Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (5) (2022), 1288 – 1303
DOI : 10.15672/hujms.960966

Research Article

# Characterization of proper curves and proper helix lying on $S_1^2(r)$

Buddhadev Pal*, Santosh Kumar

*Department of Mathematics, Institute of Science, Banaras Hindu University, Varanasi-221005, India*

## Abstract

In this paper, we analyse the proper curve $\gamma(s)$ lying on the pseudo-sphere. We develop an orthogonal frame $\{V_1, V_2, V_3\}$ along the proper curve, lying on pseudosphere. Next, we find the condition for $\gamma(s)$ to become $V_k-$ slant helix in Minkowski space. Moreover, we find another curve $\beta(\bar{s})$ lying on pseudosphere or hyperbolic plane heaving $V_2 = \bar{V_2}$ for which $\{\bar{V_1}, \bar{V_2}, \bar{V_3}\}$, an orthogonal frame along $\beta(\bar{s})$. Finally, we find the condition for curve $\gamma(s)$ to lie in a plane.

**Mathematics Subject Classification (2020).** 53C50, 53C40

**Keywords.** Minkowski space, pseudo sphere, proper helix of order 2, proper curve of order 2, $V_k$-slant helix

## 1. Introduction

The various properties of a regular curve in the Riemannian manifold can be described by constructing the Frenet frame along the curve. In 1982, K. Sakamoto [13], defined the helix of order $d$ by using helical geodesic immersion from connected complete Riemannian manifold $(N)$ to a Riemannian manifold $(M)$. Let $f : N \longrightarrow M$ be an isometric immersion and let $\gamma$ be any geodesic in $N$ and $f \circ \gamma$ be a curve of constant curvature of osculating order $d$ in $M$ then $f \circ \gamma$ is known as a helix of order $d$ in $M$. The proper curve and proper helix of order $d$ in pseudo-Riemannian manifold were defined in [14]. A general helix in Euclidean space is a curve whose ratio of curvature and torsion is constant, this condition was stated by M. A. Lancret in 1802 whereas the proof for the condition was given by B. de Saint Venant and revisited by M. Barros [6] in 1997. A curve in Euclidean space whose principal normal makes a constant angle with some fixed direction is known as a slant helix. The curve $\gamma(s)$ in Euclidean space is slant helix iff the function [10],

$$M(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}\left(\frac{\tau}{\kappa}\right)'\right)(s); \kappa(s) \neq 0, \text{ is constant.}$$

*Corresponding Author.
Email addresses: pal.buddha@gmail.com (B. Pal), thakursantoshbhu@gmail.com (S. Kumar)
Received: 01.07.2021; Accepted: 05.03.2022

The slant helix in $E_1^3$ is a curve whose principal normal have an inner product with some non zero fixed direction is constant. The necessary and sufficient conditions for curve to become the slant helix in $E_1^3$ were given in [4]. A timelike or spacelike curve with spacelike normal is said to be a slant helix in $E_1^3$ iff any one of the following two functions is constant[4],

$$K(s) = \left(\frac{\kappa^2}{(\tau^2-\kappa^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'\right)(s) \ or \ L(s) = \left(\frac{\kappa^2}{(\kappa^2-\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'\right)(s); \tau^2 - \kappa^2 \neq 0.$$

But, if the curve is spacelike and its normal is timelike vector then the function,

$$M(s) = \left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}}(\frac{\tau}{\kappa})'\right)(s); \kappa(s) \neq 0,$$

is constant. Whereas all the spacelike curves with timelike normal are slant helices in $E_1^3$. To study the position vector for slant helix in Euclidean space and Minkowski space, we refer the reader to read the papers [3, 5].

In 2009, İ. Gök, Ç. Camci and H. Hilmi Hacisalihoğlu [9], defined the $V_n-$ slant helix in Euclidean space. A curve $\gamma(s)$ in Euclidean $n-$ space heaving frenet frame $\langle V_1, V_2, ..., V_n\rangle$ and $\kappa_i(i = 1, 2, ...n)$ are non zero curvatures along $\gamma(s)$, then $\gamma(s)$ is said to be $V_n-$ slant helix if there exists a fixed direction $U$, which makes constant angle with Frenet vector field $V_n$ that is, $\langle V_n, U\rangle = \cos\psi$, $\psi \neq \frac{\pi}{2}$, $\psi =$ constant. The $V_n-$ slant helix in pseudo-Riemannian manifold was studied in [16]. Let $\gamma(s)$ be a unit speed curve in $E^3$ and $\langle T, N, B\rangle$ be Frenet frame along the curve $\gamma(s)$, then the plane spanned by $\langle T, N\rangle$, $\langle T, B\rangle$ and $\langle N, B\rangle$ are known as the osculating plane, the rectifying plane and the normal plane, respectively. If the position vector of $\gamma(s)$ lies in rectifying plane then the curve is known as rectifying curve. The conditions for position vector of Euclidean curve $\gamma(s)$ to lie in the rectifying plane was studied in [8].

Saint Venant in 1845, proposed a question, whether in the surface generated by the curve does there exist another curve whose principal normal coincides with the principal normal to the given curve. The answer for this question was given by Bertrand in 1850 by the condition that is, a curve is Bertrand curve in $E^3$ iff curvature $\kappa$ and torsion $\tau$ of curve satisfies the condition $\nu\kappa + \lambda\tau = 1$, where $\nu \neq 0$ and $\lambda$ are constants. Generalized Bertrand curves in Minkowski 3 - space was defined in [15], as a curve whose normal makes the constant angle with its Bertrand mate.

Getting motivation from the above papers [2, 12], we organize our paper as follows: In Section 2, we discuss some basic concepts and results. Section 3, we study the global view of the helix of proper order 2 lying on pseudosphere of radius $r$ from $E_1^3$. In Section 4, we develop the orthogonal frame along the proper curve of order 2 lying on pseudosphere. In section 5, we find the condition for proper curve of order 2 with non constant curvature lying on $S_1^2(r)$ to become globally a $V_k-$ slant helix in $E_1^3$. In section 6, we find the condition for a proper curve to lie in a plane spanned by the vector field constructed in Section 4. In section 7, we find the conditions for another curve whose second vector field coincides with the second vector field of the initial one.

## 2. Preliminaries and some results

A smooth manifold $R^3$ furnished with the Lorentzian metric $g = -dx_1^2 + dx_2^2 + dx_3^2$ is said to be a Minkowski 3-space $E_1^3$. A vector field $V$ in $E_1^3$ is said to be
(1) spacelike if $g(V, V) > 0$,
(2) timelike if $g(V, V) < 0$,
(3) lightlike if $g(V, V) = 0$.

A curve $\gamma : [a, b] \to E^3_1$, where $a, b \in R$ is said to be a spacelike, timelike or lightlike according to the causal character of the tangent vector field along the curve $\gamma(s)$ in $E^3_1$. The pseudo-sphere and hyperbolic plane of radius $r$ and origin 0 in $E^3_1$ are defined by $S^2_1(r) = \{X \in E^3_1 : g(X, X) = r^2\}$, $H^2_0(-r) = \{X \in E^3_1 : g(X, X) = -r^2\}$.

**Definition 2.1** ([11])**.** A surface $M$ in $E^3_1$ is said to be a spacelike surface if the metric induced on surface is Riemannian metric and the surface said to be timelike if the induced metric is Lorentz metric.

