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Research Article

Direction Curves of Spherical Indicatrices of a New Framed Curve

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Abstract

In this study, direction curves of spherical indicatrix curves for a curve in three - dimensional Euclidean space were investigated. Mentioned curve was given with a new alternative frame, which is $\{N, C, W\}$ frame. It is a new aspect that researching relations between direction curves and the main curve via that new alternative frame. The harmonic curvatures and geodesic curvatures of the spherical images of principal normal indicatrix of direction curves were obtained. The general helix, slant helix and C-slant helix characterizations of direction curves were given. Finally, these curves with their figures were exemplified.

Keywords: Direction curve, spherical indicatrix curve, alternative frame, helix

Yeni Çatılı Bir Eğrinin Küresel Göstergelerinin Yönlü Eğrileri

Öz

Bu çalışmada, 3 boyutlu Öklid uzayında bir eğrinin küresel gösterge eğrilerinin yönlü eğrileri araştırıldı. Adı geçen eğri yeni bir alternatif çatı olan $\{N, C, W\}$ çatısı ile verildi. Bu yeni alternatif çatı aracılığıyla esas eğri ve yönlü eğriler arasındaki ilişkinin araştırılması yeni bir bakış açısıdır. Yönlü eğrilerin harmonik eğrilikleri ve asli normal gösterge eğrisinin küresel resminin geodezik eğrilikleri bulundu. Genel helis, slant helis ve C-slant helis olma karakterizasyonları verildi. En sonda bu eğriler şekilleri ile birlikte örneklendirildi.

Anahtar Kelimeler: Yönlü eğri, küresel gösterge eğrisi, alternatif çatı, helis.

1. Introduction

Curve theory is the major research area of differential geometry and has been studied for long years. Curves exist in wide scope. For example, helix curves can be seen in a great deal of nature, like animal horns, seedpods, plant sprouts. Besides, the DNA molecule, which is the most significant genetic material, comprises of two helical chains. The shapes of these chains are a double helix. Also, helices exist in bacterial flagella of Salmonella and Escherichia coli. The most conversant helix example is a screw. It is a helix structure which transfers rotation to motion throughout axis direction. Helices are used in computer graphics and the industry of production of an artificial helix with different materials.

Also, elliptic curves are used in cryptography. The most important part of curves is that they play a role for surfaces. Creating surface notion, which is used in nature, physics, industry, engineering benefits from curves.

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The associated curve is one of the most attractive subjects to curve theory. When there is a mathematical relation among two or more curves, then they are called associated curves. Adjoint curves, Bertrand curve mates, involute-evolute curve mates, Mannheim curve mates are some examples of these curves.

In recent times, Choi and Kim (2012) proposed an item for the agenda, which is the new type of associated curve. They defined that curve as the integral curve of a declared curve's any Frenet vector and they called it direction curve. They gave curvature and torsion relations of associated curves. They described general and slant helices in the sense of associated curves and they gave a canonic way to construct these helices. Qian ve Kim (2015) investigated direction curves in Minkowski space. Macit and Düldül (2014) were inspired by these direction curves and identified W-direction curves with the help of the Darboux vector of a Frenet curve. Also, they defined V-direction curve of a curve which lies on a surface with the Darboux frame. Körpınar et al (2013) gave direction curves once more again by utilizing Bishop frame of a curve. Theory of direction curves was studied by Kızıltuğ and Önder (2015) in three-dimensional compact Lie groups.

Spherical indicatrix curves are also in the category of associated curves and they have been studied popularly for a long time. Kula and Yaylı (2005) investigated spherical indicatrix curves of slant helices. They obtained that these spherical indicatrices are spherical helices. Şahiner (2019) related direction curves and spherical indicatrices and studied the direction curve of tangent indicatrix. These curves are also in the class of associated curves. Certain connections among curvatures of these curves were given. Methods of acquiring slant helix and general helix out of the circle, spherical helix and spherical slant helix were obtained. Similar cases were investigated for principal normal indicatrix curves in Şahiner (2018).

On the other hand, while studying in curve theory, frames of curves are so important to examine their specifications. There are several adapted curve frame examples. The most known and applicated one is Frenet frame that is a moving frame. Despite it is popular, it is not defined when the curve has vanished second derivative and the frame rotates along the tangent of the curve. This case causes an unenviable inflection in surface modeling and motion design. Because of these reasons, some frames except Frenet frame have been using in researches. For example, Bishop frame Bishop (1975), Flc-like frame Dede (2019).

