligned} \tag{15}$$

From (15) and $s_n = x_n + \theta_n(x_n - x_{n-1})$, we have

$$\begin{aligned}\|z_n^i - x^*\|^2 &\leq \|(x_n + \theta_n(x_n - x_{n-1})) - x^*\|^2 \\ &\quad - c\ \Big(\|(x_n + \theta_n(x_n - x_{n-1})) - y_n^i\|^2 + \|y_n^i - z_n^i\|^2\Big) \\ &\leq \|(x_n - x^*) + \theta_n(x_n - x_{n-1})\|^2 \\ &\quad - c\ \Big(\|(x_n - y_n^i) + \theta_n(x_n - x_{n-1})\|^2 + \|y_n^i - z_n^i\|^2\Big) \\ &\leq \|x_n - x^*\|^2 + 2\langle \theta_n(x_n - x_{n-1}), x_n - x^* + \theta_n(x_n - x_{n-1})\rangle \\ &\quad - c\ \Big(\|x_n - y_n^i\|^2 + 2\langle \theta_n(x_n - x_{n-1}), x_n - y_n^i + \theta_n(x_n - x_{n-1})\rangle \\ &\quad + \|y_n^i - z_n^i\|^2\Big) \\ &\leq \|x_n - x^*\|^2 + 2\theta_n\langle x_n - x_{n-1}, s_n - x^*\rangle \\ &\quad - c\ \Big(\|x_n - y_n^i\|^2 + 2\theta_n\langle x_n - x_{n-1}, s_n - y_n^i\rangle + \|y_n^i - z_n^i\|^2\Big) \\ &\leq \|x_n - x^*\|^2 - c\ \Big(\|x_n - y_n^i\|^2 + \|y_n^i - z_n^i\|^2\Big) \\ &\quad + 2\theta_n\langle x_n - x_{n-1}, s_n - x^*\rangle + 2\theta_n c\langle x_n - x_{n-1}, s_n - y_n^i\rangle \\ &\leq \|x_n - x^*\|^2 - c\ \Big(\|x_n - y_n^i\|^2 + \|y_n^i - z_n^i\|^2\Big) \\ &\quad + (1+c)2\theta_n\langle x_n - x_{n-1}, y_n^i - x^*\rangle. \end{aligned} \tag{16}$$

From (12) and (16), we obtain inequality (8). □

Lemma 2.4. *Suppose that $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ generated by Algorithm 2.1. Then*

(i) $F \subset C_n \cap Q_n$ and x_{n+1} is well-defined for all $n \geq 0$.

(ii) If $\Sigma\theta_n\|x_n - x_{n-1}\| < \infty$, then for each $i = 1, ..., N$, the following relations hold:

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = \lim_{n\to\infty} \|y_n^i - x_n\| = \lim_{n\to\infty} \|z_n^i - x_n\| = \lim_{n\to\infty} \|s_n - y_n^i\| = 0.$$

Proof. (i) Since A_i is Lipschitz continuous, A_i is continuous. Thus, Lemma 2.1 of [38] ensures that $VI(A_i, C)$ is closed and convex for all $i = 1, ..., N$. Hence, F is closed and convex. From the definitions of C_n and Q_n, we see that Q_n is closed and convex and C_n is closed. On the other hand, the relation $\|\bar{z}_n - \upsilon\| \leq \|s_n - \upsilon\|$ is equivalent to

$$2\langle \upsilon, s_n - \bar{z}_n\rangle \leq \|s_n\|^2 - \|\bar{z}_n\|^2.$$

This implies that C_n is convex. Moreover, for each $u \in F$, from Lemma 2.3, we obtain $\|\bar{z}_n - u\| \leq \|s_n - u\|$. Thus, $F \subset C_n$ for all $n \geq 1$. Next, we will show that $F \subset C_n \cap Q_n$ by the induction. Indeed, $F \subset Q_n$ and so $F \subset C_n \cap Q_n$. Assume that $F \subset C_n \cap Q_n$ for some $n \geq 1$. From $x_{n+1} = P_{C_n\cap Q_n}x_1$ and the characterization of the metric projection by Lemma 2.3 (iii) of [42], we obtain

$$\langle \upsilon - x_{n+1}, x_{n+1} - x_1\rangle \geq 0, \quad \forall \upsilon \in C_n \cap Q_n.$$