Let $\gamma : [a, b] \to S^2_1(r) \subset E^3_1$ be a curve parametrised by arc length parameter s lying on $S^2_1(r)$ and $\gamma(s) = \phi$ be the position vector of curve on $S^2_1(r)$. Now, if $\tilde{\nabla}$ and $\nabla$ are Levi-Civita connections on $E^3_1$ and $S^2_1(r)$ respectively, then Gauss Wiengarten formulae are

$$\begin{cases} \tilde{\nabla}_X Y = \nabla_X Y - \frac{g(X,Y)}{r^2}\phi, \\ \tilde{\nabla}_X \phi = -A_\phi X. \end{cases} \tag{2.1}$$

**Definition 2.2** ([14])**.** A non null curve $\gamma(s)$ is said to a proper curve of order $d$ in pseudo-Euclidean space if there exists an orthogonal frame $\{V_1, ..., V_d\}$ along $\gamma(s)$ and satisfies the Frenet formula

$$\tilde{\nabla}_{V_1} V_i = -\mu_{i-2}\mu_{i-1}\kappa_{i-1}(s)V_{i-1} + \kappa_i(s)V_{i+1}, \qquad 1 \leq i \leq d,$$

where, $V_0 = V_{d+1} = 0$, $\kappa_0(s) = \kappa_d(s) = 0$ and $\tilde{\nabla}$ is Levi-Civita connection on pseudo-Euclidean space. $\kappa_i(s)$ are known as the $i^{th}$ curvatures of the curve $\gamma(s)$ and are given as,

$$\kappa_i(s) = \| \tilde{\nabla}_{V_1} V_i + \mu_{i-2}\mu_{i-1}\kappa_{i-1}V_{i+1} \|, \qquad 1 \leq i < d,$$
$$\mu_{i-1} = g'(V_i, V_i), \qquad 1 \leq i < d.$$

If the curvature functions along the proper curve $\gamma(s)$ of order $d$ are constant, then $\gamma(s)$ is known as a proper helix of order $d$ in pseudo-Euclidean space.

Now, by getting motivation from [16], we define the $V_k-$ slant helix in Minkowski space $E^3_1$.

**Definition 2.3.** A curve $\gamma(s)$ in Minkowski space $E^3_1$ corresponding to an orthogonal frame $\{V_1, V_2, V_3\}$ is said to be a $V_k-$ slant helix if there exists a vector field parallel along $\gamma(s)$ such that $g(V_k, U) = C = constant$, for $k \in \{1, 2, 3\}$. The parallel vector field $U$ is said to an axis of the $V_k$ slant helix, and if the constant $C = 0$ then the axis $U$ is said to be orthogonal to the frame field $V_k$.

**Definition 2.4** ([1])**.** A curve $\gamma(s)$ in $E^3_1$ is said to be a circle if it satisfies the differential equation $\tilde{\nabla}_X \tilde{\nabla}_X X + g(\tilde{\nabla}_X X, \tilde{\nabla}_X X)g(X, X)X = 0$, where $X$ is a tangent vector field along $\gamma(s)$ .

**Definition 2.5** ([7])**.** A curve $\gamma(s)$ in Euclidean space $E^3_1$ is said to be Bertrand curve if there exist another curve $\beta(\bar{s})$ such that the principal normal of $\gamma(s)$ is linearly dependent with the principal normal of $\beta(\bar{s})$. $\beta(\bar{s})$ is known as Bertrand mate of Bertrand curve $\gamma(s)$. The position vector for $\beta(\bar{s})$, is given as $\beta(\bar{s}) = \gamma(s) + \lambda(s)N_1(s)$, where $N_1(s)$ is a principal normal of curve $\gamma(s)$ .

## 3. Proper helix of order 2

**Theorem 3.1.** *A curve $\gamma(s)$ which is locally a geodesic in psedosphere $S^2_1(r)$ but globally a circle in $E^3_1$.*

***Proof.*** Let $\gamma(s)$ is a geodesic in pseudosphere and $\nabla$, $\tilde{\nabla}$ are Levi-Civita connections on pseudo sphere and Minkowski-3 space respectively. Then, $\gamma(s)$ satisfy the equation

$$\nabla_{V_1} V_1 = 0, \tag{3.1}$$

where $V_1$ is unit tangent vector field along $\gamma(s)$ .
**Case 1.** Let $\gamma(s)$ is a spacelike curve lying on pseudosphere. Now using set of equation (2.1) and equation (3.1), we obtain

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1 V_2, \\ \tilde{\nabla}_{V_1} V_2 = -\kappa_1 V_1. \end{cases} \tag{3.2}$$

Here $\kappa_1 = \frac{1}{r}$, $V_2 = -\frac{1}{r}\phi$ and $g(V_2, V_2) = 1$. Using equation (3.2), we get

$$\tilde{\nabla}_{V_1}\tilde{\nabla}_{V_1}V_1 + g(\tilde{\nabla}_{V_1}V_1, \tilde{\nabla}_{V_1}V_1)g(V_1, V_1)V_1 = 0.$$

**Case 2.** Let $\gamma(s)$ be a timelike curve lying on pseudosphere. Then using set of equation (2.1) and equation (3.1), we have

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1 V_2, \\ \tilde{\nabla}_{V_1} V_2 = \kappa_1 V_1. \end{cases} \tag{3.3}$$

Here $\kappa_1 = \frac{1}{r}$, $V_2 = \frac{1}{r}\phi$ and $g(V_2, V_2) = 1$. Using equation (3.3), we get

$$\tilde{\nabla}_{V_1}\tilde{\nabla}_{V_1}V_1 + g(\tilde{\nabla}_{V_1}V_1, \tilde{\nabla}_{V_1}V_1)g(V_1, V_1)V_1 = 0.$$

Thus from Case(1) and Case(2) we can say that $\gamma(s)$ is a circle in $E_1^3$. □

**Theorem 3.2.** *A curve $\gamma(s)$ with curvature $\kappa(s)$ (either $\kappa(s) > \frac{1}{r}$ or $0 < \kappa(s) < \frac{1}{r}$) which is a proper helix of order 2 locally in pseudosphere ($S_1^2(r)$) but globally a circle in $E_1^3$.*

***Proof.*** Let $\gamma(s)$ be a proper helix of order 2 with curvature function $\kappa(s) > \frac{1}{r}$ or $\kappa(s) < \frac{1}{r}$ lying on pseudosphere. If $\nabla$, $\tilde{\nabla}$ are Levi-Civita connections on $S_1^2(r)$ and $E_1^3$, then we have

$$\begin{cases} \nabla_{V_1} V_1 = \kappa Y, \\ \nabla_{V_1} Y = \kappa V_1, \end{cases} \tag{3.4}$$

where $V_1$ is a tangent vector field along $\gamma(s)$ and $Y$ is a vector field orthogonal to $V_1$.

**Case 1.** If $\gamma(s)$ is a spacelike curve with constant curvature function $\kappa(s) \neq \frac{1}{r}$. Then from equation (3.4) and (2.1), we get

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1 V_2, \\ \tilde{\nabla}_{V_1} V_2 = -\epsilon\kappa_1 V_1. \end{cases} \tag{3.5}$$

Here $\kappa_1 = \sqrt{\kappa^2 - \frac{1}{r^2}}$, $V_2 = \frac{1}{\kappa_1}(\kappa Y - \frac{1}{r^2}\phi)$ and $g(V_2, V_2) = \epsilon = \pm 1$, $\epsilon$ is positive one if the curvature function $\kappa(s) < \frac{1}{r}$ and $\epsilon$ is negative for the curvature function $\kappa(s) > \frac{1}{r}$. Now using equation (3.5), we get the equation

$$\tilde{\nabla}_{V_1}\tilde{\nabla}_{V_1}V_1 + g(\tilde{\nabla}_{V_1}V_1, \tilde{\nabla}_{V_1}V_1)g(V_1, V_1)V_1 = 0.$$

**Case 2** Now, let $\gamma(s)$ be a timelike curve with constant curvature function, then from (3.4) and (2.1), we have

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1 V_2, \\ \tilde{\nabla}_{V_1} V_2 = \kappa_1 V_1. \end{cases} \tag{3.6}$$

Here $\kappa_1 = \sqrt{\kappa^2 + \frac{1}{r^2}}$, $V_2 = \frac{1}{\kappa_1}(\kappa Y + \frac{1}{r^2}\phi)$ and $g(V_2, V_2) = 1$. Now using equation (3.6) in Definition 2.4, we get

$$\tilde{\nabla}_{V_1}\tilde{\nabla}_{V_1}V_1 + g(\tilde{\nabla}_{V_1}V_1, \tilde{\nabla}_{V_1}V_1)g(V_1, V_1)V_1 = 0.$$

Thus, from Case (1) and Case (2) we can conclude that the spacelike proper helix of order two $\gamma(s)$ with curvature function $\kappa(s) > \frac{1}{r}$ or $\kappa(s) < \frac{1}{r}$ is a circle in $E_1^3$.

