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Research Article

Estimation of stress-strength reliability for generalized Gompertz distribution under progressive type-II censoring

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Abstract

In this study, the stress-strength reliability, $R = P(Y < X)$ where Y represents the stress of a component and X represents this component's strength, is obtained when X and Y have two independents generalized Gompertz distribution with different shape parameters under progressive type-II censoring. The Bayes and maximum likelihood estimators of stress-strength reliability can not be acquired in closed forms. The approximate Bayes estimators under squared error loss function by using Lindley's approximations for stress-strength reliability are derived. A Monte Carlo simulation study is done to check performances of the approximate Bayes against performances of maximum likelihood estimators and observe the coverage probabilities and the intervals' average width. In addition, the coverage probabilities of the parametric bootstrap estimates are calculated. Two applications based on real datasets are provided.

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1. Introduction

When a component with X strength is subjected to Y stress, the stress-strength model defines the life of such a component. Under this model, the reliability of a system can be given by $R = P(Y < X)$. According to this, if stress exceeds strength $(Y > X)$, it is not possible for the component to living. Engineering, medicine, military, etc. fields benefit from stress-strength models at critical points such as the reliability of a pedestrian overpass, a carbon fiber, a bridge, an elevator. There are many studies about stress-strength model under complete and censoring sample carried out by [3, 10, 12, 17–19, 23, 26, 33, 34, 37] and so forth.

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In the reliability analysis and life test, it is not always possible to observe the failure time of all constituents composing a system. In this case, censored data are obtained. In life testing experiments, different censoring schemes such as type-I, type-II, and progressive type-II right (PTR-II) are encountered. Among these censoring schemes, PTR-II censoring scheme is the most commonly used since the PTR-II censoring scheme reduces the cost and period of an experiment (see [6, 7, 9, 35]). The following description of the PTR-II censoring scheme is clear enough to understand. Suppose that n identical units are tested and m failures will be observed. When the first failure occurs, R_1 items are randomly selected and taken out. Likewise, when the second failure occurs, R_2 of the remaining items are selected randomly and taken out, and so on. Finally, when the mth failure occurs, all the items which survive are censored.

The PTR-II censoring scheme is demonstrated with $\mathbf{R} = (R_1, R_2, \cdots, R_m)$. In this lifetime process, $\mathbf{X^R} = \left(X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \cdots, X_{m:m:n}^{R_m}\right)$ with $X_{1:m:n}^{R_1} < X_{2:m:n}^{R_2} < \cdots < X_{m:m:n}^{R_m}$ is named as PTR-II censored sample with $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ scheme. The joint probability density function (pdf) of this censored sample is given by

$$f_{X_{1:m:n}^{\mathbf{R}}, X_{2:m:n}^{\mathbf{R}}, \cdots, X_{m:m:n}^{\mathbf{R}}}(x_1, x_2, \cdots x_m) = c \prod_{i=1}^{m} f(x_i) \left[1 - F(x_i)\right]^{R_i}, \ -\infty < x_1 < x_2 < \cdots < x_m < \infty$$

where $c = n(n - R_1 - 1) \times \cdots \times (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1)$ [7]. Statistical inference under a PTR-II censored sample is studied by several authors such as [1,2,4,9,11, 13,16,22,27–30,36,38–40,42]. The SSR has been widely used in continuous distributions. In addition, it has recently been used in discrete distributions, such as geometric-Poison [31], geometric-exponential [21] and phase-type [20] distributions. Generalized Gompertz (GG) distribution, which was put forward by [15], can be used in many areas ranging from the comparison of the mortality rate of different populations to growth models and actuarial studies.

Let X random variable has GG (λ, c, θ) and the probability density, cumulative distribution, survival and hazard functions are as follows:

$$f(x) = \theta\lambda e^{cx} e^{-\frac{\lambda}{c}(e^{cx}-1)} \left[1 - e^{-\frac{\lambda}{c}(e^{cx}-1)}\right]^{\theta-1}, \ x > 0; \ \lambda, c, \theta > 0,$$

$$F(x) = \left[1 - e^{-\frac{\lambda}{c}(e^{cx}-1)}\right]^{\theta},$$

$$\bar{F}(x) = 1 - \left[1 - e^{-\frac{\lambda}{c}(e^{cx}-1)}\right]^{\theta},$$

$$h(x) = \theta\lambda e^{cx} e^{-\frac{\lambda}{c}(e^{cx}-1)} \left[1 - \left[1 - e^{-\frac{\lambda}{c}(e^{cx}-1)}\right]^{\theta}\right]^{-1},$$

respectively, where θ is shape parameter. The hazard function (hf) is increasing function if $c > 0$, the hf is constant function if $c = 0$. The hf is increasing function when $\theta > 0$ and the hf will be either decreasing if $c = 0$ or bath-tube if $c > 0$ for $\theta < 1$.

This study mainly aims to obtain the approximate Bayes estimators under square error loss functions for stress-strength reliability (SSR) of GG distribution based on PTR-II censored samples and to compare them with maximum likelihood (ML) estimators of SSR. The rest of the study is given respectively. Firstly, SSR for GG distribution is obtained. The ML estimators and the approximate Bayes estimators under squared error loss function by using Lindley's approximations for SSR are derived. Secondly, A Monte Carlo simulation study is done to compare performances of the approximate Bayes estimators with performances of ML and observe the coverage probabilities and the asymptotic confidence interval (ACI) average width. Thirdly, applications based on the real datasets are given. Finally, conclusions are given.

2. Stress-strength reliability

Let X and Y are the independent random variables which are GG distribution with $(\lambda_1, c_1, \theta_1)$ and $(\lambda_2, c_2, \theta_2)$ parameters, respectively. Suppose that X is the strength of a component and Y is the stress affecting this component, then the reliability of this system, when $\lambda_1 = \lambda_2 = \lambda$ and $c_1 = c_2 = c$, is obtained as follows:

$$\begin{aligned} R = P(Y < X) &= \int_0^\infty F_Y(t) f_X(t) \\ &= \int_0^\infty \theta_1 \lambda e^{ct} e^{-\frac{\lambda}{c}\left(e^{ct}-1\right)} \left[1 - e^{-\frac{\lambda}{c}\left(e^{ct}-1\right)}\right]^{\theta_1 - 1} \left[1 - e^{-\frac{\lambda}{c}\left(e^{ct}-1\right)}\right]^{\theta_2} dt \\ &= \frac{\theta_1}{\theta_1 + \theta_2} \int_0^\infty (\theta_1 + \theta_2) \lambda e^{ct} e^{-\frac{\lambda}{c}\left(e^{ct}-1\right)} \left[1 - e^{-\frac{\lambda}{c}\left(e^{ct}-1\right)}\right]^{\theta_1 + \theta_2 - 1} dt \\ &= \frac{\theta_1}{\theta_1 + \theta_2}. \end{aligned}$$

