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ON THE OSCILLATION OF SOLUTIONS OF A HIGHER ORDER NONLINEAR NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATION WITH AN OSCILLATING COEFFICIENT

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ABSTRACT

In this paper, we are concerned with the oscillation of the solutions of a certain more general higher order nonlinear neutral type functional differential equation with an oscillating coefficient of the form

$$\left[y(t)+\sum_{i=1}^{m} P_i(t)y(\tau_i(t))\right]^{(n)}+\sum_{i=1}^{m} Q_i(t)f_i(y(\sigma_i(t)))=0$$

where $n \geq 2$; $P_i(t), Q_i(t), \tau_i(t) \in C[t_0,+\infty)$ for $i=1,2,\cdots,m$; $P_i(t)$ is an oscillatory function for $i=1,2,\cdots,m$; $Q_i(t)$ is positive valued for $i=1,2,\cdots,m$. $\sigma_i(t) \in C'[t_0,+\infty)$, $\sigma_i'(t)>0$, $\sigma_i(t) \leq t$; $\sigma_i(t) \to +\infty$ as $t \to \infty$ for $i=1,2,\cdots,m$; $\tau_i(t) \to +\infty$ as $t \to \infty$ for $i=1,2,\cdots,m$; $f_i(u) \in C(R,R)$ is a nondecreasing function, $uf_i(u)>0$ for $u \neq 0$ and $i=1,2,\cdots,m$. We obtained two sufficient criteria for oscillatory behaviour of its solutions.

Mathematics Subject Classification: 34K11

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NEUTRAL TİPTEN BİR SALINIMLI KATSAYILI YÜKSEK MERTEBEDEN LİNEER OLMAYAN BİR FONKSİYONEL DİFERENSİYEL DENKLEMİN ÇÖZÜMLERİNİN SALINIMLILIĞI ÜZERİNE BİR ÇALIŞMA

ÖZET

Bu çalışmada; $n \geq 2$, $i = 1,2,\cdots,m$ için $P_i(t), Q_i(t), \tau_i(t) \in C[t_0,+\infty)$; $i = 1,2,\cdots,m$ için $P_i(t)$ ler salınımlı fonksiyonlar; $i = 1,2,\cdots,m$ için $Q_i(t)$ ler pozitif değerli fonksiyonlar; $\sigma_i(t) \in C'[t_0,+\infty)$, $\sigma_i'(t) > 0$, $\sigma_i(t) \leq t$ ve $t \to \infty$ iken $i = 1,2,\cdots,m$ için $\tau_i(t) \to +\infty$; $i = 1,2,\cdots,m$ için $f_i(u) \in C(R,R)$ ler azalmayan fonksiyonlar ve $u \neq 0$ iken $uf_i(u) > 0$ olmak üzere

$$\left[y(t) + \sum_{i=1}^{m} P_i(t)\, y(\tau_i(t))\right]^{(n)} + \sum_{i=1}^{m} Q_i(t)\, f_i(y(\sigma_i(t))) = 0$$

tipindeki yüksek mertebeden lineer olmayan diferensiyel denklemin çözümlerinin salınımlılığı üzerine yeter şartlı iki kriter elde edilmektedir.

1. INTRODUCTION.

We consider the higher order nonlinear differential equation of the form

$$\left[y(t) + \sum_{i=1}^{m} P_i(t)\, y(\tau_i(t))\right]^{(n)} + \sum_{i=1}^{m} Q_i(t)\, f_i(y(\sigma_i(t))) = 0 \quad (1.1)$$

where $n \geq 2$; $P_i(t), Q_i(t), \tau_i(t) \in C[t_0,+\infty)$ for $i = 1,2,\cdots,m$; $P_i(t)$ is an oscillatory function for $i = 1,2,\cdots,m$; $Q_i(t)$ is positive valued for $i = 1,2,\cdots,m$. $\sigma_i(t) \in C'[t_0,+\infty)$, $\sigma_i'(t) > 0$, $\sigma_i(t) \leq t$; $\sigma_i(t) \to +\infty$ as $t \to \infty$ for $i = 1,2,\cdots,m$; $\tau_i(t) \to +\infty$ as $t \to \infty$ for $i = 1,2,\cdots,m$; $f_i(u) \in C(R,R)$ is a nondecreasing function, $uf_i(u) > 0$ for $u \neq 0$ and $i = 1,2,\cdots,m$.