Since $F \subset C_n \cap Q_n, \langle \upsilon - x_{n+1}, x_{n+1} - x_1\rangle \geq 0$ for all $\upsilon \in F$. This together with the definition of Q_{n+1} implies that $F \subset Q_{n+1}$. Thus, by the induction $F \subset C_n \cap Q_n$ for all $n \geq 1$. Since $F \neq \phi, P_F x_1$ and $x_{n+1} = P_{C_n\cap Q_n}x_1$ are well defined.
(ii). We have $x_n = P_{Q_n}x_1$ and $F \subset Q_n$. For each $u \in F$, by the property of the projection P_{Q_n} we have

$$\|x_n - x_1\| \leq \|u - x_1\|, \quad \forall n \geq 0. \tag{17}$$

Thus, the sequence $\{\|x_n - x_1\|\}$ is bouned and so $\{x_n\}$ is also bounded. From $x_{n+1} \in Q_n$ and $x_n = P_{Q_n}x_1$, we also obtain

$$\|x_n - x_1\| \leq \|x_{n+1} - x_1\|, \quad \forall n \geq 0. \tag{18}$$

This implies that the sequence $\{\|x_n - x_1\|\}$ is nondecreasing. $\lim_{n\to\infty} \|x_n - x_1\|$ exists. It follows from $x_{n+1} \in Q_n$ and $x_n = P_{Q_n}x_1$, that

$$\|x_n - x_{n+1}\|^2 \leq \|x_{n+1} - x_1\|^2 - \|x_n - x_1\|^2.$$

From this inequality, taking $n \to \infty$, we get

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{19}$$

By the definition of C_n and $x_{n+1} \in C_n$, we have

$$\begin{aligned} \|\bar{z}_n - x_{n+1}\| &\leq \|x_{n+1} - s_n\| \\ &\leq \|x_{n+1} - x_n\| + \theta_n\|x_n - x_{n-1}\|. \end{aligned} \tag{20}$$

From the definition of $\{\theta_n\}$ in Step 1 and (19) we have

$$\lim_{n\to\infty} \|\bar{z}_n - x_{n+1}\| = 0.$$

This together with the triangle inequality $\|\bar{z}_n - x_n\| \leq \|\bar{z}_n - x_{n+1}\| + \|x_{n+1} - x_n\|$ implies that

$$\lim_{n\to\infty} \|\bar{z}_n - x_n\| = 0. \tag{21}$$

From (21) and the definition of i_n, we get

$$\lim_{n\to\infty} \|z_n^i - x_n\| = 0, \quad \forall i = 1, ..., N. \tag{22}$$

From Lemma 2.3 and the triangle inequality, for each $u \in F$, one has

$$c\, \|x_n - y_n^i\|^2 \leq \|x_n - u\|^2 - \|z_n^i - u\|^2 + (2-\mu)2\theta_n \langle x_n - x_{n-1}, y_n^i - u\rangle \tag{23}$$

From (22), (25) and the boundedness of $\{s_n\}, \{x_n\}, \{y_n^i\}, \{z_n^i\}$ and the condition $\Sigma\theta_n\|x_n - x_{n-1}\| < \infty$, we get

$$\lim_{n\to\infty} \|y_n^i - x_n\| = 0, \quad i = 1, ..., N. \tag{24}$$

From (15), we have

$$\begin{aligned} c\, \|s_n - y_n^i\|^2 &\leq \|s_n - x^*\|^2 - \|z_n^i - x^*\|^2 \\ &= \|(x_n - x^*) + \theta_n(x_n - x_{n-1})\|^2 - \|z_n^i - x^*\|^2 \\ &= \|x_n - x^*\|^2 + 2\theta_n\langle x_n - x_{n-1}, s_n - x^*\rangle - \|z_n^i - x^*\|^2. \end{aligned} \tag{25}$$

From the condition $\Sigma\theta_n\|x_n - x_{n-1}\| < \infty$ and (22), we get

$$\lim_{n\to\infty} \|s_n - y_n^i\| = 0 \tag{26}$$

for all i = 1,...,N. □

Theorem 2.5. *Let C be a closed and convex subset of a real Hilbert space H. Suppose that $\{A_i\}_{i=1}^N : H \to H$ is a finite family of monotone mappings. In addition, the solution set F is nonempty and $\Sigma\theta_n\|x_n - x_{n-1}\| < \infty$. Then, the sequences $\{x_n\}, \{y_n^i\}, \{z_n^i\}$ generated by Algorithm 2.1 converge strongly to $P_F x_1$.*

Proof. By Lemma 2.4, F, C_n, Q_n are nonempty closed and convex subsets. Besides, $F \subset C_n \cap Q_n$ for all $n \geq 1$. Therefore, $P_F x_1, P_{C_n\cap Q_n} x_1$ are well-defined. From Lemma 2.4, $\{x_n\}$ is bounded. Assume that p is a weak cluster point of $\{x_n\}$ and $\{x_{n_k}\}$ is subsequence of $\{x_n\}$ converging weakly to p. Since $\|y_{n_k}^i - x_{n_k}\| \to 0, y_{n_k}^i \rightharpoonup p$. Now we prove that $p \in F$. Indeed, Lemma 2.3 of [42] , ensures that the mapping

$$Q_i x = \begin{cases} A_i x + N_C(x) & if\ x \in C, \\ \emptyset & if\ x \notin C, \end{cases}$$

is maximal monotone, where $N_C(x)$ is the normal cone to C at $x \in C$. For all (x, y) in the graph of Q_i , i.e., $(x, y) \in G(Q_i)$, we have $y - A_i x \in N_C(x)$. By the definition of $N_C(x)$, we find that

$$\langle x - z, y - A_i x\rangle \geq 0$$

for all $z \in C$. Since $y_{n_k}^i \in C$,

$$\langle x - y_{n_k}^i, y - A_i x\rangle \geq 0.$$

Therefore,

$$\langle x - y_{n_k}^i, y\rangle \geq \langle x - y_{n_k}^i, A_i x\rangle. \tag{27}$$