One of the other new defined attractive frame is $\left\{N,C,W\right\}$ which is an alternative moving frame (Uzunoğlu et al. 2016). This frame is the rotated case of Frenet frame by taking principal normal vector N as constant. It has an advantage in regard to the Frenet frame that, the expression of characterization of slant helix is more short with the new curvatures. Also, Yılmaz (2016) indicated general form of all frames as orthonormal frame $\left\{N_1,N_2,N_3\right\}$. The $\left\{N,C,W\right\}$ frame and Frenet frame are equal to this general frame with some special options but they do not overlap. While authors were defining $\left\{N,C,W\right\}$ frame, they also defined new curvatures f and g by using curvature and torsion of the curve. They gave the C-slant helix notion on the strength of this new frame. This type of helix is the case that the tangent indicatrix and the binormal indicatrix are spherical slant helix. Certain characterizations of C-slant helix were established and the fact that C-constant precession curve is C-slant helix was shown. They also gave Frenet apparatus of tangent, principal normal and binormal indicatrix curves in the sense of $\{N, C, W\}$ frame apparatus.

In the present study, we aim to study combining spherical indicatrix curves (primarily tangent indicatrix), direction curves and $\{N, C, W\}$ frame. The characterizations of direction curves of spherical indicatrix curves are investigated.

2. Preliminaries

Curve is an object, which is similar to line but has not to be straight. It is given with definition topologically that a curve is a continuous function from an interval to topological space. In differential geometry, the differentiable function $\alpha: I \to X$ is called curve for an interval I and differentiable manifold X (Struik 1988).

The adapted frame of a space curve is the $\{v_1, v_2, v_3\}$ vector collection in three-dimensional Euclidean space. Here, v_i constitute orthonormal base of three-dimensional Euclidean space such that the tangent of α is v_1 and the vectors v_2, v_3 are selected from plane arbitrarily to be orthogonal to v_1 (Do Carmo 1976).

For a regular curve α and $\alpha' \times \alpha'' \neq 0$, three orthogonal vector fields are called Frenet vectors which are given by

$$T = \frac{\alpha'}{\|\alpha'\|}, \quad \mathrm{N} = B \times T, \ \mathrm{B} = \frac{\alpha' \times \alpha''}{\|\alpha' \times \alpha''\|}. \tag{1}$$

Here, T is tangent vector, N is principal normal vector and B is binormal vector. The curvature and torsion of the curve is calculated in order of

$$\kappa = \frac{\|\alpha' \times \alpha''\|}{\|\alpha'\|^3}, \qquad \tau = \frac{\det(\alpha', \alpha'', \alpha''')}{\|\alpha' \times \alpha''\|^2} \tag{2}$$

and Frenet formulas hold as (Struik 1988)

$$\begin{aligned} T' &= \kappa N, \\ N' &= -\kappa T + \tau B, \\ B' &= -\tau N \end{aligned} \tag{3}$$

Alternative $\{N, C, W\}$ frame, which is one of the new curve frames along a curve, is defined as

$$C = \frac{N'}{\|N'\|}, \qquad \mathrm{W} = \frac{\tau T + \kappa B}{\sqrt{\kappa^2 + \tau^2}}$$

where N is the unit principal normal vector, W is the unit Darboux vector and C is the derivative of principal normal vector. Here, f and g are new curvatures in regard to this new frame and given by

$$f=\kappa\sqrt{1+H^2}\,,\ g=\sigma f \tag{4}$$

where $H=\dfrac{\tau}{\kappa}$ is harmonic curvature of a curve and $\sigma=\dfrac{\kappa^2}{\left(\kappa^2+\tau^2\right)^{3/2}}\left(\dfrac{\tau}{\kappa}\right)'$ is geodesic curvature of spherical image of principal normal indicatrix curve. Also the relation between H and σ is constructed as (Uzunoğlu *et al*. 2016)

$$\sigma=\frac{H'}{\kappa\left(1+H^2\right)^{\frac{3}{2}}} \tag{5}$$

Besides, a curve is said to be general helix when $H=\dfrac{\tau}{\kappa}$ is constant and vice versa (Gray 2006). Also it is slant helix if and only if σ is constant (Izumiya 2004).