As is customary, a solution of Eq. (1.1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

For the sake of convenience, the function $z(t)$ is defined by,

$$z(t)=y(t)+\sum_{i=1}^{m}P_i(t)y(\tau_i(t)). \qquad (1.2)$$

2. SOME AUXILIARY LEMMAS

Lemma 2.1: Let $y(t)$ be a positive and n-times differentiable function on $[t_0,+\infty)$. If $y^{(n)}(t)$ is of constant sign ond not identically zero in any interval $[b,+\infty)$, then there exist a $t_1\geq t_0$ and an integer l, $0\leq l\leq n$ such that $n+l$ is even, if $y^{(n)}(t)$ is nonnegative, or $n+l$ odd, if $y^{(n)}(t)$ is nonpositive, and that, as $t\geq t_1$, if $l>0$, $y^{(k)}(t)>0$ for $k=0,1,2,\cdots,l-1$, and if $l\leq n-1$, $(-1)^{k+1}y^{(k)}(t)>0$ for $k=l,l+1,\cdots,n-1$ **[1]**.

Lemma 2.2: Let $y(t)$ defined Lemma 2.1. Let $y^{(n-1)}(t)y^{(n)}(t)\leq 0$ $(t\geq t_0)$ and there exists a constant $M>0$ for every $\lambda(0<\lambda<1)$, such that $y(\lambda t)\geq Mt^{n-1}\left|y^{(n-1)}(t)\right|$ for sufficiently large t **[1]**.

3. THE MAIN RESULTS

Teorem 3.1: Assume that n is odd and

C₁) $\lim_{t\to\infty}\sum_{i=1}^{m}P_1(t)=0$,

C₂) $\int_{t_0}^{+\infty}s^{n-1}\sum_{i=1}^{m}Q_i(s)\,ds=+\infty$.

Then every bounded solution of Eq. (1.1) is either oscillatory or tends to zero as $t\to+\infty$.

Proof: Assume that Eq. (1.1) has a bounded nonoscillatory solution $y(t)$. Without loss of generality, assume that $y(t)$ is eventually positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t)>0$,

$y(\tau_i(t))>0$ and $y(\sigma_i(t))>0$ for $t\geq t_1\geq t_0$ and $i=1,2,\cdots,m$. Furthermore suppose that $y(t)$ does not tend to zero as $t\to+\infty$. By (1.1) and (1.2), we have for $t\geq t_1$

$$z^{(n)}(t)=-\sum_{i=1}^{m}Q_i(t)f_i\left(y\left(\sigma_i(t)\right)\right)<0 \qquad (3.1)$$

That is, $z^{(n)}(t)<0$. It follows that, $z^{(j)}(t)$ $(j=0,1,2,\cdots,n-1)$ is strictly monotone and of constant sign eventually. Since $y(t)$ is a bounded function and $\lim_{t\to\infty}\sum_{i=1}^{m}P_1(t)=0$ for $i=1,2,\cdots,m$, there exists a $t_2\geq t_1$ such that as $t\geq t_2$ $z(t)>0$ eventually and there is a $t_3\geq t_2$ such that $z(t)$ is also bounded for $t\geq t_3$. Because of n is odd and $z(t)$ is bounded, by Lemma 2.1, when $l=0$ (otherwise $z(t)$ is not bounded) there exists a $t_4\geq t_3$ such that $(-1)^k z^{(k)}(t)>0$ $(k=0,1,2,\cdots,n-1)$ as $t\geq t_4$. In particular, since $z'(t)<0$ for $t\geq t_4$, $z(t)$ is decreasing. Since $z(t)$ is bounded, we may write $\lim_{t\to\infty}z(t)=L$ $(-\infty<L<+\infty)$. Assume that $0\leq L\leq+\infty$. Let be $L>0$. Then there exists a constant $c>0$ and a $t_5\geq t_4$ such that $z(t)>c>0$ for $t\geq t_5$. Since $y(t)$ is bounded, $\lim_{t\to\infty}\sum_{i=1}^{m}P_i(t)y(\tau_i(t))=0$ by (C_1). Therefore, there exist a constant $c_1>0$ and a $t_6\geq t_5$ such that $y(t)=z(t)-\sum_{i=1}^{m}P_i(t)y(\tau_i(t))>c_1>0$ for $t\geq t_6$. So, we may take a t_7 with the property of $t_7\geq t_6$ such that $y(\sigma_i(t))>c_1>0$ for $t\geq t_7$. From (3.1), we have