2.1. Maximum likelihood estimation of SSR

Suppose that $\mathbf{X^R} = \left(X^{R_1}_{1:m:n}, X^{R_2}_{2:m:n}, \cdots, X^{R_m}_{m:m:n}\right)$ and $\mathbf{Y^S} = \left(Y^{S_1}_{1:k:l}, Y^{S_2}_{2:k:l}, \cdots, Y^{S_k}_{k:k:l}\right)$ are progressive type-II right censored samples with R and S censoring schemes taken from $GG(\lambda, c, \theta_1)$ distribution and $GG(\lambda, c, \theta_2)$ distribution, respectively. The likelihood function L is obtained as follows:

$$\begin{aligned} L\left(\lambda, c, \theta_1, \theta_2 \mid x^R y^S\right) &= \left[c_1 \prod_{i=1}^m f(x_{i:m:n}) \left(1 - F(x_{i:m:n})\right)^{R_i}\right] \\ &\quad \times \left[c_2 \prod_{j=1}^k f(y_{j:k:l}) \left(1 - F(y_{j:k:l})\right)^{S_j}\right], \end{aligned}$$

$$\begin{aligned} L\left(\lambda, c, \theta_1, \theta_2 \mid x^R y^S\right) &= c_1 c_2 \theta_1^m \theta_2^k \lambda^m \lambda^k \exp\left[c\left(\sum_{i=1}^m x_{i:m:n} + \sum_{j=1}^k y_{j:k:l}\right)\right] \\ &\quad \times \prod_{i=1}^m \left\{ e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)} \left[1 - e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right]^{\theta_1 - 1} \right. \\ &\qquad \left. \times \left[1 - \left(1 - e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)^{\theta_1}\right]^{R_i} \right\} \\ &\quad \times \prod_{j=1}^k \left\{ e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)} \left[1 - e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right]^{\theta_2 - 1} \right. \\ &\qquad \left. \times \left[1 - \left(1 - e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)^{\theta_2}\right]^{S_j} \right\}, \end{aligned}$$

where $\mathbf{x^R} = (x_{1:m:n}, x_{2:m:n}, \cdots, x_{i:m:n})$, $\mathbf{y^S} = (y_{1:k:l}, y_{2:k:l}, \cdots, y_{k:k:l})$,

$$c_1 = (n - 1 - R_1)(n - 2 - R_1 - R_2) \cdots (n - m + 1 + R_1 - \cdots - R_{m-1})$$

and

$$c_2 = (l-1-S_1)(l-2-S_1-S_2)\cdots(l-k+1+S_1-\cdots-S_{k-1}).$$

The log-likelihood function is given by

$$\begin{aligned}\ell\left(\lambda, c, \theta_1, \theta_2 \mid x^R y^S\right) &= m\ln\theta_1 + k\ln\theta_2 + (m+k)\ln\lambda \\ &\quad + c\left(\sum_{i=1}^{m} x_{i:m:n} + \sum_{j=1}^{k} y_{j:k:l}\right) - \sum_{i=1}^{m}\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right) \\ &\quad - \sum_{j=1}^{k}\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right) + (\theta_1-1)\sum_{i=1}^{m}\ln\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right) \\ &\quad + \sum_{i=1}^{m} R_i \ln\left[1-\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)^{\theta_1}\right] \\ &\quad + (\theta_2-1)\sum_{j=1}^{k}\ln\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right) \\ &\quad + \sum_{j=1}^{k} S_j \ln\left[1-\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)^{\theta_2}\right].\end{aligned}$$

Accordingly, the partial derivatives of log-likelihood function $\ell\left(\lambda, c, \theta_1, \theta_2 \mid x^R y^S\right)$ with respect to λ, c, θ_1 and θ_2 parameters are taken and then each of them is equalized to zero.

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta_1} &= \frac{m}{\theta_1} + \sum_{i=1}^{m}\ln\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right) \\ &\quad - \sum_{i=1}^{m} R_i \frac{\left[1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right]^{\theta_1}\ln\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)}{\left[1-\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)^{\theta_1}\right]}\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta_2} &= \frac{k}{\theta_2} + \sum_{i=1}^{m}\ln\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right) \\ &\quad - \sum_{j=1}^{k} S_j \frac{\left[1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right]^{\theta_2}\ln\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)}{\left[1-\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)^{\theta_2}\right]}\end{aligned}$$

$$\begin{aligned}\frac{\partial \ln L}{\partial \lambda} &= \frac{m}{\lambda} - \frac{-m+\sum_{i=1}^{m} e^{cx_{i:m:n}}}{c} + (\theta_1-1)\sum_{i=1}^{m}\frac{\left(e^{cx_{i:m:n}}-1\right)\left(e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)}{c\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)} \\ &\quad - \sum_{i=1}^{m} R_i \frac{\left[1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right]^{\theta_1}\theta_1\left(e^{cx_{i:m:n}}-1\right)\left(e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)}{c\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)\left[1-\left(1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)^{\theta_1}\right]} + \frac{k}{\lambda} \\ &\quad - \frac{-k+\sum_{j=1}^{k} e^{cy_{j:k:l}}}{c} + (\theta_2-1)\sum_{j=1}^{k}\frac{\left(e^{cy_{j:k:l}}-1\right)\left(e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)}{c\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)} \\ &\quad - \sum_{j=1}^{k} S_j \frac{\left[1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right]^{\theta_2}\theta_2\left(e^{cy_{j:k:l}}-1\right)\left(e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)}{c\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)\left[1-\left(1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)^{\theta_2}\right]}\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \ln L}{\partial c} &= \sum_{i=1}^{m} x_{i:m:n} + \frac{\lambda\left(-m+\sum_{i=1}^{m} e^{cx_{i:m:n}}\right)}{c^2} - \frac{\lambda\left(\sum_{i=1}^{m} x_{i:m:n} e^{cx_{i:m:n}}\right)}{c} \\
&+ (\theta_1 - 1)\sum_{i=1}^{m}\left(-\frac{\left(\frac{\lambda(e^{cx_{i:m:n}}-1)}{c^2} - \frac{\lambda(x_{i:m:n}e^{cx_{i:m:n}})}{c}\right)e^{-\frac{\lambda}{c}(e^{cx_{i:m:n}}-1)}}{1-e^{-\frac{\lambda}{c}(e^{cx_{i:m:n}}-1)}}\right) \\
&+ \sum_{i=1}^{m} R_i \frac{\left[1-e^{-\frac{\lambda}{c}(e^{cx_{i:m:n}}-1)}\right]^{\theta_1}\theta_1\left(\frac{\lambda(e^{cx_{i:m:n}}-1)}{c^2} - \frac{\lambda(x_{i:m:n}e^{cx_{i:m:n}})}{c}\right)e^{-\frac{\lambda}{c}(e^{cx_{i:m:n}}-1)}}{\left(1-e^{-\frac{\lambda}{c}(e^{cx_{i:m:n}}-1)}\right)\left[1-\left(1-e^{-\frac{\lambda}{c}(e^{cx_{i:m:n}}-1)}\right)^{\theta_1}\right]} \\
&+ \sum_{j=1}^{k} y_{j:k:l} + \frac{\lambda\left(-k+\sum_{j=1}^{k} e^{cy_{j:k:l}}\right)}{c^2} - \frac{\lambda\left(\sum_{j=1}^{k} y_{j:k:l} e^{cy_{j:k:l}}\right)}{c} \\
&+ (\theta_2 - 1)\sum_{j=1}^{k}\left(-\frac{\left(\frac{\lambda(e^{cy_{j:k:l}}-1)}{c^2} - \frac{\lambda(y_{j:k:l}e^{cy_{j:k:l}})}{c}\right)e^{-\frac{\lambda}{c}(e^{cy_{j:k:l}}-1)}}{1-e^{-\frac{\lambda}{c}(e^{cy_{j:k:l}}-1)}}\right) \\
&+ \sum_{j=1}^{k} S_j \frac{\left[1-e^{-\frac{\lambda}{c}(e^{cy_{j:k:l}}-1)}\right]^{\theta_2}\theta_2\left(\frac{\lambda(e^{cy_{j:k:l}}-1)}{c^2} - \frac{\lambda(y_{j:k:l}e^{cy_{j:k:l}})}{c}\right)e^{-\frac{\lambda}{c}(e^{cy_{j:k:l}}-1)}}{\left(1-e^{-\frac{\lambda}{c}(e^{cy_{j:k:l}}-1)}\right)\left[1-\left(1-e^{-\frac{\lambda}{c}(e^{cy_{j:k:l}}-1)}\right)^{\theta_2}\right]}
\end{aligned}
$$