Taking into account $y^i_{n_k} = P_C(s_{n_k} - \lambda^i_{n_k} A_i s_{n_k})$ and Lemma 6.6 of [1], we get

$$\langle x - y^i_{n_k}, y^i_{n_k} - s_{n_k} + \lambda^i_{n_k} A_i s_{n_k}\rangle \geq 0$$
$$\langle x - y^i_{n_k}, \frac{y^i_{n_k} - s_n}{\lambda^i_{n_k}} + A_i s_{n_k}\rangle \geq 0$$

or

$$\langle x - y^i_{n_k}, A_i s_{n_k}\rangle \geq \langle x - y^i_{n_k}, \frac{s_{n_k} - y^i_{n_k}}{\lambda^i_{n_k}}\rangle \tag{28}$$

Therefore, from (27), (28) and the monotonicity of A_i , we find that

$$\begin{aligned}
\langle x - y^i_{n_k}, y\rangle &\geq \langle x - y^i_{n_k}, A_i x\rangle \\
&= \langle x - y^i_{n_k}, A_i x - A_i y^i_{n_k}\rangle + \langle x - y^i_{n_k}, A_i y^i_{n_k} - A_i s_{n_k}\rangle \\
&\quad + \langle x - y^i_{n_k}, A_i s_{n_k}\rangle \\
&= \langle x - y^i_{n_k}, A_i y^i_{n_k} - A_i s_{n_k}\rangle + \langle x - y^i_{n_k}, A_i s_{n_k}\rangle \\
&\geq \langle x - y^i_{n_k}, A_i y^i_{n_k} - A_i s_{n_k}\rangle + \langle x - y^i_{n_k}, \frac{s_{n_k} - y^i_{n_k}}{\lambda^i_n}\rangle.
\end{aligned} \tag{29}$$

Since $\|y^i_n - s_n\| \to 0$ and A_i is L-Lipschitz continuous,

$$\lim_{n\to\infty} \|A_i y^i_n - A_i s_n\| = 0. \tag{30}$$

Passing the limit in (29) as $k \to \infty$ and using (30), $y^i_{n_k} \rightharpoonup p$, we obtain $\langle x - p, y\rangle \geq 0$ for all $(x, y) \in G(Q_i)$. This together with the maximal monotonicity of Q_i implies that $p \in Q_i^{-1} 0 = VI(A_i, F)$ for all $1 \leq i \leq N$. Hence, $p \in F$.
Finally, we show that $x_n \to p = x^\dagger := P_F x_1$. From (18) and $x \in F$, we have

$$\|x_n - x_1\| \geq \|x^\dagger - x_1\|, \quad \forall n \geq 0.$$

This relation together with the lower weak semi-continuity of the norm implies that

$$\|x^\dagger - x_1\| \leq \|p - x_1\| \leq \liminf_{k\to\infty} \|s_{n_k} - x_1\| \leq \limsup_{k\to\infty} \|s_{n_k} - x_1\| \leq \|x^\dagger - x_1\|.$$

By the definition of $x^\dagger, p = x^\dagger$ and $\lim_{n\to\infty} \|x_{n_k} - x_1\| = \|x^\dagger - x_1\|$. Thus, from $x_{n_k} - x_1 \rightharpoonup x^\dagger - x_1$ and Lemma (Kadec-Klee) we obtain $x_{n_k} - x_1 \to x^\dagger - x_1$, and so $x_{n_k} \to x^\dagger$. Lemma 2.3 ensures that the sequences $\{y^i_n\}, \{z^i_n\}$ also converge strongly to $P_F x_1$. □

3. Application to Signal Recovery

Signal processing is analysis, modifying, and synthesizing signals. We can use signal processing techniques for improving transmission, storage efficiency and subjective quality and also emphasizing or defecting components of interest in a measured signal. Signal processing problem can be modeled as the following under determinate linear equation system $b = Bx+\nu$ where x is a original signal with N components to be recovered $(x \in \mathbb{R}^N), \nu, b$ are noise and the observed signal with noisy for M components respectively $(\nu, b \in \mathbb{R}^M)$ and $B : \mathbb{R}^N \to \mathbb{R}^M (M \leq N)$ is a filtering. Finding the solutions of $b = Bx + \nu$ can be seen as solving least squares (LS) problem

$$\min_{x\in\mathbb{R}^N} \frac{1}{2}\|b - Bx\|_2^2 \tag{31}$$

where $\| \,.\, \|$ is l_2-norm defined by $\| x \|= \sqrt{\sum_{i=1}^{n}|x_i|^2}$. The solution of (31) can be estimated by many well known iteration methods [13, 45]. Many algorithms based on optimization have been proposed for solving signal recovery problems 31, see in [22, 27, 28]