$$z^{(n)}(t)=-\sum_{i=1}^{m}Q_i(t)f_i(c_1)<0,\ t\geq t_7 \qquad (3.2)$$

If we multiply (3.2) by t^{n-1} and integrate from t_7 to t then we obtain

$$F(t)-F(t_7)\leq -f(c_1)\int_{t_7}^{t}\sum_{i=1}^{m}Q_i(s)s^{n-1}ds \qquad (3.3)$$

where

$$F(t)=$$
$$t^{n-1}z^{(n-1)}(t)-(n-1)t^{n-2}z^{(n-2)}(t)+(n-1)(n-2)t^{n-3}z^{(n-3)}(t)$$
$$-\cdots-(n-1)(n-2)(n-3)\cdots 3.2tz'(t)$$
$$+(n-1)(n-2)(n-3)\cdots 3.2.1z(t)$$

Since $(-1)^k z^{(k)}(t)>0$ for $k=0,1,2,\cdots,n-1$ and $t\geq t_4$, $F(t)>0$ for $t\geq t_7$. From (3.3), we have

$$-F(t_7)\leq -f(c_1)\int_{t_7}^{t}\sum_{i=1}^{m}Q_i(s)s^{n-1}ds$$

From (C_2), we obtain

$$-F(t_7)\leq -f(c_1)\int_{t_7}^{t}\sum_{i=1}^{m}Q_i(s)s^{n-1}ds=-\infty$$

at $t\to\infty$. This is a contradiction. So, $L>0$ is impossible. Therefore, $L=0$ is the only possible case. That is, $\lim_{t\to\infty}z(t)=0$. Since $y(t)$ is bounded, we obtain

$$\lim_{t\to\infty}y(t)=\lim_{t\to\infty}z(t)-\lim_{t\to\infty}\sum_{i=1}^{m}P_i(t)y(\tau_i(t))=0$$

by (C_1) and (1.2). Now let us consider the case of $y(t)<0$ for $t\geq t_1$. By (1.1) an (1.2),

$$z^{(n)}(t)=-\sum_{i=1}^{m}Q_i(t)f_i(y(\sigma_i(t)))<0,\ t\geq t_1.$$

That is, $z^{(n)}(t)>0$. It follows that, $z^{(j)}(t)$ $(j=0,1,2,\cdots,n-1)$ is strictly monotone and of constant sign eventually. Since $y(t)$ is a bounded function and $\lim_{t\to\infty}\sum_{i=1}^{m}P_i(t)=0$ for $i=1,2,\cdots,m$, there exists a $t_2\geq t_1$ such that as $t\geq t_2$ $z(t)>0$ eventually and there is a $t_3\geq t_2$ such that $z(t)$ is also bounded for $t\geq t_3$. Assume that $x(t)=-z(t)$. Then $x^{(n)}(t)=-z^{(n)}(t)$. Therefore, $x(t)>0$ and $x^{(n)}(t)<0$ for $t\geq t_3$. Hence, we observe that $x(t)$ is bounded. Since n is odd. By Lemma 2.1, there is a $t_4\geq t_3$ and $l=0$ (otherwise, $x(t)$ is not bounded) such that

$(-1)^k x^{(k)}(t) > 0$ for $k = 0,1,2,\cdots,n-1$ and $t \geq t_4$. That is $(-1)^k z^{(k)}(t) < 0$ for $k = 0,1,2,\cdots,n-1$ and $t \geq t_4$. In particular, as $t \geq t_4$, $z'(t) > 0$. Therefore, $z(t)$ is increasing. So, we can assume that $\lim_{t\to\infty} z(t) = L$ $(-\infty < L < +\infty)$. As in the proof of $y(t) > 0$, we may proof that $L = 0$. As for the rest of proof, it similar to the case of $y(t) > 0$. That is, $\lim_{t\to\infty} y(t) = 0$. Hence, the proof is completed ■

Teorem 3.2: Assume that n is even and (C_1) is held. If
C_3) there is a function $\varphi(t) \in C'[t_0, +\infty)$. Moreover

$$\limsup_{t\to\infty} \int_{t_0}^{t} \varphi(s) \sum_{i=i}^{m} Q_i(s) ds = +\infty$$

and

$$\limsup_{t\to\infty} \int_{t_0}^{t} \left[\frac{[\varphi'(s)]^2}{\varphi(s)\sigma_i'(s)\sigma_i^{n-2}(s)} \right] ds < +\infty$$

for $i = 1,2,\cdots,m$ is satisfied, then every bounded solution of Eq. (1.1) is oscillatory.

