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On Generalized Recurrent and Generalized Concircularly Recurrent Weyl Manifolds

İlhan GÜL*[1]

Abstract

In the present work, generalized recurrent and generalized concircularly recurrent Weyl manifolds are examined. Nearly quasi-Einstein Weyl manifolds are defined and it is proved that if a generalized recurrent or generalized concircularly recurrent Weyl manifold admits a special concircular vector field, then the manifold is a nearly quasi-Einstein Weyl manifold. Also, some other results are presented.

Keywords: Generalized recurrent Weyl manifold, generalized concircularly recurrent Weyl manifold, nearly quasi-Einstein Weyl manifold

1. INTRODUCTION

In 1918, H. Weyl introduced a generalization of Riemannian geometry to unify electromagnetism with gravity as a fully geometric model [1]. A Weyl manifold is a conformal manifold equipped with a torsion free connection preserving the conformal structure, called a Weyl connection.

An n-dimensional Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where ∇ denotes the Levi-Civita connection and R is the Riemannian curvature tensor. Locally symmetric manifold has been generalized to recurrent manifolds [2], generalized recurrent manifolds [3]-[4], concircularly recurrent manifolds [5], generalized concircularly recurrent manifolds [6], Ricci recurrent manifolds [7], generalized Ricci recurrent manifolds [8] etc. Recently, there are some studies on generalizations of symmetric manifolds [9-11].

On the other hand, generalized recurrent and generalized projectively recurrent Weyl manifolds are introduced by Canfes [12]; generalized concircularly recurrent and conformally recurrent Weyl manifolds are defined by Arsan and Yıldırım [13].

A non-flat Riemannian manifold of dimension (n > 2) is called a generalized recurrent manifold if its curvature tensor R^h_{ijk} satisfies the condition

$$\nabla_s R^i_{jkh} = A_s R^i_{jkh} + B_s (g_{jk} \delta^i_h - g_{jh} \delta^i_k), \qquad (1)$$

where A_s and B_s are two 1-forms, B_s is non-zero [3]. If $B_s = 0$, then the manifold is a recurrent manifold.

A non-flat Riemannian manifold of dimension (n > 2) is called a generalized concircularly recurrent manifold [6] if

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$$\nabla_s \tilde{C}^i_{jkh} = A_s \tilde{C}^i_{jkh} + B_s (g_{jk}\delta^i_h - g_{jh}\delta^i_k), \qquad (2)$$

where A_s and B_s are two 1-forms, B_s is non-zero. The concircular curvature tensor $\tilde{C}^i_{jkh}$ is defined by

$$\tilde{C}^i_{jkh} = R^i_{jkh} - \frac{r}{n(n-1)}\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right), \qquad (3)$$

where r is the scalar curvature of the manifold.

If $B_s = 0$, then the manifold reduces to a concircularly recurrent manifold.

A non-flat Riemannian manifold of dimension (n > 2) is called a generalized Ricci recurrent manifold if its Ricci tensor R_{ij} satisfies the condition

$$\nabla_s R_{ij} = A_s R_{ij} + B_s g_{ij}, \qquad (4)$$

where A_s and B_s are two 1-forms, B_s is non-zero [8]. If $B_s = 0$, then the manifold reduces to a Ricci recurrent manifold.

A non-flat Riemannian manifold (n > 2) is called a nearly quasi Einstein manifold if the components of its Ricci tensor R_{ij} are non-zero and satisfy the condition

$$R_{ij} = a g_{ij} + b E_{ij} \qquad (5)$$

where a, b are scalars of which $b \neq 0$ and E_{ij} is a symmetric (0,2) tensor [14].

The above definitions for Weyl manifolds are all given in the next section. In this paper, we investigate generalized recurrent and generalized concircularly recurrent Weyl manifolds.