Since there are no explicit forms of these equations, ML estimates of λ, c, θ_1 and θ_2 parameters are obtained by using Newton Rapson method. The ML estimator of SSR, due to invariant property of ML estimate, can be written as

$$\hat{R}_{MLE} = \frac{\hat{\theta}_1}{\hat{\theta}_1 + \hat{\theta}_2}, \tag{2.1}$$

where $\hat{\theta}_1$ and $\hat{\theta}_2$ are the ML estimators of θ_1 and θ_2, respectively, under progressive type-II censoring.

2.2. Asymptotic confidence interval

Fisher information matrix $\eta = (\lambda, c, \theta_1, \theta_2)$ parameter vector is given by

$$
I(\eta) = -E\begin{bmatrix}
\frac{\partial^2 \ln L}{\partial \lambda^2} & \frac{\partial^2 \ln L}{\partial \lambda \partial c} & \frac{\partial^2 \ln L}{\partial \lambda \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \lambda \partial \theta_2} \\
\frac{\partial^2 \ln L}{\partial c \partial \lambda} & \frac{\partial^2 \ln L}{\partial c^2} & \frac{\partial^2 \ln L}{\partial c \partial \theta_1} & \frac{\partial^2 \ln L}{\partial c \partial \theta_2} \\
\frac{\partial^2 \ln L}{\partial \theta_1 \partial \lambda} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial c} & \frac{\partial^2 \ln L}{\partial \theta_1^2} & \frac{\partial^2 \ln L}{\partial \theta_1 \partial \theta_2} \\
\frac{\partial^2 \ln L}{\partial \theta_2 \partial \lambda} & \frac{\partial^2 \ln L}{\partial \theta_2 \partial c} & \frac{\partial^2 \ln L}{\partial \theta_2 \partial \theta_1} & \frac{\partial^2 \ln L}{\partial \theta_2^2}
\end{bmatrix}.
$$

It is difficult to calculate $I(\eta)$. Therefore, observed Fisher information, $I(\hat{\eta})$ is used approximate to expected Fisher information matrix. Here, $\hat{\eta}$ is MLE of η. $I(\hat{\eta})$ is written as $I(\hat{\eta}) = -[I(\eta)]_{\eta=\hat{\eta}}$. While $m \to \infty$, $k \to \infty$ and $n = l$.

$$\sqrt{n}(\hat{\eta} - \eta) \underset{\rightarrow}{d} N\left(0, I^{-1}(\eta)\right),$$

where $I^{-1}(\eta)$ is inverse of Fisher information matrix. Let $A = \left(\frac{\partial R}{\partial \lambda}, \frac{\partial R}{\partial c}, \frac{\partial R}{\partial \theta_1}, \frac{\partial R}{\partial \theta_2}\right)$. By using delta method and invariant property of MLE, asymptotic distribution of $\hat{R}$ is as follows:

$$\sqrt{n}\left(\hat{R} - R\right) \underset{\rightarrow}{d} N\left(0, Var\left(\hat{R}\right)\right),$$

where $Var\left(\hat{R}\right) = A^T I^{-1} A$. The $100\,(1-\alpha)$ confidence interval for R can be constructed as $\left(\hat{R} - z_{\frac{\alpha}{2}} \times \sqrt{Var\left(\hat{R}\right)}, \hat{R} + z_{\frac{\alpha}{2}} \times \sqrt{Var\left(\hat{R}\right)}\right)$. Here, z_α denotes the upper αth quantile of the standard normal distribution [41].

2.3. Bootstrap confidence intervals

The bootstrap method widely used in practice is the percentile bootstrap $(Boot-p)$ proposed by [14]. The following steps briefly demonstrate how to estimate bootstrap parametric confidence intervals of P using the $Boot-p$ method [4].

Step 1. Generate PTR-II censored samples $\mathbf{X^R} = \left(X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \cdots, X_{m:m:n}^{R_m}\right)$ and $\mathbf{Y^S} = \left(Y_{1:k:l}^{S_1}, Y_{2:k:l}^{S_2}, \cdots, Y_{k:k:l}^{S_k}\right)$ taken from GG distribution with $(\lambda_1, c_1, \theta_1)$ and $(\lambda_2, c_2, \theta_2)$.
Step 2. Estimation R, say $\hat{R}_{MLE}$.
Step 3. To generate bootstrap samples $\mathbf{X^R} = (X_{1:m:n}^*, X_{2:m:n}^*, \cdots, X_{m:m:n}^*)$ and $\mathbf{Y^S} = (Y_{1:k:l}^*, Y_{2:k:l}^*, \cdots, Y_{k:k:l}^*)$, using $\hat{R}_{MLE}$, $R_1, R_2, \ldots, R_{m_1}$; $S_1, S_2, \ldots S_{m_2}$. Find the bootstrap estimate of R, say $\hat{R}_{MLE}^*$.
Step 4. Repeat Step 3 NBoot times.
Step 5. Let $F^*(x) = P\left(\hat{R}_{MLE}^* \leq x\right)$, the cumulative distribution function of $\hat{P}_{MLE}^*$. Define $\hat{R}_{MLEBoot-p(x)}^* = F^{*^{-1}}(x)$ for given x. The approximate $100\,(1-\alpha)\,\%$ confidence interval for P is given by Equation (2.1).