$\square$

**Corollary 3.3.** *Let $\gamma(s)$ be a spacelike curve and a proper helix of order 2 with curvature $\kappa(s) = \frac{1}{r}$ lying on $S_1^2(r)$ then there exist a null vector field parallel along $\gamma(s)$ globally in $E_1^3$.*

***Proof.*** Let $\gamma(s)$ is a spacelike curve and a proper helix of order 2 with constant curvature function $\kappa(s) = \frac{1}{r}$ lying on $S_1^2(r)$ then Frenet formulae along $\gamma(s)$ in $S_1^2(r)$, are given by

$$\begin{cases} \nabla_{V_1} V_1 = \frac{1}{r} Y, \\ \nabla_{V_1} Y = \frac{1}{r} V_1, \end{cases} \tag{3.7}$$

where $V_1$ is tangent vector field along $\gamma(s)$ and $Y$ is a vector field orthogonal to $V_1$. Now using equation (3.7) and equation (2.1), we get

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = V_2, \\ \tilde{\nabla}_{V_1} V_2 = 0. \end{cases} \tag{3.8}$$

Where $V_2 = \kappa Y - \frac{1}{r^2}\phi$ is a null vector field parallel along $\gamma(s)$ in $E_1^3$. $\square$

## 4. Orthogonal frame for proper curve of order 2 with non constant curvature function

**Theorem 4.1.** *Let $\gamma(s)$ be a proper curve of order two lying on $S_1^2(r)$ with non constant curvature function $\kappa(s)$ then there exist an orthogonal frame along $\gamma(s)$ in $E_1^3$, such that*

***i.*** *If $\gamma(s)$ is a spacelike proper curve of order 2 with curvature function $\kappa(s) < \frac{1}{r}$ then the orthogonal frame $\{V_1, V_2, V_3\}$, satisfies the following relations*

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1(s) V_2, \\ \tilde{\nabla}_{V_1} V_2 = -\kappa_1(s) V_1 + \kappa_2(s) V_3, \\ \tilde{\nabla}_{V_1} V_3 = \kappa_2(s) V_2. \end{cases} \tag{4.1}$$

*Here $V_1$ and $V_2$ are spacelike vector fields whereas $V_3$ is a timelike vector field. Darboux vector for corresponding frame is $D = -\kappa_2(s)V_1 + \kappa_1(s)V_3$.*

***ii.*** *If $\gamma(s)$ is a spacelike proper curve of order 2 with curvature function $\kappa(s) > \frac{1}{r}$ then the orthogonal frame $\{V_1, W_2, W_3\}$, satisfies the following relations*

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1(s) W_2, \\ \tilde{\nabla}_{V_1} W_2 = \kappa_1(s) V_1 + \kappa_2(s) W_3, \\ \tilde{\nabla}_{V_1} W_3 = \kappa_2(s) W_2, \end{cases} \tag{4.2}$$

*where $g(V_1, V_1) = g(W_3, W_3) = 1$ and $g(W_2, W_2) = -1$. Axis of rotation for orthogonal frame $\{V_1, W_2, W_3\}$ is $D = \kappa_2 V_1 - \kappa_1 W_3$*

***iii.*** *If $\gamma(s)$ is a timelike proper curve of order 2 with non constant curvature function then the orthogonal frame $\{V_1, Z_2, Z_3\}$, satisfies the following relations*

$$\begin{cases} \tilde{\nabla}_{V_1} V_1 = \kappa_1(s) Z_2, \\ \tilde{\nabla}_{V_1} Z_2 = \kappa_1(s) V_1 + \kappa_2(s) Z_3, \\ \tilde{\nabla}_{V_1} Z_3 = -\kappa_2(s) Z_2, \end{cases} \tag{4.3}$$

*where $g(Z_2, Z_2) = g(Z_3, Z_3) = 1$ and $g(V_1, V_1) = -1$. The Darboux vector for an orthogonal frame $\{V_1, Z_2, Z_3\}$ is $D = \kappa_2 V_1 + \kappa_1 Z_3$.*

*Here $V_1$ is a tangent vector field along $\gamma(s)$ and $\kappa_1(s)$, $\kappa_2(s)$ are first and second curvature functions which are different in all these three cases.*

***Proof.*** Let $\gamma(s)$ be a proper curve of order two lying on $S_1^2(r)$ with non constant curvature function $\kappa(s)$, and $\nabla$ ,$\tilde{\nabla}$ are Levi-Civita connections on $S_1^2(r)$ and $E_1^3$ respectively, then Frenet formulae along $\gamma(s)$ in $S_1^2(r)$ are

$$\begin{cases} \nabla_{V_1} V_1 = \kappa(s) Y, \\ \nabla_{V_1} Y = \kappa(s) X, \end{cases} \tag{4.4}$$

where $V_1$ is a tangent vector field along $\gamma(s)$ in $S_1^2(r)$ and $Y$ is vector field orthogonal to $V_1$.

**Case 1.** If, $\gamma(s)$ is a spacelike curve with non constant curvature function $\kappa(s) < \frac{1}{r}$ then by using equations (2.1)and (4.4), we get the equation

$$\tilde{\nabla}_{V_1} V_1 = \kappa_1(s) V_2, \tag{4.5}$$

where $\kappa_1(s) = \sqrt{\frac{1}{r^2} - \kappa^2(s)}$ and $V_2 = \frac{1}{\kappa_1(s)}(\kappa(s)Y - \frac{1}{r^2}\phi)$. The covariant derivative of vector field $V_2$, with respect to $V_1$ is

$$\tilde{\nabla}_{V_1} V_2 = -\kappa_1(s) V_1 + \kappa_2(s) V_3, \tag{4.6}$$

where $\kappa_2(s) = \frac{\kappa'(s)}{\kappa_1(s)}\sqrt{\frac{\kappa^2(s)}{\kappa_1^2(s)} + 1}$ and $V_3 = \frac{1}{\kappa_2(s)}\left(\frac{\kappa'(s)}{\kappa_1(s)} Y + \frac{\kappa(s)\kappa'(s)}{\kappa_1^2(s)} V_2\right)$. Covariant derivative of $V_3$ along $V_1$, is given by the equation

$$\tilde{\nabla}_{V_1} V_3 = \kappa_2(s) V_2. \tag{4.7}$$

From above equations we see that $g(V_1, V_1) = g(V_2, V_2) = 1$, $g(V_3, V_3) = -1$ and $g(V_1, V_2) = g(V_2, V_3) = g(V_1, V_3) = 0$. Thus $\{V_1, V_2, V_3\}$ is an orthogonal frame along $\gamma(s)$ in $E_1^3$. For positive orientation of orthogonal frame we define the cross product of vector field as $V_1 \times V_2 = -V_3, V_2 \times V_3 = V_1$ and $V_3 \times V_1 = V_2$. Let $D = a_1V_1 + a_2V_2 + a_3V_3$ be a Darboux vector for corresponding frame, then we have

$$D \times V_k = \tilde{\nabla}_{V_1} V_k, \qquad k \in \{1, 2, 3\}. \tag{4.8}$$

Now, using equation (4.8) and the cross product of vector fields we get $a_1 = -\kappa_2, a_2 = 0$ and $a_3 = \kappa_1$. Hence $D = -\kappa_2(s)V_1 + \kappa_1(s)V_3$, is a Darboux vector for orthogonal frame $\{V_1, V_2, V_3\}$ .

**Case 2.** If $\gamma(s)$ is a spacelike curve with non constant curvature function $\kappa(s) > \frac{1}{r}$ then by using equations (2.1) and (4.4), we obtain

$$\tilde{\nabla}_{V_1} V_1 = \kappa_1(s) W_2, \tag{4.9}$$

where $\kappa_1(s) = \sqrt{\kappa^2(s) - \frac{1}{r^2}}$ and $W_2 = \frac{1}{\kappa_1(s)}(\kappa(s)Y - \frac{1}{r^2}\phi)$. The covariant derivative of vector field $W_2$, with respect to $V_1$ is given as

$$\tilde{\nabla}_{V_1} W_2 = \kappa_1(s) V_1 + \kappa_2(s) W_3. \tag{4.10}$$

where $\kappa_2(s) = \frac{\kappa'(s)}{\kappa_1(s)}\sqrt{\frac{\kappa^2(s)}{\kappa_1^2(s)} - 1}$ and $W_3 = \frac{1}{\kappa_2(s)}\left(\frac{\kappa'(s)}{\kappa_1(s)} Y - \frac{\kappa(s)\kappa'(s)}{\kappa_1^2(s)} V_2\right)$. Covariant derivative of $W_3$ along $V_1$, is given by the equation

$$\tilde{\nabla}_{V_1} W_3 = \kappa_2(s) W_2. \tag{4.11}$$

From above equations we see that $g(V_1, V_1) = g(W_3, W_3) = 1$, $g(W_2, W_2) = -1$ and $g(V_1, W_2) = g(W_2, W_3) = g(V_1, W_3) = 0$. Thus $\{V_1, W_2, W_3\}$ is an orthogonal frame along $\gamma(s)$ in $E_1^3$. For positive orientation of orthogonal frame $\{V_1, W_2, W_3\}$ we define the cross product of vector field as $V_1 \times W_2 = W_3, W_2 \times W_3 = V_1$ and $W_3 \times V_1 = -W_2$. Let