For a unit speed curve α, it is given the name C-slant helix if the vector field C forms a constant angle $\theta\neq\dfrac{\pi}{2}$ with a fixed direction u, namely, $\left\langle C,u\right\rangle=\cos\theta=\mathrm{constant}$ and α is C-slant helix if and only if

$$\frac{\left(f^2+g^2\right)^{\frac{3}{2}}}{f^2\left(\dfrac{g}{f}\right)'}=\tan\theta=\mathrm{constant} \tag{6}$$

where the constant angle between fixed direction u and the vector C is θ (Uzunoğlu *et al*. 2016).

The curves, which are formed on the surface of unit sphere by translating each vector of Frenet frame of a curve to the center of unit sphere and collecting terminal points of these vectors on the surface of sphere are called indicatrix curves. Tangent vector, principal normal vector and binormal vector generate tangent indicatrix curve, principal normal indicatrix curve and binormal indicatrix curve and equations of these curves are indicated respectively as (Struik 1988)

$$\begin{aligned}\alpha_T&=T,\\ \alpha_N&=N,\\ \alpha_B&=B\end{aligned} \tag{7}$$

If $\left\{T_T, N_T, B_T, \kappa_T, \tau_T\right\}$ is Frenet apparatus of tangent indicatrix curve of α, then the relations between Frenet apparatus of curve α and tangent indicatrix are given in Uzunoğlu et al. (2016) as

$$\begin{aligned} T_T &= N, \\ N_T &= \frac{\kappa}{f}\left(-T + HB\right), \\ B_T &= \frac{\kappa}{f}\left(HT + B\right) \end{aligned} \tag{8}$$

and

$$\kappa_T = \frac{f}{\kappa}, \qquad \tau_T = \frac{g}{\kappa}. \tag{9}$$

An integral curve is the parametric curve, which gives a special solution to a differential equation or equation system. In the event X is a vector field and $\alpha(t)$ is a parametric curve, if $\alpha(t)$ is a solution of differential equation $\alpha'(t) = X(\alpha(t))$, then $\alpha(t)$ is called integral curve (Struik 1988).

If the tangent vector fields of a curve α in three dimensional Euclidean space are normal to a curve β , then α is called evolute of β (Struik 1988).

3. Main Theorem and Proof

3.1. Direction Curves

In this section, direction curves of spherical indicatrix curves are investigated by the help of $\left\{N, C, W\right\}$ frame apparatus. The results are given in terms of new curvatures f and g.

Let δ be a regular curve with Frenet frame $\left\{T, N, B\right\}$. If δ_T is tangent indicatrix curve of δ and Frenet apparatus of δ_T is $\left\{T_T, N_T, B_T, \kappa_T, \tau_T\right\}$, then the integral curves

$$\begin{aligned} \varphi_1 &= \int T_T, \\ \varphi_2 &= \int N_T, \\ \varphi_3 &= \int B_T \end{aligned} \tag{10}$$

are called direction curves of tangent indicatrix curve δ_T. The curves φ_1, φ_2 and φ_3 are special cases of direction curves given in Şahiner (2019) but we now investigate these curves and apparatus of them in terms of $\left\{N, C, W\right\}$ curvatures.

Remark 1: Let the arc length parameters of φ_1 and δ_T are s_1 and s_T , respectively. Then by differentiating first equation in (10) with respect to s_1 and taking norm, we get $s_1 = s_T$. Thus arc length parameters of direction curves φ_1, φ_2 and φ_3 are same with arc length parameter of tangent indicatrix curve δ_T.

Proposition 3.1: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_1 be T_T -direction curve of δ_T . Frenet vector fields, curvature and torsion of φ_1 are given as

$$\begin{aligned} T_{\varphi_1} &= N, \\ N_{\varphi_1} &= \frac{\kappa}{f}\left(-T + HB\right), \\ B_{\varphi_1} &= \frac{\kappa}{f}\left(HT + B\right), \\ \kappa_{\varphi_1} &= \frac{f}{\kappa} \quad \text{and} \quad \tau_{\varphi_1} = \frac{g}{\kappa}. \end{aligned} \tag{11}$$

Proof: By differentiating first equation in (10), using equations (1), (2), (3), (4) and making appropriate calculations, the results are obtained easily.

Corollary 3.1: Let δ be an s arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_1 be T_T -direction curve of δ_T. Tangent indicatrix curve of δ and T_T -direction curve of δ_T coincide.

Proof: Taking notice of equations (8), (9) and (11), Frenet apparatus of δ_T and φ_1 are equal.