2. PRELIMINARIES

An n-dimensional differentiable manifold M having a conformal metric tensor g and a torsion-free connection D satisfying the following condition

$$D_k\, g_{ij} = 2\,\omega_k\, g_{ij}, \qquad (6)$$

where ω is a 1-form, is called a Weyl manifold and it is denoted by $M_n(g,\omega)$.

If ω is locally a gradient, $M_n(g,\omega)$ is conformal to a Riemannian manifold.

Under the conformal change of the metric tensor g,

$$\tilde{g}_{ij} = \lambda^2 g_{ij},\ \lambda > 0 \text{ is a function}, \qquad (7)$$

the 1-form ω changes as follows:

$$\widetilde{\omega}_k = \omega_k + D_k \ln\lambda. \qquad (8)$$

It is not hard to see that $M_n(\tilde{g},\widetilde{\omega})$ satisfies the equation (6) and therefore, we obtain the same Weyl manifold ([15]-[17]).

Throughout the paper, the Einstein convention of summing over the repeated indices will be adopted.

We have the following basic tensors for a Weyl manifold [18]:

$$v^j\, W^p_{jkl} = (D_k D_l - D_l D_k)v^p, \qquad (9)$$

$$W_{hjkl} = g_{hp} W^p_{jkl}, \qquad (10)$$

$$W_{ij} = W^p_{ijp} = g^{hk} W_{hijk}, \qquad (11)$$

$$W = g^{ij} W_{ij}\,. \qquad (12)$$

Here, W_{ij} and W represent the Ricci and the scalar curvature tensor, respectively.

From (9) it follows that

$$W^p_{jkl} = \partial_k \Gamma^p_{jl} - \partial_l \Gamma^p_{jk} + \Gamma^p_{hk}\Gamma^h_{jl} - \Gamma^p_{hl}\Gamma^h_{jk}, \qquad (13)$$

where $\partial_k = \frac{\partial}{\partial x^k}$ and Γ^i_{kl} are Weyl connection coefficients and defined by

$$\Gamma^i_{kl} = \left\{ {i \atop kl} \right\} - g^{im}(g_{mk}\omega_l + g_{ml}\omega_k - g_{kl}\omega_m). \qquad (14)$$

Here, $\left\{ {i \atop kl} \right\}$ are the Levi-Civita connection coefficients.

The following relations hold for a Weyl manifold [18]:

$$W_{ijkl} + W_{ijlk} = 0 \quad (15)$$

$$W_{ijkl} + W_{jikl} = 4g_{ij}D_{[l}\omega_{k]} \quad (16)$$

$$W_{[ij]} = n\, D_{[i}\omega_{j]}. \quad (17)$$

Here, brackets indicate the antisymmetric parts of the corresponding tensors.

A quantity A is called a satellite of g with weight p if it admits a transformation of the form

$$\tilde{A} = \lambda^p A, \quad (18)$$

under the change (7) of the metric tensor g [15].

The prolonged covariant derivative of a satellite A of g with weight p is defined by [15],

$$\dot{D}_k A = D_k A - p\omega_k A. \quad (19)$$

From (6) and (19) it follows that $\dot{D}_k g_{ij} = 0$.

We also note that the prolonged covariant differentiation preserves the weights of the satellites of g.

If the scalar curvature of a Weyl manifold is prolonged covariantly constant i.e. $\dot{D}_i W = 0$, and since the weight of W is *-2*, we get

$$\dot{D}_i W = D_i W + 2\omega_i W = 0. \quad (20)$$

Hence, we find

$$\omega_i = -\frac{D_i W}{2W}, \quad (21)$$

which means that ω_i is locally a gradient. Therefore, the Weyl manifold is conformal to a Riemannian manifold.

Definition 1 A Weyl manifold is said to be generalized concircularly recurrent if

$$\dot{D}_s Z^i_{jkh} = A_s Z^i_{jkh} + B_s(g_{jk}\delta^i_h - g_{jh}\delta^i_k), \quad (22)$$

where A_s and B_s are 1-forms of weight *0* and *-2*, respectively [13].