$D = a_1V_1 + a_2W_2 + a_3W_3$ be a Darboux vector for an orthogonal frame $\{V_1, W_2, W_3\}$, then we have

$$D \times W_k = \tilde{\nabla}_{V_1} W_k, \qquad k \in \{1, 2, 3\}, W_1 = V_1. \tag{4.12}$$

Now using equation (4.12) and the cross product of vector fields we get $a_1 = \kappa_2, a_2 = 0$ and $a_3 = -\kappa_1$. Hence $D = \kappa_2(s)V_1 - \kappa_1(s)W_3$, is a Darboux vector of the corresponding orthogonal frame.

**Case 3.** If $\gamma(s)$ is a timelike curve with non constant curvature function $\kappa(s)$ then by using equations (2.1) and (4.4), we get the equation

$$\tilde{\nabla}_{V_1} V_1 = \kappa_1(s) Z_2, \tag{4.13}$$

where $\kappa_1(s) = \sqrt{\kappa^2(s) + \frac{1}{r^2}}$ and $Z_2 = \frac{1}{\kappa_1(s)}(\kappa(s)Y + \frac{1}{r^2}\phi)$. The covariant derivative of vector field $Z_2$, with respect to $V_1$ is given by

$$\tilde{\nabla}_{V_1} Z_2 = \kappa_1(s)V_1 + \kappa_2(s)Z_3, \tag{4.14}$$

where $\kappa_2(s) = \frac{\kappa'(s)}{\kappa_1(s)}\sqrt{1 - \frac{\kappa^2(s)}{\kappa_1^2(s)}}$ and $Z_3 = \frac{1}{\kappa_2(s)}\left(\frac{\kappa'(s)}{\kappa_1(s)}Y - \frac{\kappa(s)\kappa'(s)}{\kappa_1^2(s)}V_2\right)$. Covariant derivative of $Z_3$ along $V_1$, is given by the equation

$$\tilde{\nabla}_{V_1} Z_3 = -\kappa_2(s) Z_2. \tag{4.15}$$

From above equations we see that $g(V_1, V_1) = -1$, $g(Z_3, Z_3) = g(Z_2, Z_2) = 1$ and $g(V_1, Z_2) = g(Z_2, Z_3) = g(V_1, Z_3) = 0$. Thus $\{V_1, Z_2, Z_3\}$ is an orthogonal frame along $\gamma(s)$ in $E_1^3$. For positive orientation of orthogonal frame $\{V_1, Z_2, Z_3\}$ we define the cross product of vector field as $V_1 \times Z_2 = Z_3, Z_2 \times Z_3 = -V_1$ and $Z_3 \times V_1 = Z_2$. Let $D = a_1V_1 + a_2Z_2 + a_3Z_3$ be a Darboux vector for an orthogonal frame $\{V_1, Z_2, Z_3\}$, then we have

$$D \times W_k = \tilde{\nabla}_{V_1} Z_k, \qquad k \in \{1, 2, 3\}, Z_1 = V_1. \tag{4.16}$$

Now using equation (4.16) and the cross product of vector fields we get $a_1 = \kappa_2, a_2 = 0$ and $a_3 = \kappa_1$. Hence $D = \kappa_2(s)V_1 + \kappa_1(s)Z_3$, is a Darboux vector of the corresponding orthogonal frame. □

**Note.** In theorem 4.1 we get the different orthogonal frame depending on the causal character and curvature function $\kappa(s)$ of $\gamma(s)$. So for any proper curve of order 2 which is lying on pseudosphere to study its characterizations, we have to discuss the three cases. To eliminate these cases, we are going to use symbols $S_2$ and $S_3$, where $S_2 \in \{V_2, W_2, Z_2\}$ and $S_3 \in \{V_3, W_3, Z_3\}$, such that if $S_2 = V_2$ then $S_3$ must be $V_3$. Similarly if $S_2 = Z_2$, then $S_3$ must be $Z_3$ and for $S_2 = W_2$ implies $S_3 = W_3$. Same argument we will use for the symbols $A_2$ and $A_3$.

## 5. Proper curve of order 2 with non constant curvature function

**Theorem 5.1.** *Let $\gamma(s)$ be a proper curve of order 2 with non constant positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ be globally a $V_1-$ slant helix in $E_1^3$ with an axis $U$ not orthogonal to $V_1$ corresponding to orthogonal frame constructed in Theorem 4.1, iff the ratio of first curvature function and second curvature function is constant.*

***Proof.*** Let $\gamma(s)$ is $V_1-$ slant helix with non constant curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) in $E_1^3$ lying on $S_1^2(r)$, then according to Theorem 4.1 there exist an orthogonal frame along $\gamma(s)$ and the axis $U$ of $V_1-$ slant helix in $E_1^3$ with respect to the corresponding frame can be written as

$$U = \lambda V_1 + \nu_1 S_2 + \nu_2 S_3,$$

where $\lambda$ is non zero constant and $\nu_1(s), \nu_2(s)$ are functions of parameter s. Here $S_2 \in \{V_2, W_2, Z_2\}$ and $S_3 \in \{V_3, W_3, Z_3\}$ according to the orthogonal frame defined in Theorem

4.1. Taking covariant derivative of vector field $U$ with respect to $V_1$ and comparing the equation on both sides , we obtain

$$\begin{cases} \nu_1' + \epsilon\lambda\kappa_1 + \nu_2\kappa_2 = 0, \\ \nu_1\kappa_1 = 0, \\ \nu_2' + \nu_1\kappa_2 = 0. \end{cases} \tag{5.1}$$

Here $\epsilon$ is negative if the curve $\gamma(s)$ is timelike otherwise $\epsilon$ is positive. Since $\kappa_1 \neq 0$, therefore from first and third part of the equation (5.1) , we get $\nu_2 = C = -\epsilon\lambda\frac{\kappa_1}{\kappa_2}$ where $C$ is some constant. Thus, for $V_1-$ slant helix $\gamma(s)$ the ratio of first curvature function and second curvature function is constant.

Conversely, assume that $\gamma(s)$ is proper curve of order 2 with curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$, and the ratio of first curvature function and second curvature function of curve $\gamma(s)$ is constant. According to Theorem 4.1 there exist an orthogonal frame along $\gamma(s)$ in $E_1^3$ and we take a vector field $U$ in $E_1^3$ such that $U = \lambda V_1 - \epsilon\lambda\frac{\kappa_1}{\kappa_2}S_3$. Now taking the covariant derivative of $U$ with respect to $V_1$, we get $\tilde{\nabla}_{V_1}U = 0$. Thus vector field $U$ is parallel along $\gamma(s)$ such that $g(V_1, U) = \lambda =$ constant. Hence $\gamma(s)$ is $V_1-$ slant helix in $E_1^3$ with axis $U$ not orthogonal to $V_1$.