Proof: Assume that Eq. (1.1) has a bounded nonoscillatory solution $y(t)$. Without loss af generality, assume that $y(t)$ is eventully positive (the proof is similar when $y(t)$ is eventually negative). That is, $y(t) > 0$, $y(\tau_i(t)) > 0$ and $y(\sigma_i(t)) > 0$ for $t \geq t_1 \geq t_0$ and $i = 1,2,\cdots,m$. By (1.1) and (1.2), we have for $t \geq t_1$

$$z^{(n)}(t) = -\sum_{i=1}^{m} Q_i(t) f_i(y(\sigma_i(t))) < 0 \qquad (3.4)$$

That is, $z^{(n)}(t) < 0$. It follows that, $z^{(j)}(t)$ $(j = 0,1,2,\cdots,n-1)$ is strictly monotone and of constant sign eventually. Since $y(t)$ is a bounded function and $\lim_{t\to\infty} \sum_{i=1}^{m} P_1(t) = 0$ for $i = 1,2,\cdots,m$, there exists a $t_2 \geq t_1$ such that as $t \geq t_2$ $z(t) > 0$ eventually and there is a $t_3 \geq t_2$ such that $z(t)$ is also bounded for $t \geq t_3$. Because of n is even, by Lemma 2.1, when $l=1$ (otherwise $z(t)$ is not bounded) there exists a $t_4 \geq t_3$ such that as $t \geq t_4$

$$(-1)^{k+1} z^{(k)}(t) > 0 \quad (k=0,1,2,\cdots,n-1) \tag{3.5}$$

In particular, since $z'(t)>0$ for $t \geq t_4$, $z(t)$ is increasing. Since $y(t)$ is bounded, $\lim_{t\to\infty} \sum_{i=1}^{m} P_i(t) y(\tau_i(t)) = 0$ by (C$_1$). Then, there exists a $t_5 \geq t_4$ and δ positive integer,

$$y(t) = z(t) - \sum_{i=1}^{m} P_i(t) y(\tau_i(t)) > \frac{1}{\delta} z(t) > 0$$

for $t \geq t_5$ by (1.2). We may get $t_6 \geq t_5$ such that for $t \geq t_6$ and $i=1,2,\cdots,m$

$$y(\sigma_i(t)) > \frac{1}{\delta} z(\sigma_i(t)) > 0 \tag{3.6}$$

From (3.4), (3.6) and the properties of f, we have

$$\begin{aligned} z^{(n)}(t) &\leq -\sum_{i=1}^{m} Q_i(t) f_i\left(\frac{1}{\delta} z(\sigma_i(t))\right) \\ &= -\sum_{i=1}^{m} Q_i(t) \frac{f_i\left(\frac{1}{\delta} z(\sigma_i(t))\right)}{z(\sigma_i(t))} z(\sigma_i(t)) \end{aligned} \tag{3.7}$$

for $t \geq t_6$. Since $z(t)>0$ is bounded and increasing, $\lim_{t\to\infty} z(t) = L$ $(-\infty < L < +\infty)$. By the continuity of f, we have

$$\lim_{t\to\infty} \frac{f_i\left(\frac{1}{\delta} z(\sigma_i(t))\right)}{z(\sigma_i(t))} = \frac{f_i\left(\frac{L}{\delta}\right)}{L} > 0.$$

Then, there is a $t_7 \geq t_6$ such that as $t \geq t_7$ for $i=1,2,\cdots,m$

$$\lim_{t\to\infty} \frac{f_i\left(\frac{1}{\delta} z(\sigma_i(t))\right)}{z(\sigma_i(t))} = \frac{f_i\left(\frac{L}{\delta}\right)}{2L} = \alpha > 0. \tag{3.8}$$

By (3.7) and (3.8), we obtain

$$z^{(n)}(t) \leq -\alpha \sum_{i=1}^{m} Q_i(t) z(\sigma_i(t)) \text{ for } t \geq t_7. \tag{3.9}$$