In the real, the observation of signal may be disturbed by some filters and noises. The goal in this paper is to find the original signal without knowing the type of filter and noise. Thus, we can consider this problem in the following problem system.

$$\min_{x\in\mathbb{R}^N}\frac{1}{2}\|B_1x-b_1\|_2^2,\ \min_{x\in\mathbb{R}^N}\frac{1}{2}\|B_2x-b_2\|_2^2, ...,\ \min_{x\in\mathbb{R}^N}\frac{1}{2}\|B_Nx-b_N\|_2^2, \tag{32}$$

where x is an original signal, B_i is a bounded linear operator and b_i is an observed signal with noisy for all $i = 1, 2, ..., N$. We can apply the Algorithm 2.1 to solve the problem (32) by setting $A_ix = B_i^T(B_ix - b_i)$ for all $i = 1, 2, ..., N$ and $C = \mathbb{R}^N$.

Algorithm 3.1. ***Initialization:*** *Take $\rho > 0, \mu \in (0, 1), \theta \in [0, 1)$ and $\{\theta_n\} \subseteq [0, \theta]$. Select arbitrary points $x_0, x_1 \in H$. For i = 1, 2, ..., N set $n := 1$*

Step 1. *Compute S_n,*

$$s_n = x_n + \theta_n(x_n - x_{n-1}).$$

Step 2. *Compute y_n,*

$$y_n^i = P_C(s_n - \lambda_n^i B_i^T(B_is_n - b_i)),$$

where $\lambda_n^i = \rho^{l^i}$ and l^i is the smallest nonegative integer such that

$$\rho^{l^i}\|B_i^T(B_i(s_n - y_n^i)\| \le \mu\|s_n - y_n^i\|.$$

Step 3. *Compute z_n^i,*

$$z_n^i = P_{T_n^i}(s_n - \lambda_n^i B_i^T(B_iy_n^i - b_i)),\ i = 1, ..., N,$$

where $T_n^i = \{v \in H : \langle s_n - \lambda_n^i B_i^T(B_is_n - b_i)) - y_n^i, v - y_n^i\rangle \le 0\}$.

Step 4. *Compute $\overline{z}_n$, i.e.,*

$$\bar{z}_n = argmax\{\|z_n^i - s_n\| : i = 1, ..., N\}.$$

Step 5. *Compute $x_{n+1} = P_{C_n\cap Q_n}x_1$,where*

$$C_n = \{v \in H : \|\bar{z}_n - v\| \le \|s_n - v\|\},$$

and

$$Q_n = \{v \in H : \langle v - x_n, x_n - x_0\rangle \ge 0\}.$$

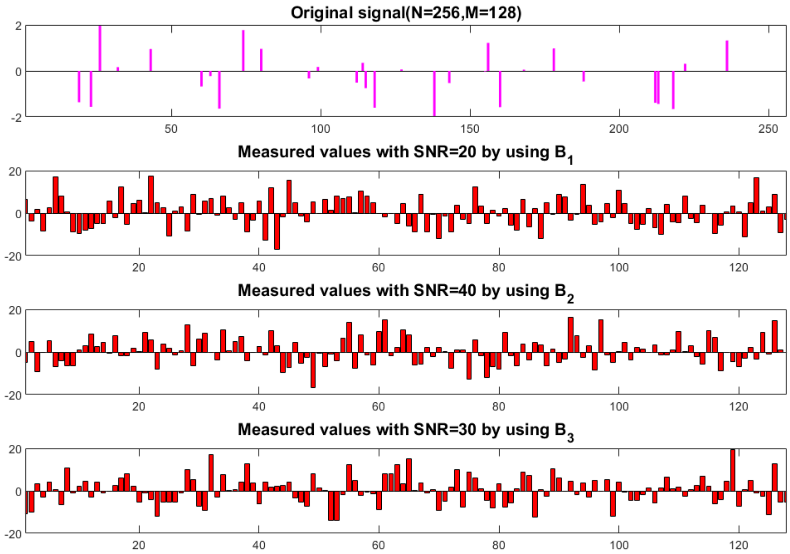
Step 6. *Set $n := n + 1$ and back to **Step 1.***