□

**Corollary 5.2.** *Let $\gamma(s)$ be a proper curve of order 2 with non constant curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ and $\gamma(s)$ is a $V_1-$ slant helix in $E_1^3$ then axis of $\gamma(s)$ can never be orthogonal to $V_1$.*

***Proof.*** Let $\gamma(s)$ is $V_1-$ slant helix with non constant curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) in $E_1^3$ lying on $S_1^2(r)$, then according to Theorem 4.1 there exist an orthogonal frame along $\gamma(s)$ and the axis $U$ orthogonal to vector field $V_1$, of $V_1-$ slant helix in $E_1^3$ with respect to the corresponding frame can be written as

$$U = \nu_1 S_2 + \nu_2 S_3,$$

where $\nu_1(s), \nu_2(s)$ are functions of some parameter s and $S_2 \in \{V_2, W_2, Z_2\}$ and $S_3 \in \{V_3, W_3, Z_3\}$ according to the orthogonal frame defined in Theorem 4.1. Taking covariant derivative of vector field $U$ with respect to $V_1$ and comparing the equation on both sides , we obtain

$$\begin{cases} \nu_1' + \nu_2\kappa_2 = 0, \\ \nu_1\kappa_1 = 0, \\ \nu_2' + \nu_1\kappa_2 = 0. \end{cases} \tag{5.2}$$

On solving the equation (5.2), we get $\nu_1 = \nu_2 = 0$ which implies that $U = 0$ but according to the definition 2.3 $U$ must be non zero vector field. Hence we can conclude that axis of $V_1-$ slant helix can never be orthogonal to tangent vector field along $\gamma(s)$. □

**Corollary 5.3.** *Let $\gamma(s)$ be a proper curve of order 2 with non constant curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$, then $\gamma(s)$ be a $V_1-$ slant helix in $E_1^3$ iff $\gamma(s)$ is a $V_3-$ slant helix in $E_1^3$.*

***Proof.*** Let $\gamma(s)$ be a proper curve of order 2 with non constant curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$, and $\gamma(s)$ is a $V_1-$ slant helix in $E_1^3$, then according to Theorem 5.1 an axis for $\gamma(s)$ is $U = \lambda V_1 - \epsilon\lambda\frac{\kappa_1}{\kappa_2}S_3$, where ratio of first curvature function and second curvature function is constant. Because $g(U, S_3) = constant$, which implies that $\gamma(s)$ is $S_3-$ slant helix in $E_1^3$.

To prove the converse part we consider the axis $U = \nu_1 V_1 + \nu_2 S_2 + \lambda S_3$ for $S_3-$ slant helix and follow the same steps which we used to prove the above argument. Here $\nu_1, \nu_2$ are functions of some parameter s and $\lambda$ is some constant. □

**Theorem 5.4.** *Let $\gamma(s)$ be a proper curve of order 2 with non constant positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$. Then with respect to orthogonal frame constructed in Theorem 4.1, we have*

***i.*** *The spacelike curve $\gamma(s)$ with curvature function $\kappa(s) < \frac{1}{r}$ will be a $V_2-$ slant helix if and only if the curvature functions $k_1(s)$ and $k_2(s)$ of $\gamma(s)$ in $E_1^3$ are given by*

$$\kappa_1(s) = f'(s)cosh(f(s)) \text{ and } \kappa_2(s) = f'(s)sinh(f(s)),$$

*for some smooth function $f(s)$ on $S_1^2(r)$. And the axis for $V_2-$ slant helix is*

$$U = \lambda sinh(f(s))V_1 + \lambda V_2 - \lambda cosh(f(s))V_3,$$

*where $0 \neq \lambda \in R$ .*

***ii.*** *The spacelike curve $\gamma(s)$ with curvature function $\kappa(s) > \frac{1}{r}$ will be a $W_2-$ slant helix if and only if the curvature functions $k_1(s)$ and $k_2(s)$ of $\gamma(s)$ in $E_1^3$ are given by*

$$\kappa_1(s) = g'(s)cos(g(s)) \text{ and } \kappa_2(s) = g'(s)sin(g(s)),$$

*for some smooth function $g(s)$ on $S_1^2(r)$. An axis for $W_2-$ slant helix is given by $U = -\lambda sin(g(s))dsV_1 + \lambda W_2 + \lambda cos(g(s))W_3$, where $\lambda$ is non zero constant.*

***iii.*** *The timelike curve $\gamma(s)$ will be a $Z_2-$ slant helix in $E_1^3$ if and only if the curvature functions $k_1(s)$ and $k_2(s)$ of $\gamma(s)$, are given by*

$$\kappa_1(s) = f'(s)cosh(f(s)) \text{ and } \kappa_2(s) = f'(s)sinh(f(s)),$$

*for some smooth function $f(s)$ on $S_1^2(r)$. And an axis for $Z_2-$ slant helix not orthogonal to $Z_2$ is given by $U = -\lambda sinh(f(s))V_1 + \lambda Z_2 - \lambda cosh(f(s))V_3$, for some $\lambda = contant \neq 0$.*

***Proof.*** **Case 1.** Let $\gamma(s)$ is spacelike $V_2-$ slant helix with curvature function $\kappa(s) < \frac{1}{r}$ in $E_1^3$ lying on $S_1^2(r)$, then according to Theorem 4.1 there exist an orthogonal frame along $\gamma(s)$. The axis $U$ of $V_2-$ slant helix in $E_1^3$ with respect to the corresponding frame can be written as

$$U = \nu_1 V_1 + \lambda V_2 + \nu_2 V_3,$$

where $\lambda$ is non zero constant and $\nu_1(s), \nu_2(s)$ are functions of parameter s. Taking covariant derivative of vector field $U$ with respect to $V_1$ and comparing the equation on both sides , we obtain,

$$\begin{cases} \nu_1' - \lambda\kappa_1 = 0, \\ \nu_1\kappa_1 + \nu_2\kappa_2 = 0, \\ \nu_2' + \lambda\kappa_2 = 0. \end{cases} \tag{5.3}$$

From first and third part of the equation (5.3) , we obtain the following set of solutions

$$\begin{cases} \nu_1 = \lambda \int \kappa_1(s)ds, \\ \nu_2 = -\lambda \int \kappa_2(s)ds. \end{cases} \tag{5.4}$$

Substituting $\nu_1$ and $\nu_3$ from set of solutions of equation (5.4) into second part of set of equation (5.3), we get

$$\kappa_1(s) \int \kappa_1(s)ds - \kappa_2(s) \int \kappa_2(s)ds = 0. \tag{5.5}$$

Then, from the equation (5.5), we obtain

$$\kappa_1(s) = f'(s)cosh(f(s)) \text{ and } \kappa_2(s) = f'(s)sinh(f(s)),$$

where $f(s)$ is some smooth function on $S_1^2(r)$.

Conversely, assume that $\gamma(s)$ be a spacelike proper curve of order 2 with curvature function $\kappa(s) < \frac{1}{r}$ lying on $S_1^2(r)$, and $\gamma(s)$ have curvature functions in $E_1^3$ are, $\kappa_1(s) = f'(s)cosh(f(s))$ and $\kappa_2(s) = f'(s)sinh(f(s))$, for some smooth function $f(s)$ in $S_1^2(r)$. Let us assume a vector field $U = \lambda sinh(f(s))V_1 + \lambda V_2 - \lambda cosh(f(s))V_3$, where $\lambda = constant \neq 0$. Taking the covariant derivative of $U$ with respect to tangent vector field $V_1$ along $\gamma(s)$, we obtain, $\tilde{\nabla}_{V_1}U = 0$. Thus $U$ is parallel along $\gamma(s)$, and $g(V_2, U) = \lambda$. Hence, $\gamma(s)$ is a $V_2-$ slant helix in $E_1^3$ and the axis is $U$ not orthogonal to $V_2$.

**Case 2.** Let $\gamma(s)$ is spacelike $W_2-$ slant helix with curvature function $\kappa(s) > \frac{1}{r}$ in $E_1^3$ lying on $S_1^2(r)$, then according to Theorem 4.1 there exist an orthogonal frame along $\gamma(s)$ and the axis $U$ of $W_2-$ slant helix in $E_1^3$ with respect to the corresponding frame can be written as

$$U = \nu_1 V_1 + \lambda W_2 + \nu_2 W_3,$$

where $\lambda$ is non zero constant and $\nu_1(s), \nu_2(s)$ are functions of some parameter s. Taking covariant derivative of vector field $U$ with respect to $V_1$ and comparing the equation on both sides , we obtain

$$\begin{cases} \nu_1' + \lambda\kappa_1 = 0, \\ \nu_1\kappa_1 + \nu_2\kappa_2 = 0, \\ \nu_2' + \lambda\kappa_2 = 0. \end{cases} \tag{5.6}$$

From first and third part of the equation (5.6) , we obtain the following set of solutions

$$\begin{cases} \nu_1 = -\lambda \int \kappa_1(s)ds, \\ \nu_2 = -\lambda \int \kappa_2(s)ds. \end{cases} \tag{5.7}$$

Substituting $\nu_1$ and $\nu_3$ from set of solutions of equation (5.7) into second part of set of equations (5.6), we get

$$\kappa_1(s)\int \kappa_1(s)ds + \kappa_2(s)\int \kappa_2(s)ds = 0. \tag{5.8}$$

Thus, from equation (5.8), we have

$$\kappa_1(s) = g'(s)cos(g(s)) \ \ and \ \ \kappa_2(s) = g'(s)sin(g(s)),$$

where $g(s)$ is a some smooth function on $S_1^2(r)$.