Define

$$w(t)=\frac{z^{(n-1)}(t)}{z\left(\frac{1}{\delta}(\sigma_i(t))\right)}.$$

We know from (3.5) that, there is a $t_8 \geq t_7$ such that $w(t)>0$ for sufficiently large $t \geq t_8$. Since $z(t)>0$ is increasing, there exists a $t_9 \geq t_8$ such that $z(\sigma_i(t)) \geq z\left(\frac{1}{\delta}\sigma_i(t)\right)>0$ for sufficiently large $t \geq t_9$. We may get a result together with (3.9) such that

$$w'(t)=\frac{z\left(\frac{1}{\delta}\sigma_i(t)\right)z^{(n)}(t)-z'\left(\frac{1}{\delta}\sigma_i(t)\right)z^{(n-1)}(t)\frac{\sigma_i'(t)}{\delta}}{z^2\left(\frac{1}{\delta}\sigma_i(t)\right)} \quad (3.10)$$

$$=\frac{z^{(n)}(t)}{z\left(\frac{1}{\delta}\sigma_i(t)\right)}-\frac{1}{\delta}w(t)\frac{z'\left(\frac{1}{\delta}\sigma_i(t)\right)}{z\left(\frac{1}{\delta}\sigma_i(t)\right)}\sigma_i'(t).$$

We know from (3.5) that, $z'(t)>0$ and $z^{(n-1)}(t)>0$ for $t \geq t_9$. Since $\sigma_i(t) \leq t$ and $\sigma_i'(t)>0$, there exists a constant $M>0$ and a $t_{10} \geq t_9$ such that

$$z'\left(\frac{1}{\delta}\sigma_i(t)\right) \geq M\sigma_i^{n-2}(t)\sigma_i'(t)z^{(n-1)}(\sigma_i(t)) \geq M\sigma\sigma_i^{n-2}(t)\sigma_i'(t)z^{(n-1)}(t)$$

for $\lambda=\frac{1}{\delta}$ and $z'(t)$ and $t \geq t_{10}$ by Lemma 2.2. Therefore, we may get a result together with (3.10)

$$w'(t) \leq -\alpha\sum_{i=1}^{m}Q_i(t)-\frac{M}{\delta}w^2(t)\sigma_i^{n-2}(t)\sigma_i'(t) \quad (3.11)$$

From (3.11), we have

$$\alpha\sum_{i=1}^{m}Q_i(t) \leq -w'(t)-\frac{M}{\delta}w^2(t)\sigma_i^{n-2}(t)\sigma_i'(t),\ (t \geq t_{10}) \quad (3.12)$$

If we multiply (3.12) by $\varphi(t)$ and integrate it from t_{10} to t, we obtain

$$\alpha \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s) ds \leq -\int_{t_{10}}^{t} \varphi(s) w'(s) ds$$

$$-\frac{M}{\delta} \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma_i'(s) ds$$

$$= -\varphi(t) w(t) + \varphi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \varphi'(s) w(s) ds$$

$$-\frac{M}{\delta} \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma_i'(s) ds$$

$$\leq \varphi(t_{10}) w(t_{10}) - \frac{M}{\delta} \int_{t_{10}}^{t} \varphi(s) w^2(s) \sigma_i^{n-2}(s) \sigma_i'(s)$$

$$\times \left[w(s) - \frac{\delta \varphi'(s)}{2M\varphi(s) \sigma_i^{n-2}(s) \sigma_i'(s)} \right]^2 ds$$

$$+ \int_{t_{10}}^{t} \frac{\delta \left[\varphi'(s) \right]^2}{4M\varphi(s) \sigma_i^{n-2}(s) \sigma_i'(s)} ds$$

$$\leq \varphi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \frac{\delta [\varphi'(s)]^2}{4M\varphi(s) \sigma_i'(s) \sigma^{n-2}{}_i(s)} ds < +\infty .$$

Therefore, we have

$$+\infty = \alpha \limsup_{t \to \infty} \int_{t_{10}}^{t} \varphi(s) \sum_{i=1}^{m} Q_i(s) ds$$

$$\leq \varphi(t_{10}) w(t_{10}) + \int_{t_{10}}^{t} \frac{\delta \left[\varphi'(s) \right]^2}{4M\varphi(s) \sigma_i^{n-2}(s) \sigma_i'(s)} ds < +\infty$$

for $i = 1, 2, \cdots, m$ by (C_3). This is a contradiction. If we assume that $y(t) < 0$ then we may prove when $y(t) < 0$ as in Teorem 3.1. Hence, the proof is completed.

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