In this experiment, the parameters ρ_n, θ_n, and μ on an implemented algorithm in solving the image deblurring is set as equation (7). The Cauchy error and the signal error are measured by using second norm $\|x_n - x_{n-1}\|_2$ and $\|x_n - x\|_2$ respectively. The performance of the proposed method at n^{th} iteration is measured quantitatively by the means of the signal-to-ratio (SNR), which is defined by

$$SNR(x_n) = 20log_{10}\Big(\frac{\|x\|_2}{\|x_n - x\|_2}\Big),$$

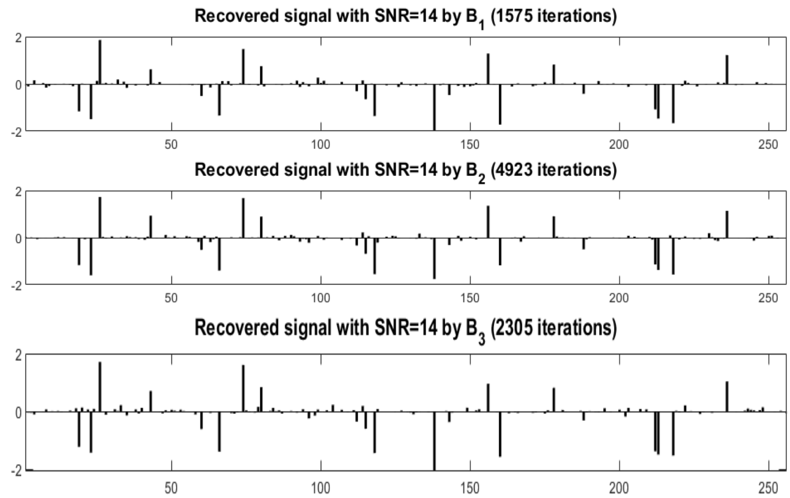
where x_n is the recovered signal at n^{th} iteration by using the proposed method.

The original signal x with $N = 256$, $M = 128$ is generated by the uniform distribution in the interval $[-2, 2]$ with $m = 40$ nonzero element. The matrix B_1, B_2 and B_3 are generated by the Gaussian matrix generated by the MATLAB routine randn(M, N). The observation b_1, b_2 and b_3 with $M = 128$ are generated by white Gaussian noise with signal-to-noise ratio $SNR = 20(\text{For}B_1), SNR = 40(\text{For}B_2)$ and $SNR = 30(\text{For}B_3)$, respectively. The process is started with signal initial data x_1 with $N = 256$ are picked randomly.



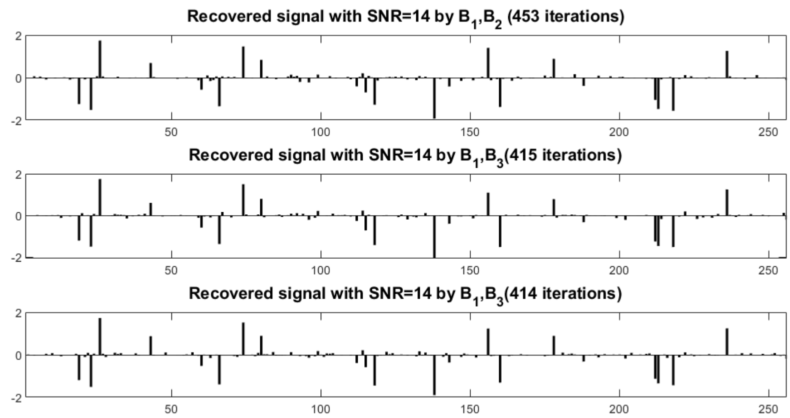
Figures 1-4 : The original signal, observation data using $SNR = 20(\text{For}B_1), SNR = 40(\text{For}B_2)$ and $SNR = 30(\text{For}B_3)$, respectively.

Next, we aim to find the solutions of signal recovery problem (32) with N = 1 by using the our Algorithm 3.1. We show the performance of B_1, B_2 and B_3 with N = 256, M = 128.



Figures 5-7 : Recovering Signal based on SNR = 14 quality by B_1, B_2 and B_3.

Next, we aim to find the solutions of signal recovery problem (31) with N = 2 by using Algorithm 3.1. We show the performance of B_1, B_2 and B_1, B_3 and B_2, B_3 with N = 256, M = 128.



Figures 8-10 : Recovering Signal based on SNR = 14 quality by B_1, B_2 and B_1, B_3 and B_2, B_3.

Next, we aim to find the solutions of signal recovery problem (31) with N = 3 by using Algorithm 3.1. We show the performance of B_1, B_2, B_3 with N = 256, M = 128.