Here, Z^i_{jkh} is the concircular curvature tensor of a Weyl manifold which is given by [19]:

$$Z^i_{jkh} = W^i_{jkh} - \frac{W}{n(n-1)}(g_{jk}\delta^i_h - g_{jh}\delta^i_k). \quad (23)$$

If $B_s = 0$ in (22), then the manifold is concircularly recurrent.

Using (23) in (22), we have the following equation:

$$\dot{D}_s W^i_{jkh} = A_s W^i_{jkh} + \left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)\left(B_s - \frac{A_s W}{n(n-1)} + \frac{\dot{D}_s W}{n(n-1)}\right). \quad (24)$$

Definition 2 A Weyl manifold is said to be generalized Ricci recurrent if

$$\dot{D}_s W_{ij} = A_s W_{ij} + B_s g_{ij}, \quad (25)$$

where A_s and B_s are 1-forms of weight *0* and *-2*, respectively [12]. If $B_s = 0$ in (25), the manifold reduces to a Ricci recurrent manifold.

Definition 3 A Weyl manifold is said to be generalized recurrent if

$$\dot{D}_s W^i_{jkh} = A_s W^i_{jkh} + B_s(g_{jk}\delta^i_h - g_{jh}\delta^i_k), \quad (26)$$

where A_s and B_s are 1-forms of weight *0* and *-2*, respectively [12]. In particular, if $B_s = 0$ the manifold is recurrent.

Definition 4 A Weyl manifold $M_n(g,\omega)$ is said to be a nearly quasi-Einstein Weyl manifold if $W_{(ij)}$, the symmetric part of W_{ij}, satisfies the condition

$$W_{(ij)} = \alpha g_{ij} + \beta E_{ij}, \quad (27)$$

where α is a function of weight *-2*, and the sum of the weight of the function β and the symmetric tensor E_{ij} is *0*.

Example 1 Consider a *3*-dimensional Weyl manifold M_3 with a metric by,

$ds^2 = g_{ij}dx^i dx^j = e^{x^1}[(dx^1)^2 + (dx^2)^2] + (dx^3)^2$ and a 1-form $\omega = e^{x^1}dx^2 + dx^3$. The nonzero Weyl connection coefficients are

$$\Gamma^1_{11} = \frac{1}{2}, \qquad \Gamma^1_{12} = \Gamma^1_{21} = -e^{x^1},$$

$$\Gamma^1_{13} = \Gamma^1_{31} = -1, \qquad \Gamma^1_{22} = -\frac{1}{2}$$

$$\Gamma^2_{11} = e^{x^1}, \qquad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{2},$$

$$\Gamma^2_{22} = -e^{x^1}, \qquad \Gamma^2_{23} = \Gamma^2_{32} = -1,$$

$$\Gamma^2_{33} = 1, \qquad \Gamma^3_{11} = \Gamma^3_{22} = e^{x^1},$$

$$\Gamma^3_{23} = \Gamma^3_{32} = -e^{x^1}, \qquad \Gamma^3_{33} = -1.$$

It is easy to see that $M_3(g,\omega)$ is a Weyl manifold.

We can find the nonzero components of the Ricci tensor as follows:

$$W_{11} = e^{x^1}\left(1+e^{x^1}\right), W_{12} = -W_{21} = \frac{3}{2}e^{x^1},$$

$$W_{22} = e^{x^1}, W_{23} = W_{32} = -e^{x^1}, W_{33} = e^{x^1}.$$

Moreover, we have

$$W_{(11)} = e^{x^1}\left(1+e^{x^1}\right), \qquad W_{(22)} = e^{x^1},$$

$$W_{(23)} = -e^{x^1}, W_{(33)} = e^{x^1},$$

$$W = 2\left(1+e^{x^1}\right).$$

If $\alpha = 1+e^{x^1}$, $\beta = -(1+e^{x^1})$ and the components of the symmetric (0,2) tensor E_{ij} are

$$E_{11} = E_{12} = E_{13} = E_{21} = E_{31} = 0,$$

$$E_{22} = \frac{e^{2x^1}}{1+e^{x^1}}, E_{33} = \frac{1}{1+e^{x^1}},$$

$$E_{23} = E_{32} = \frac{e^{x^1}}{1+e^{x^1}},$$

then, (27) holds.