Beside this, taking derivative of first equations of (7) and (10), using Frenet formulas and $\frac{ds}{ds_T} = \frac{1}{\kappa}$, we have the result.

Theorem 3.1: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_1 be T_T -direction curve of δ_T. The curve δ is slant helix if and only if T_T -direction curve of δ_T is general helix.

Proof: By the equations (4) and (11), the harmonic curvature of φ_1 is found as $H_{\varphi_1} = \sigma$ in which σ is the geodesic curvature of spherical image of principal normal indicatrix curve of δ.

Theorem 3.2: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_1 be T_T -direction curve of δ_T. The curve δ is C-slant helix if and only if T_T -direction curve of δ_T is slant helix.

Proof: The geodesic curvature of spherical image of principal normal indicatrix curve of φ_1 is computed by

$$\sigma_{\varphi_1} = \frac{\kappa_{\varphi_1}{}^2}{\left(\kappa_{\varphi_1}{}^2 + \tau_{\varphi_1}{}^2\right)^{3/2}} \frac{d}{ds_1}\left(\frac{\tau_{\varphi_1}}{\kappa_{\varphi_1}}\right). \tag{12}$$

Inserting curvature and torsion in equation (11) to equation (12) and considering equation (6), $s_1 = s_T$ and $\frac{ds}{ds_T} = \frac{1}{\kappa}$, we get $\sigma_{\varphi_1} = \frac{1}{\tan\theta}$, where θ is the angle with the vector field C and fixed direction u.

When the curve δ is C-slant helix, then θ is constant. Thus σ_{φ_1} is constant and vice versa.

Theorem 3.3: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_1 be T_T -direction curve of δ_T . The curve δ is general helix if and only if T_T -direction curve of δ_T is planar.

Proof: Taking into account equations (4) and (5), the torsion of φ_1 in equation (11) is calculated as $\tau_{\varphi_1} = \frac{H'}{\kappa\left(1+H^2\right)}$.

Hence the result is apparent.

Proposition 3.2: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_2 be N_T -direction curve of δ_T. Frenet vector fields, curvature and torsion of φ_2 are given as

$$\begin{aligned}
T_{\varphi_2} &= \frac{\kappa}{f}\left(-T + HB\right),\\
N_{\varphi_2} &= \frac{\sigma H}{\sqrt{H^2+1}\sqrt{\sigma^2+1}}T - \frac{1}{\sqrt{\sigma^2+1}}N\\
&\quad + \frac{\sigma}{\sqrt{H^2+1}\sqrt{\sigma^2+1}}B,\\
B_{\varphi_2} &= \frac{H}{\sqrt{H^2+1}\sqrt{\sigma^2+1}}T + \frac{\sigma}{\sqrt{\sigma^2+1}}N\\
&\quad + \frac{1}{\sqrt{H^2+1}\sqrt{\sigma^2+1}}B,
\end{aligned} \tag{13}$$

$$\kappa_{\varphi_2} = \frac{\sqrt{f^2+g^2}}{\kappa} \quad \text{and} \quad \tau_{\varphi_2} = \frac{\sigma'}{\kappa\left(\sigma^2+1\right)}.$$

Proof: By differentiating second equation in (10), using equations (1), (2), (3), (4) and making appropriate calculations, the results are obtained easily.

Theorem 3.4: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_2 be N_T -direction curve of δ_T . The curve δ is slant helix if and only if N_T -direction curve of δ_T is planar.

Proof: The torsion of φ_2 in equation (13) is obtained as $\tau_{\varphi_2} = \dfrac{\sigma'}{\kappa\left(\sigma^2+1\right)}$.

When the curve δ is slant helix, then $\sigma = \text{constant}$. Thus the proof is completed.

Corollary 3.2: If the curve δ is general helix, then N_T -direction curve of δ_T is also planar.

Proof: Considering the equation (5) and τ_{φ_2} the result is obvious.

Theorem 3.5: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_2 be N_T -direction curve of δ_T . If the curve δ is slant helix, then there does not exist general helix and slant helix N_T -direction curve.