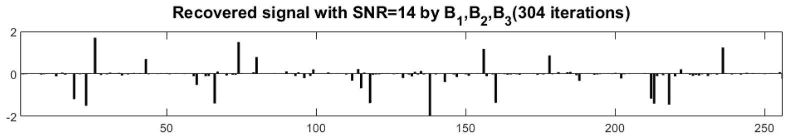
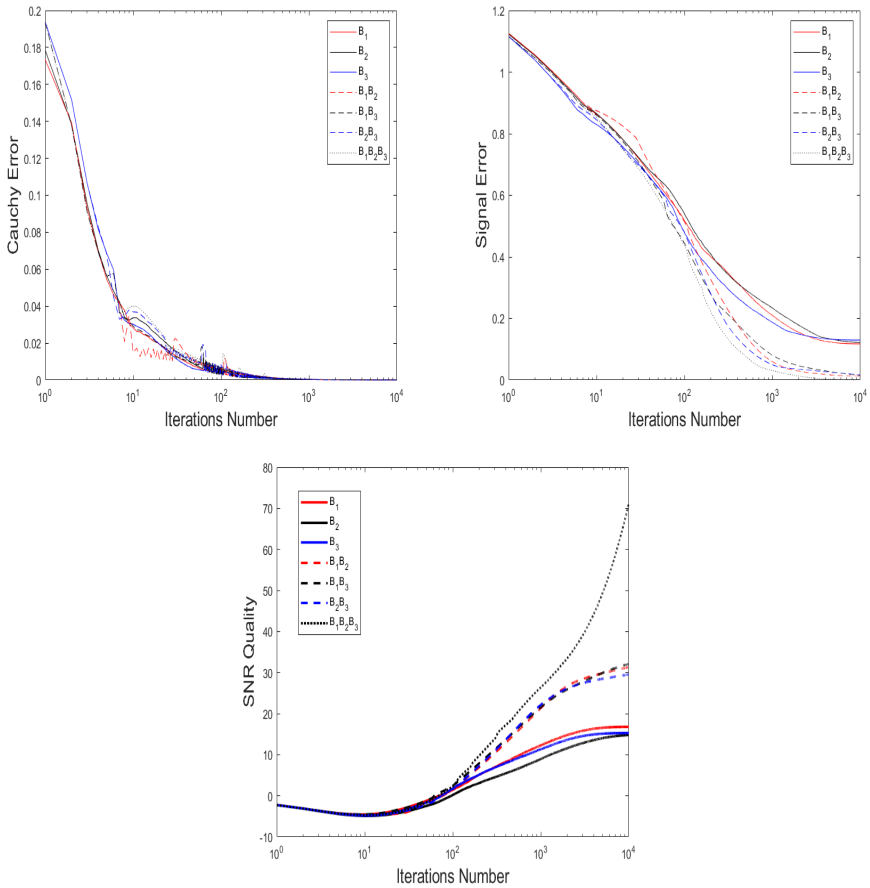


Figure 11 : Recovering Signal based on SNR = 14 quality by B_1, B_2, B_3.

The Cauchy error, signal error and SNR quality of the proposed method for recovering the degraded signal are shown in Figures 12-14. The Cauchy error shows that the proposed method can be applied to signal recovering problem. And, the signal error confirms the convergence of the implemented algorithm.



Figures 12-14 Cauchy Error, Signal Error and SNR Quality of the proposed methods in recovering the observed signal.

It is clearly seen that the solution of the signal recovering problem solved by the proposed algorithm get the quality improvements of the observed signal.

4. Application to Image Recovery Problem

Image restoration is the process of recovering an unknown image by denoising and deblurring of image. The image restoration problem can be considered in the following linear equation system:

$$b = Bx + v, \tag{33}$$

where $x \in \mathbb{R}^{n\times 1}$ is an original image, $b \in \mathbb{R}^{m\times 1}$ is the unknown image which is by blurred by matrix $B \in \mathbb{R}^{m\times n}$ and added by noise v. One technique in order to solve problem (33) is the inverse filtering when the image is blurred by a know blurring matrix B some case the inverse of blurring matrix B is difficult to fined, the convex, minimization is use, which is known as the following least squares (LS) problem (31).

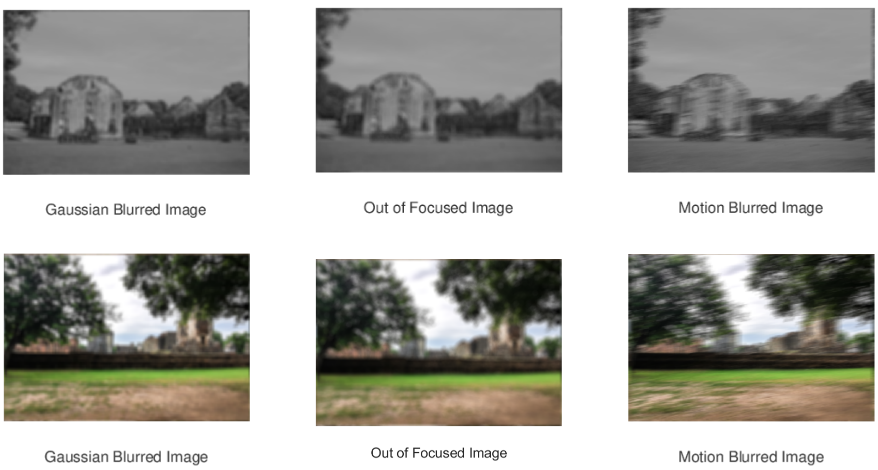
In the real, we do not know the blurring matrix of any unknown image in general. So, the goal of solving image restoration is deblurring the image without knowing which is in the blurring operator. This problem can be considered in the problem system 32 where x is the original true image, B_i is the blurred matrix, b_i is the blurred image by the blurred matrix B_i for all $i = 1, 2, ..., N$. We know that $B_i^T(B_ix - b_i)$ is Lipschitz continuous for each $i = 1, 2, ..., N$, thus we can apply our Algorithm 3.1 to solve the problem (32) in the area of image restoration problem.