Thus, $M_3(g,\omega)$ is a nearly quasi-Einstein Weyl manifold.

3. GENERALIZED RECURRENT AND GENERALIZED CONCIRCULARLY RECURRENT WEYL MANIFOLDS

A generalized recurrent Weyl manifold is concircularly recurrent [13]. Conversely, assume that $M_n(g,\omega)$ is concircularly recurrent i.e.

$$\dot{D}_s Z^i_{jkh} = A_s Z^i_{jkh}. \tag{28}$$

Then, using (23) in (28), we get

$$\dot{D}_s\left(W^i_{jkh} - \frac{W}{n(n-1)}\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)\right) = A_s\left(W^i_{jkh} - \frac{W}{n(n-1)}\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)\right). \tag{29}$$

Now, the above equation can be written

$$\dot{D}_s W^i_{jkh} = A_s W^i_{jkh} + B_s\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right), \tag{30}$$

where $B_s = \frac{\dot{D}_s W - A_s W}{n(n-1)}$. Therefore, we can state the following theorem:

Theorem 1 A necessary and sufficient condition for $M_n(g,\omega)$ to be generalized recurrent is that the $M_n(g,\omega)$ is concircularly recurrent.

Theorem 2 A generalized concircularly recurrent Weyl manifold is generalized recurrent.

Proof. Assume that $M_n(g,\omega)$ is generalized concircularly recurrent. Then (24) can be written

$$\dot{D}_s W^i_{jkh} = A_s W^i_{jkh} + C_s\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right), \tag{31}$$

where $C_s = B_s - \frac{\dot{D}_s W - A_s W}{n(n-1)}$ from which we conclude that $M_n(g,\omega)$ is generalized recurrent.

Theorem 3 If a generalized recurrent Weyl manifold admits a special concircular vector field of weight *-2*, then the manifold is a nearly quasi-Einstein Weyl manifold.

Proof. A vector field ρ of weight *-2* defined by $A_j = \rho^i g_{ij}$ is said to be special concircular vector field if

$$\dot{D}_i A_j = \alpha g_{ij}, \tag{32}$$

where α is a function of weight *-2*. It is easy to see that the weight of the 1-form A_j is *0*.

Assume that a generalized recurrent Weyl manifold admits a special concircular vector field as defined in (32). Then applying the Ricci identity to (32) gives

$$A_s W^s_{kij} = \dot{D}_i \dot{D}_j A_k - \dot{D}_j \dot{D}_i A_k = \dot{D}_i(\alpha g_{jk}) - \dot{D}_j(\alpha g_{ik}) = g_{jk}\dot{D}_i\alpha - g_{ik}\dot{D}_j\alpha. \quad (33)$$

Transvecting (33) with g^{jk}, we get

$$A_s W^s_{kij} g^{jk} = (n-1)\dot{D}_i\alpha. \quad (34)$$

Now, taking the covariant derivative of (34), we have

$$(n-1)\dot{D}_r\dot{D}_i\alpha = \dot{D}_r\left(A_s W^s_{kij} g^{jk}\right) = A_s g^{jk}\dot{D}_r W^s_{kij} + W^s_{kij} g^{jk}\dot{D}_r A_s. \quad (35)$$