Proof: By using curvature and torsion of φ_2 in equation (13), harmonic curvature and geodesic curvature of spherical image of principal normal indicatrix curve of φ_2 are obtained respectively as

$$H_{\varphi_2} = \frac{\sigma'}{f\left(\sigma^2+1\right)^{\frac{3}{2}}}$$

and

$$\sigma_{\varphi_2} = \frac{f^2\left(\sigma^2+1\right)^4}{\left(f^2\left(\sigma^2+1\right)^3+\left(\sigma'\right)^2\right)^{\frac{3}{2}}}\left(\frac{\sigma'}{f\left(\sigma^2+1\right)^{\frac{3}{2}}}\right)'.$$

In here, if the curve δ is slant helix, then $H_{\varphi_2} = 0$ and $\sigma_{\varphi_2} = 0$. Thus the result is obvious.

Proposition 3.3: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_3 be B_T -direction curve of δ_T . Frenet vector fields, curvature and torsion of φ_3 are given as

$$\begin{aligned} T_{\varphi_3} &= \frac{\kappa}{f}\left(HT + B\right), \\ N_{\varphi_3} &= \frac{\kappa}{f}\left(T - HB\right), \\ B_{\varphi_3} &= N, \\ \kappa_{\varphi_3} &= \frac{g}{\kappa} \quad \text{and} \quad \tau_{\varphi_3} = -\frac{f}{\kappa}. \end{aligned} \tag{14}$$

Proof: By differentiating third equation in (10), using equations (1), (2), (3), (4) and making appropriate calculations, the results are obtained easily.

Theorem 3.6: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_3 be B_T -direction curve of δ_T. There does not exist planar B_T -direction curve of δ_T .

Proof: Since the curvature $\kappa \neq 0$ and so the curvature $f \neq 0$, the torsion of φ_3 in equation (14) is not equal to zero.

Theorem 3.7: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_3 be B_T -direction curve of δ_T. The curve δ is slant helix if and only if B_T -direction curve of δ_T is general helix.

Proof: By the equations (4) and (14), the harmonic curvature of φ_3 is found as

$$H_{\varphi_3} = -\frac{1}{\sigma} .$$

If the curve δ is slant helix, then $\sigma = \text{constant}$ and thus $H_{\varphi_3} = \text{constant}$ and also vice versa.

Theorem 3.8: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_3 be B_T -direction curve of δ_T. The curve δ is general helix if and only if B_T -direction curve of δ_T is line.

Proof: Considering the curvature of φ_3 in equation (14), equations (4) and (5), we have

$$\kappa_{\varphi_3} = \frac{H'}{\kappa\left(1 + H^2\right)} .$$

Therefore, the curve δ is general helix ($H =$constant) if and only if $\kappa_{\varphi_3} = 0$ which means φ_3 is line.

Theorem 3.9: Let δ be an arc length parametrised curve, δ_T be tangent indicatrix of δ and φ_3 be B_T -direction curve of δ_T. The curve δ is C-slant helix if and only if B_T -direction curve of δ_T is slant helix.

Proof: The geodesic curvature of spherical image of principal normal indicatrix curve of φ_3 is computed by

$$\sigma_{\varphi_3} = \frac{{\kappa_{\varphi_3}}^2}{\left({\kappa_{\varphi_3}}^2 + {\tau_{\varphi_3}}^2\right)^{3/2}} \frac{d}{ds_3}\left(\frac{\tau_{\varphi_3}}{\kappa_{\varphi_3}}\right). \tag{15}$$

Inserting curvature and torsion in equation (14) to equation (15) and considering equation (6), $s_3 = s_T$ and $\dfrac{ds}{ds_T} = \dfrac{1}{\kappa}$, we get $\sigma_{\varphi_3} = \dfrac{1}{\tan\theta}$,

where θ is the angle with the vector field C and fixed direction u. Thus the proof is completed.

Now, by comparing Frenet vector fields of δ_T, φ_1, φ_2 and φ_3 in equations (8), (11), (13), (14), following corollaries are given simply.

Corollary 3.3: The tangent indicatrix curve δ_T is evolute of N_T -direction curve of δ_T (φ_2) and B_T -direction curve of δ_T (φ_3).

Corollary 3.4: T_T -direction curve of δ_T (φ_1) is evolute of N_T -direction curve of δ_T (φ_2) and B_T -direction curve of δ_T (φ_3).