For showing the advantage of our Algorithm (3.1), we will use the following different three types of blurred matrices:
(1) Gaussian blur of filter size 9×9 with standard deviation s = 4 (B_1).
(2) Out of focus blur (Disk) with radius r = 6 (B_2).
(3) Motion blur specifying with motion length of 21 pixels (len = 21) and motion orientation $11°(\theta = 11)(B_3)$.

We will test these different three blur matrices with the following original Grey and RGB images.

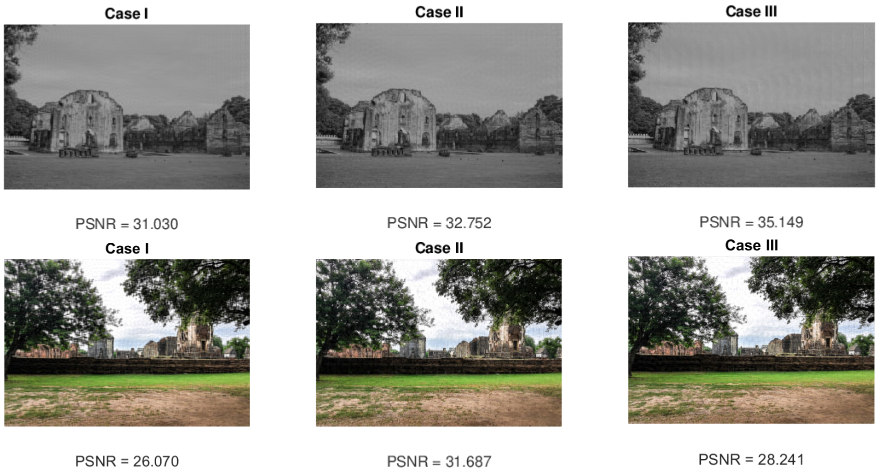
Figures 15-16 : The original Grey and RGB image of sizes 320×480 and $323 \times 475 \times 3$, respectively.

Three different types of blurred Grey and RGB images degraded by the blurring matrixes B_1, B_2 and B_3 are shown in Figures 17-22.



Figures 17-22 : Three degraded Grey and RGB images by blurred matrices B_1, B_2 and B_3, respectively.

To show the first efficiency of our Algorithm 3.1 we put one by one of the blurring matrices B_1, B_2 and B_3 when 10000^{th} iterations is the stoping of the Algorithm:
Case I: Inputting B_1 on the Algorithm 3.1;
Case II: Inputting B_2 on the Algorithm 3.1;
Case III: Inputting B_3 on the Algorithm 3.1;
are shown in Figures 23 -28 that becomposed of the restored image and its PSNR.



Figures 23 -28 : The reconstructed Grey and RGB images with their PSNR for different three cases being used the proposed algorithm presented in 10000^{th} iterations, respectively.

Next, we put two different blurred matrixes into our Algorithm 3.1, so we can split testing into following there cases when 10000^{th} iterations is the stoping of the Algorithm:
Case IV: Inputting B_1 and B_2 on the Algorithm 3.1;
Case V: Inputting B_1 and B_3 on the Algorithm 3.1;
Case VI: Inputting B_2 and B_3 on the Algorithm 3.1.

Figures 29-34 : The reconstructed Grey and RGB images with their PSNR for different three cases being used the proposed algorithm presented in 10000^{th} iterations, respectively.

It can be seen from Figures 29-34 that the quality of restoration by using the Algorithm 3.1 when two different blurring matrixes are used (N = 2) has improved compare with the previous result for every case, see on Figures 23-28.

The last case is inputting three different blurring matrixes B_1, B_2 and B_3 in Algorithm (3.1). The stopping of the algorithm is 10000^{th} iterations. The result are shown in the following figures.

Figures 35-36 : The reconstructed Grey and RGB images from the blurring operators B_1, B_2 and B_3 being used the proposed algorithm presented in 10000^{th} iterations, respectively.