If we use (26) and (32) in the above equation, we obtain

$$\begin{aligned}(n-1)\dot{D}_r\dot{D}_i\alpha &= A_s g^{jk}\left(A_r W^s_{kij} + B_r\left(g_{ik}\delta^s_j - g_{kj}\delta^s_i\right)\right) + W^s_{kij} g^{jk}(\alpha g_{rs}) \\ &= A_r A_s W^s_{kij} g^{jk} + A_s B_r g^{jk} g_{ik}\delta^s_j - A_s B_r g^{jk} g_{kj}\delta^s_i + \alpha g_{rs} W^s_{kij} g^{jk} \\ &= A_r(n-1)\dot{D}_i\alpha + (1-n)A_i B_r + \alpha g^{jk} g_{rs} W^s_{kij}, \end{aligned} \quad (36)$$

and hence we get

$$(n-1)(\dot{D}_r\dot{D}_i\alpha - A_r\dot{D}_i\alpha + A_i B_r) = \alpha g^{jk} W_{rkij}. \quad (37)$$

Now, transvecting (16) with g^{jk} gives

$$W_{rkij} g^{jk} = -W_{krij} g^{jk} + 4 g_{kr} D_{j[}\omega_{i]}\, g^{jk} = -W_{ri} + 4D_{[r}\omega_{i]} \quad (38)$$

Also, we have

$$A_r\dot{D}_i\alpha = \dot{D}_i(\alpha A_r) - \alpha\dot{D}_i A_r = \dot{D}_i(\alpha A_r) - \alpha^2 g_{ir}. \quad (39)$$

Using (38) and (39) in (37), we obtain

$$(n-1)\left(\dot{D}_r\dot{D}_i\alpha - \dot{D}_i(\alpha A_r) + \alpha^2 g_{ir} + A_i B_r\right) = \alpha\left(-W_{ri} + 4D_{[r}\omega_{i]}\right), \quad (40)$$

from which we get

$$W_{ri} = (1-n)\alpha g_{ir} + \left(\frac{1-n}{\alpha}\right)\left(\dot{D}_r\dot{D}_i\alpha - \dot{D}_i(\alpha A_r) + A_i B_r\right) + 4D_{[r}\omega_{i]}. \quad (41)$$

If we define a (0,2) tensor E_{ri} such that $E_{ri} = \dot{D}_r\dot{D}_i\alpha - \dot{D}_i(\alpha A_r) + A_i B_r$, then (41) can be written as follows:

$$W_{ri} = (1-n)\alpha g_{ir} + \left(\frac{1-n}{\alpha}\right)E_{ri} + 4D_{[r}\omega_{i]}.$$

Taking the symmetric part of the above equation, we find that

$$W_{(ri)} = \varphi g_{ri} + \phi E_{(ri)}, \quad (42)$$

where $\varphi = (1-n)\alpha$, $\phi = \frac{1-n}{\alpha}$ are functions of weight *-2* and *2*, respectively and $E_{(ri)}$ is the symmetric part of E_{ri} with weight *-2*. Hence, the manifold is a nearly quasi-Einstein.

Lemma 1 Scalar curvature tensor of a generalized concircularly recurrent Weyl manifold is prolonged covariantly constant if and only if $A_s W = \frac{n}{2}\left(A^j W_{js} + A_i g^{jk} W^i_{jks}\right) - \frac{n(n-1)(n-2)}{2} B_s$.

Proof. Suppose that the Weyl manifold is generalized concircularly recurrent. Permuting (24) cyclically with respect to *s*, *k*, *h*, we obtain two more equations such that

$$\dot{D}_k W^i_{jhs} = A_k W^i_{jhs} + (g_{jh}\delta^i_s - g_{js}\delta^i_h)(B_k - \frac{A_k W}{n(n-1)} + \frac{\dot{D}_k W}{n(n-1)}, \quad (43)$$

$$\dot{D}_h W^i_{jsk} = A_h W^i_{jsk} + (g_{js}\delta^i_k - g_{jk}\delta^i_s)(B_h - \frac{A_h W}{n(n-1)} + \frac{\dot{D}_h W}{n(n-1)}, \quad (44)$$