Conversely, assume that $\gamma(s)$ be a proper curve of order 2 with curvature function $\kappa(s) > \frac{1}{r}$ lying on $S_1^2(r)$, and the curvature functions along $\gamma(s)$ in $E_1^3$ are, $\kappa_1(s) = g'(s)cos(g(s)) \ \ and \ \ \kappa_2(s) = g'(s)sin(g(s))$, for some smooth function $g(s)$ in $S_1^2(r)$. Consider a vector field $U = -\lambda sin(g(s))V_1 + \lambda W_2 + \lambda cos(g(s))W_3$, where $\lambda$ is some non zero constant. Taking the covariant derivative of $U$ with respect to tangent vector field $V_1$ along $\gamma(s)$, we obtain, $\tilde{\nabla}_{V_1}U = 0$. Therefore, $U$ is parallel along $\gamma(s)$ and $g(W_2, U) = \lambda$. Thus, $\gamma(s)$ is a $W_2-$ slant helix in $E_1^3$ and the axis $U$ is not orthogonal to $V_2$.

The third part of the theorem can be proved in similar way as we prove first and second part of the theorem. □

**Corollary 5.5.** *Let $\gamma(s)$ be a $S_2-$ slant helix with non constant positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ with the axis $U$ is not orthogonal to $S_2$ then $\gamma(s)$ is neither $V_1$ nor $S_3-$ slant helix.*

***Proof.*** Here, $S_2 \in \{V_2, W_2, Z_2\}$ and $S_3 \in \{V_3, W_3, Z_3\}$ according to the orthogonal frame defined in Theorem 4.1. Let $\gamma(s)$ be a $S_2-$ slant helix with non constant positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ with an axis $U$ not orthogonal to $S_2$. Then by Theorem 5.4, we have

$$\begin{cases} U^{V_2} = \lambda sinh(f(s))V_1 + \lambda V_2 - \lambda cosh(f(s))V_3, \\ U^{W_2} = -\lambda sin(g(s))dsV_1 + \lambda W_2 + \lambda cos(g(s))W_3, \\ U^{Z_2} = -\lambda sinh(f(s))V_1 + \lambda Z_2 - \lambda cosh(f(s))Z_3, \end{cases} \tag{5.9}$$

where $\lambda$ is some constant and $U^{S_2}$ is the axis of $S_2-$ slant helix. Then

$$\begin{cases} g(U^{S_2}, V_1) \neq constant, \\ g(U^{S_2}, S_3) \neq constant. \end{cases} \tag{5.10}$$

These implies, if $\gamma(s)$ be a $S_2-$ slant helix with axis $U^{S_2}$ is not orthogonal to $S_2$ then $\gamma(s)$ is neither $V_1$ nor $S_3-$ slant helix. □

**Corollary 5.6.** *Let $\gamma(s)$ be a $S_2-$ slant helix with non constant positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ with the axis $U$ is orthogonal to $S_2$ then $\gamma(s)$ is also $V_1$ and $S_3-$ slant helix with same axis.*

***Proof.*** Let $\gamma(s)$ is $S_2-$ slant helix with non constant positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) in $E_1^3$ lying on $S_1^2(r)$, then according to Theorem 4.1 there exist an orthogonal frame along $\gamma(s)$ and the axis $U$ orthogonal to vector field $S_2$, of $S_2-$ slant helix in $E_1^3$ with respect to the corresponding frame can be written as

$$U = \nu_1 V_1 + \nu_2 S_3,$$

where $\nu_1(s), \nu_2(s)$ are functions of parameter s. Taking covariant derivative of vector field $U$ with respect to $V_1$ and comparing the equation on both sides, we obtain

$$\begin{cases} \nu_1' = 0, \\ \nu_1\kappa_1 + \epsilon\nu_2\kappa_2 = 0, \\ \nu_2' = 0. \end{cases} \tag{5.11}$$

Here, $\epsilon$ will be positive if $\gamma(s)$ is a spacelike curve and $\epsilon$ will be negative if $\gamma(s)$ is a timelike curve. From set of equation (5.11) we find that ratio of first curvature function and second curvature function is constant. Thus from Theorem 5.1 and Corollary 5.3, we can conclude that $\gamma(s)$ is a $V_1$ and $S_3-$ slant helix. □

## 6. Conditions for proper curve of order 2 with non constant curvature function to lie in plane.

**Theorem 6.1.** *Let $\gamma(s)$ be a proper curve of order 2 with positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$. If the position vector of $\gamma(s)$ is lying on some plane then $\gamma(s)$ will never lie in the plane spanned by a tangent vector field along the curve $\gamma(s)$.*

***Proof.*** Let $\gamma(s)$ be a proper curve of order 2 with positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ and the position vector of $\gamma(s)$ is lying on the plane spanned by a tangent vector fields along $\gamma(s)$, then the position vector of $\gamma(s)$ can be written as either $\gamma(s) = \nu_1 V_1 + \nu_2 S_2$ or $\gamma(s) = \nu_1 V_1 + \nu_3 S_3$, where $\nu_1$ and $\nu_2$ are smooth functions of parameter s.

**Case 1.** Let $\gamma(s)$ is lying on $\{V_1, S_2\}$ plane, then position vector of $\gamma(s)$ is

$$\gamma(s) = \nu_1 V_1 + \nu_2 S_2. \tag{6.1}$$

Taking the covariant derivative on (6.1) with respect to vector filed $V_1$ and comparing the equation on both sides, we get

$$\begin{cases} \nu_1' + \epsilon\nu_2\kappa_1 = 1, \\ \nu_2' + \nu_1\kappa_1 = 0, \\ \nu_2\kappa_2 = 0, \end{cases} \tag{6.2}$$

where $\epsilon = \pm 1$, $\epsilon = 1$ for $S_2 = V_2$ and $\epsilon = -1$ for $S_2 = W_2 = Z_2$. But set of equation (6.2) have no solution for non zero first and second curvature functions.

**Case 2.** Let $\gamma(s)$ is lying on $\{V_1, S_3\}$ plane, then position vector of $\gamma(s)$ can be written as follows

$$\gamma(s) = \nu_1 V_1 + \nu_2 S_3. \tag{6.3}$$

Taking the covariant derivative on (6.1) with respect to vector filed $V_1$ and comparing the equation on both sides, we get the following set of equations

$$\begin{cases} \nu_1' = 1, \\ \nu_2' = 0. \\ \nu_1\kappa_1 + \epsilon\nu_2\kappa_2 = 0, \end{cases} \tag{6.4}$$

where $\epsilon = \pm 1$, $\epsilon$ is negative for $S_3 = Z_2$ and $\epsilon$ will positive for $S_3$ is either of $W_3$ or $V_3$. So from set of equation (6.4), we get $\nu_1 = S + C_1$ and $\nu_2 = C_2$ for some constant $C_1$ and $C_2$. Substituting $\nu_1$ and $\nu_2$ into equation (6.3), we get

$$\gamma(s) = (S + c_1)V_1 + C_2 S_2.$$

Since $\gamma(s)$ lying on $S_1^2(r)$, therefore we have

$$(S + C_1)^2 + C_2^2 = r^2, \tag{6.5}$$

but equation (6.5) is not possible because left side of the equation is variable and right side of the equation is constant.
Thus from Case 1. and Case 2, we can conclude that $\gamma(s)$ will never lie in the plane spanned by tangent vector fields along the curve $\gamma(s)$. $\square$

**Theorem 6.2.** *Let $\gamma(s)$ be a proper curve of order two in $S_1^2(r) \subset E_1^3$ then with respect to an orthogonal frame constructed in Theorem 4.1.,*

***i.*** *If $\gamma(s)$ is a spacelike curve with curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) and lying on some plane, then the first and second curvature functions of $\gamma(s)$ satisfy the following relation*

$$\epsilon(\frac{1}{\kappa_1(s)})' - \epsilon\kappa_2(s)\int \frac{\kappa_2(s)}{\kappa_1(s)}ds = 0.$$