$$\left(\hat{R}_{MLEBoot-p(x)}^*\left(\frac{\alpha}{2}\right), \hat{R}_{MLEBoot-p(x)}^*\left(1-\left(\frac{\alpha}{2}\right)\right)\right)$$

2.4. Bayes estimation

Assuming that $\mathbf{X^R} = \left(X_{1:m:n}^{R_1}, X_{2:m:n}^{R_2}, \cdots, X_{m:m:n}^{R_m}\right)$ and $\mathbf{Y^S} = \left(Y_{1:k:l}^{S_1}, Y_{2:k:l}^{S_2}, \cdots, Y_{k:k:l}^{S_k}\right)$ are PTR-II censoring samples having GG distribution with $(\lambda_1, c_1, \theta_1)$ and $(\lambda_2, c_2, \theta_2)$ parameters, respectively, λ, c, θ_1 and θ_2 parameters are independent random variables with prior Gamma distributions. Then the density functions for these parameters are given by

$$\pi(\lambda) = \frac{\delta_1^{\gamma_1}}{\Gamma(\gamma_1)} \lambda^{\gamma_1 - 1} e^{-\delta_1 \lambda}, \delta_1, \gamma_1, \lambda > 0,$$

$$\pi(c) = \frac{\delta_2^{\gamma_2}}{\Gamma(\gamma_2)} c^{\gamma_2 - 1} e^{-\delta_2 c}, \delta_2, \gamma_2, c > 0,$$

$$\pi(\theta_1) = \frac{\psi^\rho}{\Gamma(\rho)} \theta_1^{\rho - 1} e^{-\psi \theta_1}, \psi, \rho, \theta_1 > 0,$$

$$\pi(\theta_2) = \frac{\varsigma^\tau}{\Gamma(\tau)} \theta_2^{\tau - 1} e^{-\varsigma \theta_2}, \varsigma, \tau, \theta_2 > 0,$$

respectively. The joint prior density function can be written as follows:

$$\pi(\lambda, c, \theta_1, \theta_2) \infty \delta_1^{\gamma_1} \delta_2^{\gamma_2} \psi^\rho \varsigma^\tau \lambda^{\gamma_1 - 1} c^{\gamma_2 - 1} \theta_1^{\rho - 1} \theta_2^{\tau - 1} e^{-\delta_1 \lambda} e^{-\delta_2 c} e^{-\psi \theta_1} e^{-\varsigma \theta_2},$$

where $\lambda > 0$, $c > 0, \theta_1 > 0, \theta_2 > 0$. Then, the joint posterior density function of λ, c, θ_1 and θ_2 is obtained by

$$P\left(\lambda, c, \theta_1, \theta_2 \mid \mathbf{X^R Y^S}\right) = \frac{A}{B},$$

where

$$A = \theta_1^m \theta_2^k \lambda^{m+k} e^{c\left(\sum_{i=1}^{m} x_{i:m:n} + \sum_{j=1}^{m} y_{j:k:l}\right)} k_1(x_{i:m:n}; \lambda, c, \theta_1)\, k_2(y_{j:k:l}; \lambda, c, \theta_2)\, \pi(\lambda, c, \theta_1, \theta_2)$$

and

$$B = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty \theta_1^m\theta_2^k\lambda^{m+k}e^{c\left(\sum_{i=1}^m x_{i:m:n}+\sum_{j=1}^m y_{j:k:l}\right)} k_1\left(x_{i:m:n};\lambda,c,\theta_1\right)k_2\left(y_{j:k:l};\lambda,c,\theta_2\right) \times\pi\left(\lambda,c,\theta_1,\theta_2\right)d\lambda dc d\theta_1 d\theta_2.$$

Here,

$$k_1\left(x_{i:m:n};\lambda,c,\theta_1\right)=\prod_{i=1}^m e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\left[1-e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right]^{\theta_1-1}\left[\left(e^{-\frac{\lambda}{c}\left(e^{cx_{i:m:n}}-1\right)}\right)^{\theta_1}\right]^{R_i}$$

and

$$k_2\left(y_{j:k:l};\lambda,c,\theta_2\right)=\prod_{j=1}^k e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\left[1-e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right]^{\theta_2-1}\left[\left(e^{-\frac{\lambda}{c}\left(e^{cy_{j:k:l}}-1\right)}\right)^{\theta_2}\right]^{S_j}.$$

Thus, the Bayes estimate of $u\left(\lambda,c,\theta_1,\theta_2\right)$ that is any function of λ,c,θ_1 and θ_2 under a squared error loss function can be given as follows:

$$\begin{aligned}\hat{u}_B\left(\lambda,c,\theta_1,\theta_2\right) &= E\left(u\left(\lambda,c,\theta_1,\theta_2\mid \mathbf{X}^{\mathbf{R}}\mathbf{Y}^{\mathbf{S}}\right)\right)\\ &= \frac{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty u\left(\lambda,c,\theta_1,\theta_2\right)e^{\left[l\left(\lambda,c,\theta_1,\theta_2\mid x^Ry^S\right)+G\left(\lambda,c,\theta_1,\theta_2\right)\right]}d\lambda dc d\theta_1 d\theta_2}{\int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty e^{\left[l\left(\lambda,c,\theta_1,\theta_2\mid x^Ry^S\right)+G\left(\lambda,c,\theta_1,\theta_2\right)\right]}d\lambda dc d\theta_1 d\theta_2},\end{aligned}$$

where

$$G\left(\lambda,c,\theta_1,\theta_2\right)\propto\left(\gamma_1-1\right)\ln\lambda+\left(\gamma_2-1\right)\ln c+\left(\rho-1\right)\ln\theta_1+\left(\tau-1\right)\ln\theta_2-\delta_1\lambda-\delta_2 c-\psi\theta_1-\varsigma\theta_2,$$

in which $G\left(\lambda,c,\theta_1,\theta_2\right)=\ln\pi\left(\lambda,c,\theta_1,\theta_2\right)$ is the logarithm of prior density function. As it is, however, not possible to obtain the explicit form of $\hat{u}_B\left(\lambda,c,\theta_1,\theta_2\right)$, the approximate Bayes estimators of SSR are acquired with the help of Lindley's approximation under the squared error loss function.