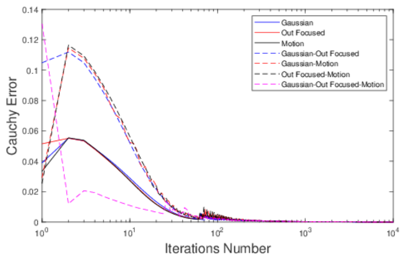
Figures 35-36 show the reconstructed Grey and RGB images with thousand iteration. It has been found that the quality of the recovered Grey and RGB images obtained by this algorithm is highest compared to the previous two algorithms.

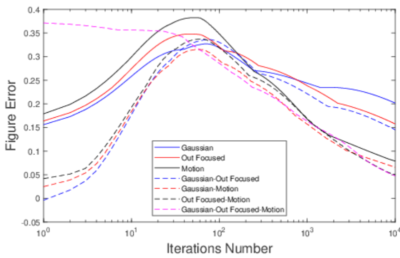
The Cauchy error define as $\|x_n - x_{n-1}\| < 10^{-5}$. The Figure error is defined as $\|x_n - x\|$ where x is the original image. The performance of the proposed at x_n on image restoring process is measured quantitatively by the means of the peak signal-to-noise ratio (PSNR), which is defined by

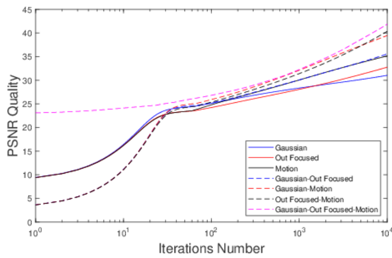
$$PSNR(x_n) = 20 \log_{10}\left(\frac{255^2}{MSE}\right),$$

where $MSE = \|x_n - x\|^2$, $\|x_n - x\|$ is the second norm of $vec(x_n - x)$.

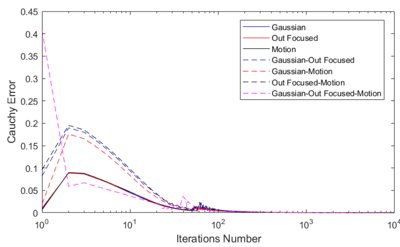
The Cauchy error plot is shown for Algorithm 3.1 the validity while the Figures error plot is shown to confirms the convergence of the proposed method and the PSNR quality plot is shown for the measured quantitatively of the image.

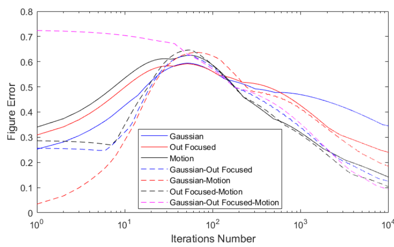


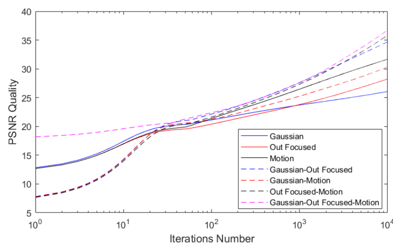




Figures 37-39 : Cauchy error, Figure error and PSNR quality plots of the proposed iteration in all cases of Grey images.







Figures 40-42 : Cauchy error, Figure error and PSNR quality plots of the proposed iteration in all cases of RGB images.

From Figures 37-42, it is clearly seen that the common solution of deblurring problem with $(N \geq 2)$ get the quality improvements of the reconstructed Grey and RGB images. Another advantage of the proposed method when the common solution of two or more image deblurring problem has been used to restored

image is that the received image is more consistent than usual (See on Figures 43-56). Figures 43-56 show the reconstructed Grey and RGB images by using the proposed algorithm in getting the common solution of the following problem with the same PSNR.
(1) Deblurring by inputting B_1, B_2 and B_3 on the Algorithm 3.1, respectively.
(2) Deblurring by inputting B_1 and B_2, B_1 and B_3, B_2 and B_3 on the Algorithm 3.1, respectively.
(3) Deblurring by inputting B_1, B_2 and B_3 on the Algorithm 3.1.

Figures 43-49 : The reconstructed Grey images of all cases being used proposed Algorithm 3.1 with PSNR = 31.

Figures 50-56 : The reconstructed RGB images of all cases being used proposed Algorithm 3.1 with PSNR = 29.

5. Conclusions

In this paper, we solve common variational inequality problems by building the algorithm using the inertial technique with a parallel CQ subgradient extragradient method. We show the strong convergence of the algorithm under some suitable assumptions on the monotone and $L-$ Lipschitz continuous operator with constant L is unknown. We also apply our proposed algorithm to solve signal and image recovery. We obtain that our algorithm gets increased efficiency when the subproblems are increasing in both signal and image recovery, see in Figures 5 -14 (signal recovery) and Figures 23 - 56 (image recovery).

6. Acknowledgement

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