*The position vector for $\gamma(s)$ in a plane is given by $\gamma(s) = \epsilon\frac{1}{\kappa_1(s)}A_2 - \epsilon\kappa_2(s)\int\frac{\kappa_2(s)}{\kappa_1(s)}dsA_3$, where $A_2 \in \{V_2, W_2\}$ and $A_3 \in \{V_3, W_3\}$ such that if $A_2 = V_2$ then $A_3 = V_3$ and if we take $A_2 = W_2$ then $A_3 = W_3$ . Here $\epsilon = \pm 1$, $\epsilon$ is positive if $\gamma(s)$ is with curvature function $\kappa(s) > \frac{1}{r}$ and is negative if $\gamma(s)$ have curvature function is $\kappa(s) < \frac{1}{r}$.*

***ii.*** *If $\gamma(s)$ is a timelike curve and lying in the plane not spanned by a tangent vector field along the curve $\gamma(s)$, then the first and second curvature function of the curve satisfy the condition*

$$(\frac{1}{\kappa_1(s)})' + \kappa_2(s)\int \frac{\kappa_2(s)}{\kappa_1(s)}ds = 0.$$

*The position vector for $\gamma(s)$ in the plane is given by $\gamma(s) = \frac{1}{\kappa_1(s)}Z_2 + \kappa_2(s)\int\frac{\kappa_2(s)}{\kappa_1(s)}dsZ_3$.*

***Proof.*** Let $\gamma(s)$ be a proper curve of order 2 with positive curvature function (either $\kappa(s) < \frac{1}{r}$ or $\kappa(s) > \frac{1}{r}$) lying on $S_1^2(r)$ and the position vector of $\gamma(s)$ is lying on the plane not spanned by the tangent vector fields along $\gamma(s)$. Then the position vector of $\gamma(s)$ is either $\gamma(s) = \nu_2 A_2 + \nu_3 A_3$ or $\gamma(s) = \nu_2 Z_2 + \nu_3 Z_3$, where $\nu_2$ and $\nu_3$ are functions of parameter s. Here $A_2 \in \{V_2, W_2\}$ and $A_3 \in \{V_3, W_3\}$ such that if $A_2 = V_2$ then $A_3 = V_3$ and if we take $A_2 = W_2$ then $A_3 = W_3$.

**Case 1.** Let $\gamma(s)$ is spacelike with position vector in $\{A_2, A_3\}$ plane, then

$$\gamma(s) = \nu_2 A_2 + \nu_3 A_3. \tag{6.6}$$

Taking the covariant derivative of (6.6) with respect to vector filed $V_1$ and comparing the equation on both sides, we get the following set of equations

$$\begin{cases} \nu_2 \kappa_1 = \epsilon, \\ \nu_2' + \nu_3 \kappa_2 = 0, \\ \nu_3' + \nu_2 \kappa_2 = 0, \end{cases} \tag{6.7}$$

where ($\epsilon = \pm 1$), $\epsilon$ is positive if $\gamma(s)$ is with curvature function $\kappa(s) > \frac{1}{r}$ and is negative if $\gamma(s)$ have curvature function is $\kappa(s) < \frac{1}{r}$. From first and third part of the set of equation (6.7), we get the following set of solutions

$$\begin{cases} \nu_2 = \epsilon \frac{1}{\kappa_1}, \\ \nu_3 = -\epsilon \int \frac{\kappa_2}{\kappa_1} ds. \end{cases} \tag{6.8}$$

By using (6.8), in (6.6) and in second part of (6.7), we obtain the required condition and position vector for $\gamma(s)$ in $\{A_2, A_3\}$ plane.

**Case 2.** Let $\gamma(s)$ is timelike and lying on $\{Z_2, Z_3\}$ plane then position vector of $\gamma(s)$, can be define as follows

$$\gamma(s) = \nu_2 Z_2 + \nu_3 Z_3. \tag{6.9}$$

Taking covariant derivative on (6.9) with respect to vector filed $V_1$ and comparing the equation on both sides, we get the following set of equations

$$\begin{cases} \nu_2 \kappa_1 = 1, \\ \nu_2' - \nu_3 \kappa_2 = 0, \\ \nu_3' + \nu_2 \kappa_2 = 0. \end{cases} \tag{6.10}$$

From first and third part of (6.10), we get

$$\begin{cases} \nu_2 = \frac{1}{\kappa_1}, \\ \nu_3 = -\int \frac{\kappa_2}{\kappa_1} ds. \end{cases} \tag{6.11}$$

Substituting $\nu_2$ and $\nu_3$ from (6.11) into second part of (6.10) and in (6.9), we obtain the required condition and position vector for $\gamma(s)$ in $\{Z_2, Z_3\}$ plane. $\square$

**Note.** To discuss the solutions of the above differential equations, given in Theorem 6.2, we obtain the following possibilities:

***i.*** In that case, we may take $\kappa_1(s) = \frac{1}{sinh(\int \kappa_2(s) ds)}$ or $\kappa_1(s) = e^{\int \kappa_2(s) ds}$.

***ii.*** In that case, we can consider the solution

$$\kappa_1(s) = \frac{C_2}{cos(C_1 - \int \kappa_2(s) ds)},$$

where $C_1$, $C_2 \neq 0$, are constants.

## 7. Conditions for another curve $\beta(\bar{s})$ whose $\bar{V_2}$ vector field coincide with $V_2$ vector field of $\gamma(s)$ .

**Theorem 7.1.** *Let $\gamma(s)$ be a curve with curvature function $\kappa(s) > \frac{1}{r}$ lying on $S_1^2(r)$, and if there exist a regular curve $\beta(\bar{s}) = \gamma(s) + \eta(s)W_2$ in $E_1^3$ such that $W_2 = \bar{W_2}$, where $\bar{s} = \bar{s}(s)$, $\bar{W}_2 = \frac{1}{\bar{\kappa}_1}\tilde{\nabla}_{\bar{V}_1}\bar{V_1}$ and $\tilde{\nabla}_{V_1}\beta = \bar{V_1}\frac{d\bar{s}}{ds}$ then the image of curve will lie in either $S_1^2(a)$ or $H_0^2(b)$ for some constant $a$ and $b$.*

***Proof.*** Let $\gamma(s)$ be a curve with curvature function $\kappa(s) > \frac{1}{r}$ is lying on $S_1^2(r) \subset E_1^3$, and $\beta(\bar{s}) = \gamma(s) + \eta(s)W_2$ is regular curve lying in $E_1^3$ such that the $W_2 = \bar{W_2}$ where $\bar{s} = \bar{s}(s)$, $\bar{W}_2 = \frac{1}{\bar{\kappa}_1}\tilde{\nabla}_{\bar{V}_1}\bar{V_1}$ and $\tilde{\nabla}_{V_1}\beta = \bar{V_1}\frac{d\bar{s}}{ds}$.

$$\beta(\bar{s}) = \gamma(s) + \eta(s)W_2. \tag{7.1}$$

Taking the covariant derivative of equation (7.1) along the vector field $V_1$, we obtain

$$\bar{V_1}\frac{d\bar{s}}{ds} = V_1 + \eta'(s)W_2 + \eta(s)(\kappa_1(s)V_1 + \kappa_2(s)W_3, \tag{7.2}$$

since $W_2 = \bar{W_2}$, taking the inner product of equation (7.2) with $W_2$, we get

$$\eta'(s) = 0 \Rightarrow \eta(s) = C_1 = constant. \tag{7.3}$$

Using equation (7.3) and equation (7.1), we obtain

$$g(\beta(\bar{s}), \beta(\bar{s})) = r^2 - C_1^2. \tag{7.4}$$

**Case 1.** If $C_1^2 > r^2$ implies that there exist a hyperbolic plane $H_0^2(b)$ where $b = C_1^2 - r^2$, such that curve $\beta(\bar{s})$ will lie in $H_0^2(b)$. From equation (7.2), we get $\beta(\bar{s})$ is a spacelike curve and $\{\bar{V_1}, \bar{W_2}, \bar{W_3}\}$ is an orthogonal frame along $\beta(\bar{s})$ where $\bar{V_1}$ and $\bar{W_3}$ are spacelike vector fields and $\bar{W_2}$ is timelike vector field.