2.4.1. Lindley's approximation. Lindley's approximation which is an approximation of the Bayes estimate was put forward by [25]. The formulas as regards Lindley's approximation are given by

$$\begin{aligned}u\left(\hat{\lambda},\hat{c},\hat{\theta}_1,\hat{\theta}_2\right)_{BAYES} &= E\left[\lambda,c,\theta_1,\theta_2\mid\mathbf{X}\right]\\ &\approx \left\{u\left(\hat{\lambda},\hat{c},\hat{\theta}_1,\hat{\theta}_2\right)_{MLE}+\frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4\left(u_{ij}+2u_ig_j\right)\sigma_{ij}\right.\\ &\quad\left.+\frac{1}{2}\sum_{i=1}^4\sum_{j=1}^4\sum_{j=1}^4\sum_{j=1}^4\left(l_{ijk}\sigma_{ij}\sigma_{kl}u_l\right)\right\}.\end{aligned}$$

$u_i,\ i=1,2,3,4$ are the unary partial derivatives and $u_{ij},\ i,j=1,2,3,4$ are the binary partial derivatives of $u\left(\lambda,c,\theta_1,\theta_2\right)$ with respect to λ, c, θ_1 and θ_2 parameters, respectively. $\ell_{ij},\ i,j=1,2,3,4$ are the binary partial derivatives and $\ell_{ijk},\ i,j,k=1,2,3,4$ are the trinary partial derivatives of log-likelihood function $\ell\left(\lambda,c,\theta_1,\theta_2\right)$with respect to λ, c, θ_1 and θ_2 parameters, respectively. In addition to this, $\left[-\ell_{ij}\right]^{-1}=\left[\sigma_{ij}\right]$, $i,j=1,2,3,4$ and σ_{ij} is the (i,j)th element of the matrix $\left[\sigma_{ij}\right]$. Then, $g_i, i=1,2,3,4$ can be seen in the formulas below:

$$g_1=\frac{\partial G\left(\lambda,c,\theta_1,\theta_2\right)}{\partial\lambda}=\frac{\left(\gamma_1-1\right)}{\lambda}-\delta_1,$$

$$g_2=\frac{\partial G\left(\lambda,c,\theta_1,\theta_2\right)}{\partial c}=\frac{\left(\gamma_2-1\right)}{\lambda}-\delta_2,$$

$$g_3 = \frac{\partial G(\lambda, c, \theta_1, \theta_2)}{\partial \theta_1} = \frac{(\rho - 1)}{\theta_1} - \psi,$$
$$g_4 = \frac{\partial G(\lambda, c, \theta_1, \theta_2)}{\partial \theta_2} = \frac{(\varsigma - 1)}{\theta_2} - \tau.$$

If $u(\lambda, c, \theta_1, \theta_2) = \frac{\theta_1}{\theta_1+\theta_2}$, the following approximate Bayes estimator under the squared error loss function of SSR for GG distribution based on PTR-II censored samples can be acquired, $u_1 = 0, u_2 = 0, u_{11} = u_{12} = u_{13} = u_{14} = 0, u_{21} = u_{22} = u_{23} = u_{24} = 0, u_3 = \frac{\theta_2}{(\theta_1+\theta_2)^2}, u_4 = \frac{\theta_1}{(\theta_1+\theta_2)^2}, u_{33} = \frac{2\theta_2}{(\theta_1+\theta_2)^3}, u_{34} = u_{43} = \frac{\theta_1-\theta_2}{(\theta_1+\theta_2)^3}, u_{41} = u_{42} = 0, u_{31} = u_{32} = 0, u_{44} = \frac{2\theta_1}{(\theta_1+\theta_2)^3}$.

$$\hat{R}_{BAYES} = \frac{\hat{\theta}_{1MLE}}{\hat{\theta}_{1MLE} + \hat{\theta}_{2MLE}} + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}(u_{ij} + 2u_i g_j)\sigma_{ij} + \frac{1}{2}\sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{j=1}^{4}\sum_{j=1}^{4}(\ell_{ijk}\sigma_{ij}\sigma_{kl}u_l)$$

3. Simulation study

In this section, a Monte Carlo simulation study is conducted to check performances of ML estimator against performances of approximate Bayes estimator calculated by using informative priors (Prior 1) $(\gamma_1 = 2, \delta_1 = 2)$, $(\gamma_2 = 2, \delta_2 = 2)$, $(\rho = 2, \psi = 2)$ and $(\varsigma = 2, \tau = 2)$ and non-informative priors (Prior 2) $(\gamma_1 = 0.0001, \delta_1 = 0.0001)$, and $(\gamma_2 = 0.0001, \delta_2 = 0.0001)$, $(\rho = 0.0001, \psi = 0.0001)$ and $(\varsigma = 0.0001, \tau = 0.0001)$ for SSR of GG distribution under PTR-II censoring. The results are shown in Tables 1–3.

Table 1. *Risk and mean values for ML and approximate Bayes Estimates of R Prior:* $(\gamma_1 = 1, \delta_1 = 2, \gamma_2 = 2, \delta_2 = 1, \rho = 3, \psi = 2, \varsigma = 3, \tau = 2)$.

		CS		ML estimation		Bayes estimation	
(n, m)	(l, k)	R	S	Risk	Mean	Risk	Mean
10,5	10,5	B	B	0.0086	0.3490	0.0111	0.3578
10,5	10,5	B	C	0.0098	0.3440	0.0122	0.3749
10,5	10,5	C	B	0.0101	0.4700	0.0108	0.4902
10,5	10,5	C	C	0.0106	0.3992	0.0106	0.4197
10,5	10,5	D	D	0.0094	0.4085	0.0118	0.4017
10,10	10,10	I	I	0.0088	0.3800	0.0084	0.3815
20,5	20,5	B	B	0.0054	0.3280	0.0084	0.3509
20,5	20,5	B	C	0.0068	0.3938	0.0088	0.3848
20,5	20,5	C	B	0.0074	0.5072	0.0085	0.5196
20,5	20,5	C	C	0.0081	0.3876	0.0085	0.4125
20,5	20,5	D	D	0.0060	0.3744	0.0083	0.3738
20,10	20,10	B	B	0.0045	0.3571	0.0053	0.3548
20,10	20,10	B	C	0.0054	0.3588	0.0058	0.3594
20,10	20,10	C	B	0.0060	0.4780	0.0062	0.4888
20,10	20,10	C	C	0.0064	0.3857	0.0065	0.3964
20,10	20,10	D	D	0.0052	0.4291	0.0057	0.4178
20,15	20,15	B	B	0.0045	0.4286	0.0049	0.4173
20,15	20,15	B	C	0.0048	0.3608	0.0048	0.3657
20,15	20,15	C	B	0.0052	0.4775	0.0053	0.4566
20,15	20,15	C	C	0.0054	0.4033	0.0054	0.4110
20,15	20,15	D	D	0.0047	0.4246	0.0048	0.4208
20,20	20,20	I	I	0.0044	0.4342	0.0046	0.4331
30,5	30,5	B	B	0.0046	0.3333	0.0072	0.3515

Table 1. (continued)