Now taking the sum of (24), (43), (44) and then applying second Bianchi Identity, we have

$$0 = A_s W^i_{jkh} + A_k W^i_{jhs} + A_h W^i_{jsk}$$
$$+\left(g_{jk}\delta^i_h - g_{jh}\delta^i_k\right)\left(B_s - \frac{A_s W}{n(n-1)} + \frac{\dot{D}_s W}{n(n-1)}\right)$$
$$+\left(g_{jh}\delta^i_s - g_{js}\delta^i_h\right)\left(B_k - \frac{A_k W}{n(n-1)} + \frac{\dot{D}_k W}{n(n-1)}\right)$$
$$+\left(g_{js}\delta^i_k - g_{jk}\delta^i_s\right)\left(B_h - \frac{A_h W}{n(n-1)} + \frac{\dot{D}_h W}{n(n-1)}\right). \quad (45)$$

Contracting the above equation with respect to i and h, we get

$$0 = A_s W_{jk} - A_k W_{js} + A_i W^i_{jsk} + (n-2)\, g_{jk}\left(B_s - \frac{A_s W}{n(n-1)} + \frac{\dot{D}_s W}{n(n-1)}\right) + (2-n) g_{js}\left(B_k - \frac{A_k W}{n(n-1)} + \frac{\dot{D}_k W}{n(n-1)}\right) \quad (46)$$

Transvecting (46) with g^{jk} and using $W^i_{jsk} = -W^i_{jks}$ we find that

$$0 = A_s W - A^j W_{js} - g^{jk} A_i W^i_{jks} + (n-1)(n-2)\left(B_s - \frac{A_s W}{n(n-1)} + \frac{\dot{D}_s W}{n(n-1)}\right). \quad (47)$$

After rearranging the terms, we obtain the following equation

$$A_s W = \frac{n}{2}\left(A^j W_{js} + A_i g^{jk} W^i_{jks}\right) - \frac{n(n-1)(n-2)}{2} B_s - \left(\frac{n-2}{2}\right)\dot{D}_s W. \quad (48)$$

By hypothesis to be prolonged covariantly constant i.e. $\dot{D}_s W = 0$, we conclude the proof.

Theorem 4 If the scalar curvature of a generalized concircularly recurrent Weyl manifold is prolonged covariantly constant, then the manifold reduces to a generalized Ricci recurrent manifold.

Proof. Contracting (24) with respect to i and h, we find that

$$\dot{D}_s W_{jk} = A_s W_{jk} + (n-1) g_{jk}\left(B_s - \frac{A_s W}{n(n-1)} + \frac{\dot{D}_s W}{n(n-1)}\right). \quad (49)$$

Using (48) and $\dot{D}_s W = 0$ in the above equation, we get

$$\dot{D}_s W_{jk} = A_s W_{jk} + g_{jk}\left(\frac{n(n-1)}{2} B_s - \frac{1}{2}\left(A^j W_{js} + g^{jk} A_i W^i_{jks}\right)\right). \quad (50)$$

Hence the above equation can be written

$$\dot{D}_s W_{jk} = A_s W_{jk} + C_s g_{jk},$$

where $C_s = \frac{n(n-1)}{2} B_s - \frac{1}{2}\left(A^j W_{js} + g^{jk} A_i W^i_{jks}\right)$ from which we conclude that the manifold is Ricci recurrent.

Here, we also note that the manifold under consideration is conformal to Riemannian manifold, since $\dot{D}_s W = 0$.

Theorem 5 If a generalized concircularly recurrent Weyl manifold admits a special concircular vector field of weight *-2*, then the manifold is a nearly quasi-Einstein Weyl manifold.