Example 1: Consider the curve

$$\delta(s) = \left(\cos\left(\frac{s}{\sqrt{5}}\right), \sin\left(\frac{s}{\sqrt{5}}\right), \frac{2}{\sqrt{5}}s\right)$$

which is an arc length parametrised general helix. The tangent indicatrix of δ is

$$\delta_T(s) = \frac{1}{\sqrt{5}}\left(-\sin\left(\frac{s}{\sqrt{5}}\right), \cos\left(\frac{s}{\sqrt{5}}\right), 2\right) .$$

Then T_T -direction curve of δ_T, N_T -direction curve of δ_T and B_T -direction curve of δ_T are computed respectively as

$$\varphi_1(s) = \left(-\frac{\sqrt{5}}{5}\sin\left(\frac{s}{\sqrt{5}}\right) + c_1, \frac{\sqrt{5}}{5}\cos\left(\frac{s}{\sqrt{5}}\right) + c_2, \frac{c_3}{5}\right)$$

$$\varphi_2(s) = \left(-\frac{\sqrt{5}}{5}\cos\left(\frac{s}{\sqrt{5}}\right) + c_4, -\frac{\sqrt{5}}{5}\sin\left(\frac{s}{\sqrt{5}}\right) + c_5, \frac{c_6}{5}\right)$$

$$\varphi_3(s)=\left(c_7, c_8, \frac{s}{5}+c_8\right).$$

where $c_i \in R, \quad 1 \le \mathrm{i} \le 9$.

It is easy to see that the curve φ_1 and φ_2 are planar and the curve φ_3 is line. Therefore this example confirms Corollary 3.2, Theorem 3.3 and Theorem 3.8.

The figures are drawn with Mathematica programme and the intervals for curves are given as: $-\pi \le s \le \pi$ for Figure 1, 2, 3 and 4. For Figure 5, the interval is $-1 \le s \le 1$.

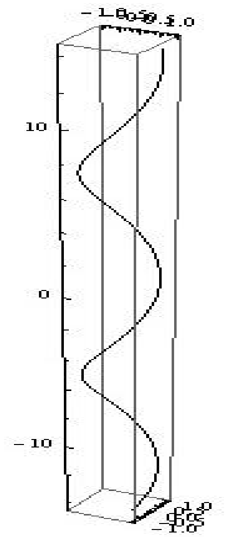

Figure 1. The curve δ .

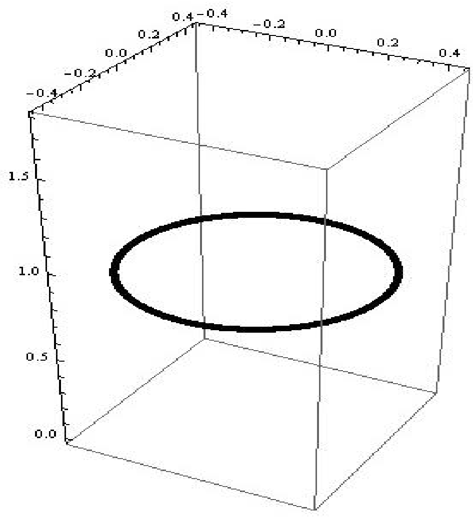

Figure 2. The curve δ_T .

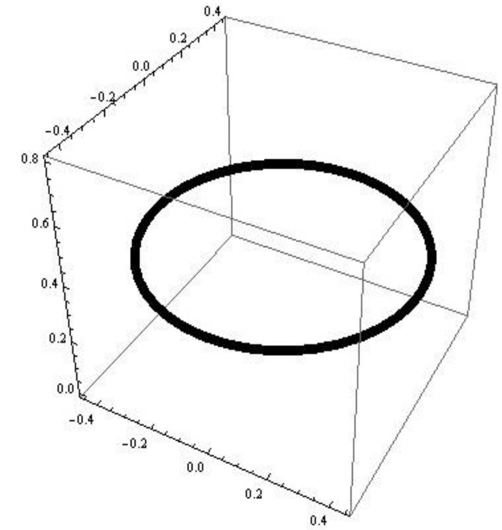

Figure 3. The curve φ_1 .

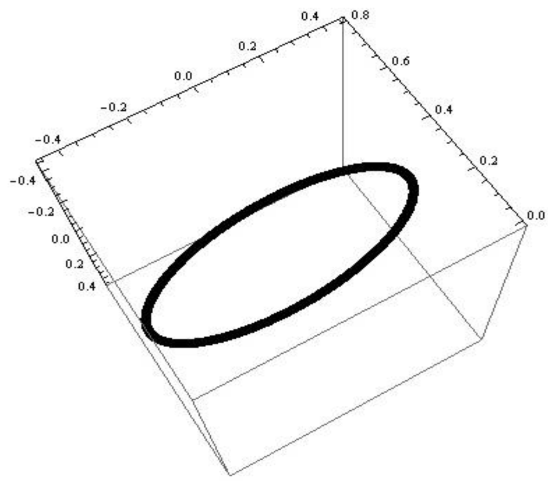

Figure 4. The curve φ_2 .