		CS		ML estimation		Bayes estimation	
(n,m)	(l,k)	R	S	Risk	Mean	Risk	Mean
30,5	30,5	B	C	0.0056	0.3977	0.0074	0.3908
30,5	30,5	C	B	0.0060	0.5195	0.0068	0.5266
30,5	30,5	C	C	0.0067	0.3891	0.0074	0.4090
30,5	30,5	D	D	0.0049	0.3907	0.0077	0.3805
30,10	30,10	B	B	0.0034	0.3226	0.0044	0.3392
30,10	30,10	B	C	0.0043	0.3910	0.0050	0.3789
30,10	30,10	C	B	0.0049	0.4913	0.0052	0.4947
30,10	30,10	C	C	0.0056	0.4104	0.0058	0.4099
30,10	30,10	D	D	0.0038	0.4033	0.0045	0.3940
30,15	30,15	B	B	0.0030	0.3348	0.0034	0.3423
30,15	30,15	B	C	0.0039	0.3811	0.0041	0.3734
30,15	30,15	C	B	0.0042	0.4884	0.0043	0.4876
30,15	30,15	C	C	0.0045	0.4266	0.0046	0.4246
30,15	30,15	D	D	0.0034	0.4224	0.0038	0.4149
30,20	30,20	B	B	0.0031	0.4043	0.0034	0.3971
30,20	30,20	B	C	0.0035	0.3866	0.0036	0.3818
30,20	30,20	C	B	0.0037	0.4778	0.0037	0.4763
30,20	30,20	C	C	0.0040	0.4323	0.0040	0.4307
30,20	30,20	D	D	0.0034	0.4161	0.0034	0.4129
50,5	50,5	B	B	0.0037	0.4292	0.0080	0.4019
50,5	50,5	B	C	0.0043	0.3864	0.0063	0.3852
50,5	50,5	C	B	0.0050	0.5268	0.0056	0.5311
50,5	50,5	C	C	0.0054	0.3851	0.0062	0.4010
50,5	50,5	D	D	0.0037	0.4650	0.0084	0.4244
50,10	50,10	B	B	0.0024	0.3431	0.0035	0.3467
50,10	50,10	B	C	0.0036	0.3983	0.0041	0.3863
50,10	50,10	C	B	0.0040	0.4920	0.0042	0.4945
50,10	50,10	C	C	0.0046	0.4191	0.0048	0.4166
50.10	50,10	D	D	0.0028	0.4113	0.0036	0.4002
50,15	50,5	B	B	0.0022	0.3433	0.0027	0.3461
50,15	50,5	B	C	0.0030	0.3916	0.0032	0.3827
50,15	50,5	C	B	0.0034	0.4871	0.0035	0.4864
50,15	50,5	C	C	0.0040	0.4307	0.0041	0.4282
50,15	50,5	D	D	0.0026	0.4202	0.0028	0.4140
50,20	50,10	B	B	0.0020	0.3518	0.0023	0.3517
50,20	50,10	B	C	0.0027	0.3854	0.0028	0.3797
50,20	50,10	C	B	0.0030	0.4831	0.0030	0.4823
50,20	50,10	C	C	0.0034	0.4284	0.0035	0.4265
50.20	50,10	D	D	0.0023	0.4165	0.0024	0.4160
50,30	50,30	B	B	0.0019	0.4098	0.0021	0.4041
50,30	50,30	B	C	0.0023	0.3890	0.0023	0.3864
50,30	50,30	C	B	0.0024	0.4740	0.0024	0.4743
50,30	50,30	C	C	0.0027	0.4296	0.0027	0.4287
50,30	50,30	D	D	0.0020	0.4220	0.0020	0.4220
50,50	50,50	I	I	0.0019	0.4125	0.0019	0.4125

In this table, true parameters are received as $\lambda = 0.4$, $c = 0.3$, $\theta_1 = 0.6$, $\theta_2 = 0.5$. In Tables 1–3, λ, c, θ_1, θ_2 are generated from Gamma distribution in each replicate. In Table 3, ACIs and Bootstrap confidence intervals of R are calculated.

Table 2. *The MSE and Mean bias values for ML and approximate Bayes estimates of SSR* ($\lambda = 0.4, c = 0.3, \theta_1 = 0.6, \theta_2 = 0.5$).

		CS		ML estimation		Bayes estimation			
						Prior I		Prior II	
(n,m)	(l,k)	R	S	Bias	MSE	Bias	MSE	Bias	MSE
20,10	20,10	B	B	-0.0508	0.0077	-0.0047	0.0037	-0.0099	0.0040
20,10	20,10	C	C	-0.0271	0.0081	0.0176	0.0051	0.0131	0.0053
20,10	20,10	D	D	-0.0272	0.0081	0.0175	0.0054	0.0131	0.0053
30,20	30,20	B	B	-0.0169	0.0043	-0.0012	0.0034	-0.0023	0.0035
30,20	30,20	C	C	-0.0148	0.0046	0.0033	0.0036	0.0020	0.0037
30,20	30,20	D	D	-0.0151	0.0042	0.0006	0.0033	-0.0006	0.0035
30,30	30,30	I	I	0.0117	0.0036	0.0117	0.0036	0.0110	0.0037
50,20	50,20	B	B	-0.0350	0.0052	-0.0180	0.0032	-0.0156	0.0035
50,20	50,20	C	C	0.0400	0.0055	0.0172	0.0034	0.0169	0.0036
50,20	50,20	D	D	-0.0480	0.0053	-0.0195	0.0027	-0.0236	0.0030
50,30	50,30	B	B	-0.0106	0.0026	-0.0004	0.0022	-0.0013	0.0023
50,30	50,30	C	C	-0.0109	0.0028	0.0014	0.0023	0.0004	0.0024
50,30	50,30	D	D	-0.0107	0.0024	-0.0003	0.0024	-0.0012	0.0025
50,50	50,50	I	I	0.0105	0.0023	0.0105	0.0023	0.0104	0.0023

Table 3. *Confidence average width and of coverage probability for ML estimates of SSR* ($\lambda = 0.2, c = 0.6, \theta_1 = 0.4, \theta_2 = 0.8$).

		CS			ML estimates				
(n,m)	(l,k)	R	S	R	$\hat{R}$	Low. Bound	Upp. Bound	ACI width	Cov. Prob.
20,10	20,10	B	B	0.3333	0.3304	0.1617	0.4991	0.3373	0.9316
20,10	20,10	C	C		0.3305	0.1947	0.4663	0.2716	0.9356
20,10	20,10	D	D		0.3306	0.1837	0.4558	0.2536	0.9365
20,20	20,20	I	I		0.3327	0.1917	0.4737	0.2820	0.9326
30,20	30,20	B	B		0.3344	0.2233	0.4465	0.2242	0.9338
30,20	30,20	C	C		0.3368	0.2245	0.4453	0.2258	0.9460
30,20	30,20	D	D		0.3319	0.2149	0.4491	0.2341	0.9444
50,20	50,20	B	B		0.3356	0.2147	0.4569	0.2422	0.9468
50,20	50,20	C	C		0.3386	0.2336	0.4237	0.1901	0.9312
50,20	50,20	D	D		0.3325	0.2345	0.4325	0.1925	0.9315
50,30	50,30	B	B		0.3316	0.2439	0.4193	0.1753	0.9496
50,30	50,30	C	C		0.3318	0.2413	0.4271	0.1803	0.9479
50,30	50,30	D	D		0.3321	0.2441	0.4253	0.1815	0.9588
20,10	20,10	B	B		0.3303	0.1668	0.5067	0.3399	0.9468
20,10	20,10	C	C		0.3231	0.1668	0.4700	0.2916	0.9436
20,10	20,10	D	D		0.3315	0.1784	0.4945	0.3049	0.9478
30,20	30,20	B	B		0.3321	0.1924	0.4490	0.2297	0.9416
30,20	30,20	C	C		0.3325	0.2192	0.4463	0.2315	0.9578
30,20	30,20	D	D		0.3288	0.2132	0.4523	0.2432	0.9580
50,20	50,20	B	B		0.3366	0.2172	0.4604	0.2395	0.9484
50,20	50,20	C	C		0.3223	0.2209	0.4225	0.2025	0.9264
50,20	50,20	D	D		0.3325	0.2200	0.4326	0.2025	0.9265
50,30	50,30	B	B		0.3399	0.2285	0.4201	0.1783	0.9476
50,30	50,30	C	C		0.3315	0.2418	0.4250	0.1820	0.9485
50,30	50,30	D	D		0.3325	0.2446	0.4230	0.1826	0.9589
50,50	50,50	I	I		0.3321	0.2315	0.4245	0.1811	0.9572