Proof. Suppose that a generalized concircularly recurrent Weyl manifold admits a special concircular vector field of weight *-2*, then we have

$$\dot{D}_i A_j = \alpha g_{ij},$$

where α is a function of weight *-2*. As in the proof of Theorem 3, if we apply the Ricci identity to the above equation, and then transvecting the resulted equation with g^{jk}, we get

$$A_s W^s_{kij} g^{jk} = (n-1)\dot{D}_i \alpha. \quad (51)$$

Now, the covariant derivative of the above equation gives

$$(n-1)\dot{D}_r \dot{D}_i \alpha = \dot{D}_r\left(A_s W^s_{kij} g^{jk}\right) = A_s g^{jk} \dot{D}_r W^s_{kij} + W^s_{kij} g^{jk} \dot{D}_r A_s.$$

If we use (24) in the above equation, we obtain

$$(n-1)\dot{D}_r\dot{D}_i\alpha = A_s g^{jk}\left(A_r W^s_{kij} + \left(g_{ki}\delta^s_j - g_{kj}\delta^s_i\right)\left(B_r - \frac{A_r W}{n(n-1)} + \frac{\dot{D}_r W}{n(n-1)}\right)\right) + W^s_{kij} g^{jk}(\alpha g_{rs})$$

$$= A_r(n-1)\dot{D}_i\alpha + (1-n)A_i\left(B_r - \frac{A_r W}{n(n-1)} + \frac{\dot{D}_r W}{n(n-1)}\right) + \alpha g_{rs} W^s_{kij} g^{jk},$$

and hence we get

$$(n-1)\left[\dot{D}_r\dot{D}_i\alpha - A_r\dot{D}_i\alpha + A_i\left(B_r - \frac{A_r W}{n(n-1)} + \frac{\dot{D}_r W}{n(n-1)}\right)\right] = \alpha g^{jk} W_{rkij}. \quad (52)$$

Using (38) and (39) in (52), we obtain

$$(n-1)\left(\dot{D}_r\dot{D}_i\alpha - \dot{D}_i(\alpha A_r) + \alpha^2 g_{ri} + A_i\left(B_r - \frac{A_r W}{n(n-1)} + \frac{\dot{D}_r W}{n(n-1)}\right)\right) = \alpha\left(-W_{ri} + 4D_{[r}\omega_{i]}\right),$$

from which we get

$$W_{ri} = (1-n)\alpha g_{ir} + \left(\frac{1-n}{\alpha}\right)\left(\dot{D}_r\dot{D}_i\alpha - \dot{D}_i(\alpha A_r) + A_i\left(B_r - \frac{A_r W}{n(n-1)} + \frac{\dot{D}_r W}{n(n-1)}\right)\right) + 4D_{[r}\omega_{i]}. \quad (53)$$

If we define a (0,2) tensor $\bar{E}_{ri}$ such that $\bar{E}_{ri} = \dot{D}_r\dot{D}_i\alpha - \dot{D}_i(\alpha A_r) + A_i\left(B_r - \frac{A_r W}{n(n-1)} + \frac{\dot{D}_r W}{n(n-1)}\right)$,

then (53) can be written as follows:

$$W_{ri} = (1-n)\alpha g_{ir} + \left(\frac{1-n}{\alpha}\right)\bar{E}_{ri} + 4D_{[r}\omega_{i]}.$$

Taking the symmetric part of the above equation, we find that

$$W_{(ri)} = \varphi g_{ri} + \phi\, \bar{E}_{(ri)},$$

where $\varphi = (1-n)\alpha$, $\phi = \frac{1-n}{\alpha}$ are functions of weight *-2* and *2*, respectively and $\bar{E}_{(ri)}$ is the symmetric part of $\bar{E}_{ri}$ with weight *-2*. Hence, the manifold is a nearly quasi-Einstein.

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The Declaration of Ethics Committee Approval

This study does not require ethics committee permission or any special permission.

The Declaration of Research and Publication Ethics

The author of the paper declare that he comply with the scientific, ethical and quotation rules of SAUJS in all processes of the paper and that he does not make any falsification on the data collected. In addition, he declare that Sakarya University Journal of Science and its editorial board have no responsibility for any ethical violations that may be encountered, and that this study has not been evaluated in any academic publication environment other than Sakarya University Journal of Science.

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