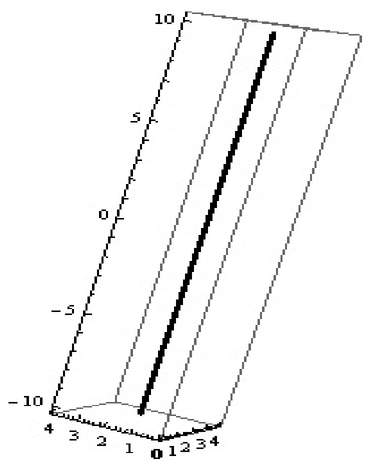

Figure 5. The curve φ_3 .

Example 2: Consider arc length parametrised slant helix

$$\beta(s)=\left(-\frac{8}{6}\left(\frac{\cos(4s)}{16}+\frac{\cos(2s)}{4}\right),-\frac{8}{6}\left(\frac{\sin(4s)}{16}+\frac{\sin(2s)}{4}\right),\frac{\sqrt{8}}{3}\cos s\right).$$

The tangent indicatrix of β is

$$\beta_T(s)=\left(\frac{8}{6}\left(\frac{\sin(4s)}{4}+\frac{\sin(2s)}{2}\right),-\frac{8}{6}\left(\frac{\cos(4s)}{4}+\frac{\cos(2s)}{2}\right),-\frac{\sqrt{8}}{3}\sin s\right)$$ and

T_T -direction curve of β_T is

$$\varphi_1(s)=\left(\frac{\sin(4s)}{3}+\frac{2\sin(2s)}{3}+c_1,-\frac{\cos(4s)}{3}-\frac{2\cos(2s)}{3}+c_2,-\frac{\sqrt{8}}{3}\sin s+c_3\right)$$

where c_1, c_2, c_3 are constant real numbers.

The curve φ_1 is general helix and this example confirms Theorem 3.1.

In Figure 8, there is a single curve and it is the curve φ_1 of tangent indicatrix β_T.

The interval $-\pi \le s \le \pi$ is used for Figure 6, 7 and 8.

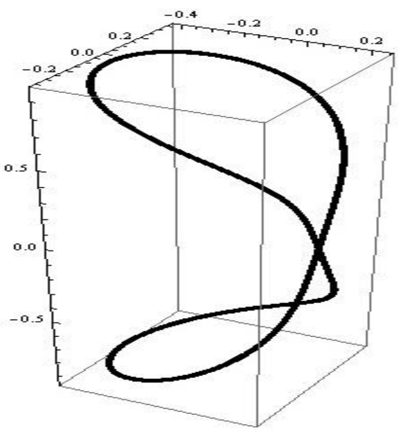

Figure 6. The curve β .

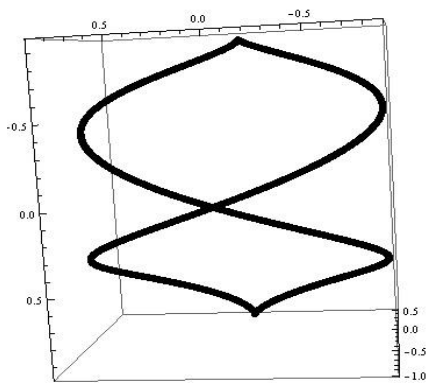

Figure 7. The curve β_T .

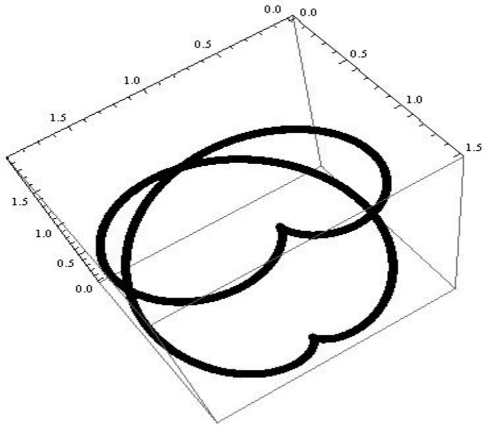

Figure 8. The curve φ_1 for β_T .

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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