The following algorithm proposed by [8] is used for obtaining PTR-II censored samples with GG distribution. Steps of this algorithm are given below.

(1) $W_1, W_2, \cdots, W_m$ are m-sized samples produced from Uniform$(0,1)$ distribution.

(2) $V_i = W_i^{\left(i+\sum\limits_{j=m-i+1}^{m} R_j\right)^{-1}}$ and $V_j = W_j^{\left(j+\sum\limits_{j=k-i+1}^{k} S_j\right)^{-1}}$ are computed by replacing $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, k$ where $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ and $\mathbf{S} = (S_1, S_2, \cdots, S_k)$ are known.

(3) $U^R_{i:m:n} = 1 - V_m V_{m-1} \cdots V_{m-i+1}$ and $U^S_{j:k:l} = 1 - V_k V_{k-1} \cdots V_{k-j+1}$ are obtained by replacing $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, k$. Thus $U^R_{1:m:n} < U^R_{2:m:n} < \cdots < U^R_{m:m:n}$ and $U^S_{1:k:l} < U^R_{2:k:l} < \cdots < U^R_{k:k:l}$ are PTR-II censored samples with censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ and $\mathbf{S} = (S_1, S_2, \cdots, S_k)$ taken from *Uniform(0,1)* distribution.

(4) Finally, $X^R_{i:m:n} = \frac{1}{c}\left\{1 + \ln\frac{1}{\lambda}\left[\ln\left(-\exp\left(\frac{\ln\left(U^R_{i:m:n}\right)}{\theta}\right) + 1\right) c\right]\right\}$ $i = 1, 2, \cdots, m$ and $Y^S_{j:k:l} = \frac{1}{c}\left\{1 + \ln\frac{1}{\lambda}\left[\ln\left(-\exp\left(\frac{\ln\left(U^S_{j:k:l}\right)}{\theta}\right) + 1\right) c\right]\right\}$ $j = 1, 2, \cdots, k$ are PTR-II censored ith and jth order statistics with censoring scheme $\mathbf{R} = (R_1, R_2, \cdots, R_m)$ taken from $GG(\lambda, c, \theta_1)$ distribution and censoring scheme $\mathbf{S} = (S_1, S_2, \cdots, S_k)$ taken from $GG(\lambda, c, \theta_2)$ distribution, respectively.

According to different censoring schemes and sample sizes, risk and mean estimation values for ML and approximate Bayes estimates of R over 10000 replications are shown in Table 1. In Table 2, for different censoring schemes and sample sizes, bias and mean square error (MSE) values for ML and approximate Bayes estimates of R over 10000 replications are shown. In Table 3, coverage probabilities, lengths, lower and upper bounds for ACI and Bootstrap confidence intervals over 5000 replicates with 500 n-boot are shown.

I: Complete sample ($R = (0, 0, \cdots, 0)$ and $S = (0, 0, \cdots, 0)$).
B: The censoring at the end of the experiment ($R = (0, 0, \cdots, n-m)$ and $S = (0, 0, \cdots, k-l)$).
C: The censoring at the beginning of the experiment ($R = (n-m, 0, \cdots, 0)$ and $S = (k-l, 0, \cdots, 0)$).
D: Other the censoring schemes ($R = (1, 1, \cdots, 1)$ and $S = (1, 1, \cdots, 1)$).

The simulated bias (s-Bias) and mean square errors (s-MSEs) are calculated by

$$s - Bias_\theta = \frac{\sum\limits_{i=1}^{5000}\left(\hat{\theta} - \theta\right)}{5000}$$

and

$$s - MSE_\theta = \frac{\sum\limits_{i=1}^{5000}\left(\hat{\theta} - \theta\right)^2}{5000},$$

where θ stands for $\lambda, c, \theta_1, \theta_2$ and $\hat{\theta}_i$ stands for $\hat{\lambda}_i, \hat{c}_i, \hat{\theta}_{i1}$ and $\hat{\theta}_{i2}$ respectively.

It is observed from Tables 1 and 2 that the bias, MSEs and risks of all estimates are given for different n, l, m and k values. The bias, MSEs and risks of the ML and Bayes estimates exhibit almost the same performances. It can also be said that all the estimates are asymptotically unbiased, when the n, l, m and k values increase, the bias, MSEs and risks of the estimates decrease to zero. Furthermore, as seen from Table 3, when the n, l, m and k values increase, the coverage probabilities decrease and reach the desired level as expected. For different n, l, m and k values, the coverage probabilities of ACIs and Bootstrap confidence intervals approximate to $1 - \alpha = 0.95$.

4. Practical data analysis

In this section, parameter estimates and confidence intervals for the two estimation methods are obtained and then the performances of ML and Bayes estimation methods are compared using two different real datasets and one simulated dataset. We modified the goodness-of-fit of censored data for the GG distribution using approximate KS test statistics proposed by [32]. The modified test statistics KS and the corresponding p-value are calculated via *R* software using parametric bootstrap for censored data sets.

Real Data Study 1: In this subsection, the datasets consist of the waiting times (in minutes) before the service of the customers of two banks A (data 1) and B (data 2) were discussed by [24].

Data 1. X $(n = 100)$.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3
3.5	3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4	4.6	4.7	4.7	4.8
4.9	4.9	5.0	5.3	5.5	5.7	5.7	6.1	6.2	6.2	6.2	6.3	6.7	6.9
7.1	7.1	7.1	7.1	7.4	7.6	7.7	8.0	8.2	8.6	8.6	8.6	8.8	8.8
8.9	8.9	9.5	9.6	9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5
11.9	12.4	12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3
17.3	18.2	18.2	18.4	18.9	19.0	19.9	20.6	21.3	21.4	21.9	23.0	27.0	31.6
33.1	38.5												

Data 2. Y $(n = 60)$.