**Case 2.** If $C_1^2 < r^2$ implies that there exist a pseudosphere $S_1^2(a)$ where $a = r^2 - C_1^2$, such that curve $\beta(\bar{s})$ will lie in $S_1^2(a)$. From equation (7.2), we have $\beta(\bar{s})$ is a spacelike curve with orthogonal frame $\{\bar{V_1}, \bar{W_2}, \bar{W_3}\}$ along $\beta(\bar{s})$. Where $\bar{V_1}$ and $\bar{W_3}$ are spacelike vector fields and $\bar{W_2}$ be a timelike vector field. □

**Theorem 7.2.** *Let $\gamma(s)$ be a curve with curvature function $\kappa(s) < \frac{1}{r}$ and lying on $S_1^2(r)$, and if there exist a regular curve $\beta(\bar{s}) = \gamma(s) + \eta(s)V_2$ in $E_1^3$ such that $V_2 = \bar{V_2}$, where $\bar{s} = \bar{s}(s)$, $\bar{V}_2 = \frac{1}{\bar{\kappa}_1}\tilde{\nabla}_{\bar{V}_1}\bar{V_1}$ and $\tilde{\nabla}_{V_1}\beta = \bar{V_1}\frac{d\bar{s}}{ds}$ then the image of curve will lie in $S_1^2(a)$ for some constant $a$. The curve $\beta(\bar{s})$ can be spacelike or timelike depends on the following conditions*

*(i) If $|\eta(s| < \frac{1}{|\kappa_2(s)| - |\kappa_1(s)|}$ then curve $\beta(\bar{s})$ will be spacelike in $S_1^2(a) \subset E_1^3$,*

*(ii) If $|\eta(s| > \frac{1}{|\kappa_2(s)| + |\kappa_1(s)|}$ then curve $\beta(\bar{s})$ will be timelike in $S_1^2(a) \subset E_1^3$.*

***Proof.*** Let $\gamma(s)$ is a curve with curvature function $\kappa(s) < \frac{1}{r}$ and lying on $S_1^2(r)$, and $\beta(\bar{s}) = \gamma(s) + \eta(s)V_2$ is regular curve lying in $E_1^3$ such that the $V_2 = \bar{V_2}$ where $\bar{s} = \bar{s}(s)$, $\bar{V}_2 = \frac{1}{\bar{\kappa}_1}\tilde{\nabla}_{\bar{V}_1}\bar{V_1}$ and $\tilde{\nabla}_{V_1}\beta = \bar{V_1}\frac{d\bar{s}}{ds}$. Now

$$\beta(\bar{s}) = \gamma(s) + \eta(s)V_2. \tag{7.5}$$

Taking the covariant derivative of equation (7.5) along the vector field $V_1$, we obtain

$$\bar{V_1}\frac{d\bar{s}}{ds} = V_1 + \eta'(s)V_2 + \eta(s)(-\kappa_1(s)V_1 + \kappa_2(s)V_3, \tag{7.6}$$

since, $V_2 = \bar{V_2}$, taking the inner product of equation (7.6) with $V_2$, we get

$$\eta'(s) = 0 \Rightarrow \eta(s) = C_1 = constant. \tag{7.7}$$

Using equation (7.7) and equation (7.6), we obtain

$$g(\beta(\bar{s}), \beta(\bar{s})) = r^2 + C_1^2. \tag{7.8}$$

From equation (7.6), we get

$$g(\bar{V_1}, \bar{V_1})(\frac{d\bar{s}}{ds})^2 = (1 - \eta(s)\kappa_1(s))^2 - (\eta(s)\kappa_2(s))^2. \tag{7.9}$$

Thus from (7.8) we can conclude that the curve $\beta(\bar{s})$ will lie in $S_1^2(a)$ , where $a = r^2 + C_1^2$. Also from equation 7.9 we see that the $\beta(\bar{s})$ is spacelike curve in $S_1^2(r)$ if $|\eta(s| < \frac{1}{|\kappa_2(s)|-|\kappa_1(s)|}$ and the curve is timlike when $|\eta(s| > \frac{1}{|\kappa_2(s)|+|\kappa_1(s)|}$. □

**Corollary 7.3.** *Let $\gamma(s)$ be a timelike and proper curve of order 2 is lying on $S_1^2(r)$, and if there exist a regular curve $\beta(\bar{s}) = \gamma(s) + \eta(s)Z_2$ in $E_1^3$ such that the $Z_2 = \bar{Z}_2$, where $\bar{s} = \bar{s}(s)$, $\bar{Z}_2 = \frac{1}{\bar{\kappa}_1}\tilde{\nabla}_{\bar{V}_1}\bar{V}_1$ and $\tilde{\nabla}_{V_1}\beta = \bar{V}_1\frac{d\bar{s}}{ds}$ then the image of curve will lie in $S_1^2(a)$ for some constant $a$. The curve $\beta(\bar{s})$ can be spacelike or timelike depends on the following conditions*

*(i) If $|\eta(s| < \frac{1}{|\kappa_2(s)|-|\kappa_1(s)|}$ then curve $\beta(\bar{s})$ will be timelike in $S_1^2(a) \subset E_1^3$,*

*(ii) If $|\eta(s| > \frac{1}{|\kappa_2(s)|+|\kappa_1(s)|}$ then curve $\beta(\bar{s})$ will be spacelike in $S_1^2(a) \subset E_1^3$.*

***Proof.*** Using third part of Theorem 4.1 and Theorem 7.2, one can prove Corollary 7.3 easily. □

**Acknowledgment.** The authors wish to express their sincere thanks and gratitude to the referees for their valuable suggestions toward the improvement of the paper. The second author is supported by the UGC JRF of india, Ref. No. 1068/ (CSIR - UGC NET JUNE 2019).

## References

[1] N. Abe, Y. Nakanishi and S. Yamaguchi, *Circles and spheres in pseudo-Riemannian geometry*, Aequationes Mathematicae, **39**, 134-145, 1990.

[2] T. Adachi, *Kähler magnetic flows for a manifold of constant holomorphic sectional curvature*, Tokyo J. Math. **18**(2), 473-483, 1995.

[3] A.T. Ali, *Position vectors of slant helices in Euclidean 3-space*, J. Egypt. Math. Soc. **20**, 1-6, 2012.

[4] A.T. Ali and R. López, *On slant helices in Minkowski space $E_1^3$*, J. Korean Math. Soc. **48**, 159-167, 2011.

[5] A.T. Ali and M. Turgut, *Position vector of a time-like slant helix in Minkowski 3-space*, J. Math. Anal. Appl. **365**, 559-569, 2010.

[6] M. Barros, *General helices and a theorem of Lancret*, Proc. Amer. Math. Soc. **125**, 1503-1509, 1997.

[7] B. Bukcu, M.K. Karacan and N. Yuksel, *New characterizations for Bertrand curves in Minkowski 3 - space*, International J. Math. Combin. **2**, 98-103, 2011.

[8] B.Y. Chen, *When does the position vector of a space curve always lie in its rectifying plane?*, Amer. Math. Monthly, **110**, 147-152, 2003.

[9] İ. Gök, Ç. Camci and H. Hilmi Hacisalihoğlu, *$V_n$ - slant helices in Euclidean $n$ - space $E^n$*, Math. Commun. **14**, 317-329, 2009.

[10] S. Izumiya and N. Takeuchi, *New special curves and developable surfaces*, Turk. J. Math. **28**, 153-163, 2004.

[11] S. Kiziltuğ and Y. Yayli, *Timelike curves on Timelike parallel surfaces in Minkowski 3-space $E_1^3$*, Mathematica Aeterna **2**, 689-700, 2012.

[12] S. Maeda and B. Hak Kim, *A certain two-parameter family of helices of order 6 in Euclidean sphere*, Differ. Geom. Appl. **35**, 117-124, 2014.
[13] K. Sakamoto, *Helical immersions into a unit sphere*, Math. Ann. **261**, 63-80, 1982.
[14] H.H. Song, T. Kimura and N. Koike, *On proper helices and extrinsic spheres in pseudo-Riemannian geometry*, Tsukuba J. Math. **20**, 263-280, 1996.
[15] C. Zhang and D. Pei, *Generalized bertrand curves in minkowski 3-space*, Mathematics MDPI, **8**(12), 2199, 2020.
[16] E. Zıplar, Y. Yaylı and İ. Gök, *A New Approach on Helices in Pseudo-Riemannian Manifold*, Abstr. Appl. Anal. **2014**, ID 718726, 2014.