0.1	0.2	0.3	0.7	0.9	1.1	1.2	1.8	1.9	2.0	2.2	2.3	2.3	2.3
2.5	2.6	2.7	2.7	2.9	3.1	3.1	3.2	3.4	3.4	3.5	3.9	4.0	4.2
4.5	4.7	5.3	5.6	5.6	6.2	6.3	6.6	6.8	7.3	7.5	7.7	7.7	8.0
8.0	8.5	8.5	8.7	9.5	10.7	10.9	11.0	12.1	12.3	12.8	12.9	13.2	13.7
14.5	16.0	16.5	28.0										

The censored data obtained according to the censoring schemes are given below:

Censored Data 1. X $(m = 25)$.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1	3.2	3.3
3.5	3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4	4.6			

Censored Data 2. Y $(k = 25)$.

0.1	0.2	0.3	0.7	0.9	1.1	1.2	1.8	1.9	2.0	2.2	2.3	2.3	2.3
2.5	2.6	2.7	2.7	2.9	3.1	3.1	3.2	3.4	3.4	3.5			

The approximate KS and the corresponding p-value (in parentheses) using parametric bootstrap for censored data 1 and censored data 2 are 0.5853 (0.9433) and 0.6706 (0.7700), respectively. These results are displayed in Table 4. Accordingly, it is seen that both censored datasets fit the GG distribution.

Table 4. *Results of the KS test for the real datasets given in study 1.*

Model		ML estimates			KS	P-value
GG	X (Censored Data1)	$\hat{\theta} = 2.2835$	$\hat{\lambda} = 0.1664$	$\hat{c} = 0.0005$	0.5853	0.9433
	Y (Censored Data2)	$\hat{\theta} = 1.2432$	$\hat{\lambda} = 0.1664$	$\hat{c} = 0.0005$	0.6706	0.7700

Then, the following ML and approximate Bayes estimates for SSR under PTR-II censoring are acquired. In Table 5, $\hat{R}_{MLE}$ and $\hat{R}_{BAYES}$ are given. Besides, in Table 6 the asymptotic and bootstrap confidence intervals for SSR are given (0.5366–0.7547) and (0.5733–0.7581), respectively.

Table 5. *The ML and approximate Bayes estimates for SSR in real data study 1.*

		CS		$\hat{R}_{MLE}$	$\hat{R}_{BAYES}$	
(n, m)	(l, k)	R	S		Prior 1	Prior 2
100,25	60,25	$(24 \times 0, 75)$	$(24 \times 0, 35)$	0.6462	0.6198	0.6195

Table 6. *The asymptotic and bootstrap confidence intervals for SSR in real data study 1.*

		CS		ML estimates			Boot ML estimation		
(n, m)	(l, k)	R	S	$\hat{R}_{MLE}$	Low. Lim	Upp. Lim	$\hat{R}_{MLE}$	Low. Lim	Upp. Lim
100,25	60,25	$(24 \times 0, 75)$	$(24 \times 0, 35)$	0.6462	0.5366	0.7547	0.6457	0.5733	0.7581

Real Data Study 2: We regard two datasets declared by [5], on failure stresses of single carbon fibers of lengths 20 mm and 50 mm, respectively. The datasets are shown as follows:

Length 20 mm $Xn = 69$:

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.994	1.958	1.966	1.997	2.006
2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270	2.272	2.274
2.301	2.301	2.359	2.382	3.382	2.426	2.434	2.435	2.478	2.490	2.511	2.514	2.535
2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684	2.697	2.726	2.770	2.773
2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012	3.067	3.084	3.090	3.096	3.128
3.233	3.433	3.585	3.585									

and

Length 50 mm $Y\,(l = 65)$:

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.764	1.807	1.812	1.840	1.852
1.852	1.862	1.864	1.931	1.952	1.974	2.019	2.051	2.055	2.058	2.088	2.125	2.162
2.171	2.172	2.180	2.194	2.211	2.270	2.272	2.280	2.299	2.308	2.335	2.349	2.356
2.386	2.390	2.410	2.430	2.431	2.458	2.471	2.497	2.514	2.558	2.577	2.593	2.601
2.604	2.620	2.633	2.670	2.682	2.699	2.705	2.735	2.785	3.020	3.042	3.116	3.174

The censored data obtained according to the censoring schemes are given below.

Censored Length 20 mm $X\,(m = 30)$

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.994	1.958	1.966	1.997	2.006
2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270	2.272	2.274
2.301	2.301	2.359	2.382									

Censored Length 50 mm $Y\,(k = 30)$

1.339	1.434	1.549	1.574	1.589	1.613	1.746	1.753	1.764	1.807	1.812	1.840	1.852
1.852	1.862	1.864	1.931	1.952	1.974	2.019	2.051	2.055	2.058	2.088	2.125	2.162
2.171	2.172	2.180	2.194									

The approximate KS and the corresponding p-value (in parentheses) using parametric bootstrap for censored length 20 mm and censored length 50 mm are 0.7621(0.5567) and 0.9121(0.1900), respectively. These results are displayed in Table 7. Accordingly, it is seen that both censored datasets fit the GG distribution.

Table 7. *Results of the KS test for the real datasets given in study 2.*

MODEL		ML estimates			KS	P-value
GG	X (Length 20mm)	$\hat{\theta} = 9.2343$	$\hat{\lambda} = 0.3536$	$\hat{c} = 0.8088$	0.7621	0.5567
	Y (Length 50mm)	$\hat{\theta} = 6.8118$	$\hat{\lambda} = 0.3536$	$\hat{c} = 0.8088$	0.9121	0.1900

Then, the following ML and approximate Bayes estimates for SSR under PTR-II censoring are acquired. In Table 8, $\hat{R}_{MLE}$ and $\hat{R}_{BAYES}$ are given. Besides, Table 9 the asymptotic and bootstrap confidence intervals for SSR are given (0.4904–0.6746) and (0.4041–0.6454), respectively.

Table 8. *The ML and approximate Bayes estimates for SSR in real data study 2.*

		CS		$\hat{R}_{MLE}$	$\hat{R}_{BAYES}$	
(n, m)	(l, k)	R	S		Prior 1	Prior 2
69,30	65,30	$(29 \times 0, 39)$	$(29 \times 0, 35)$	0.5864	0.5293	0.5201

Table 9. *The asymptotic and bootstrap confidence intervals for SSR in real data study 2.*

		CS		ML estimates			Boot ML estimation		
(n, m)	(l, k)	R	S	$\hat{R}_{MLE}$	Low. Lim	Upp Lim	$\hat{R}_{MLE}$	Low. Lim	Upp Lim
69,30	65,30	$(29 \times 0, 39)$	$(29 \times 0, 35)$	0.5864	0.4904	0.6746	0.5496	0.4041	0.6454

5. Conclusion

In this study, we have acquired the ML and the approximate Bayes estimators of SSR for GG distribution based on PTR-II censored samples. Then we have compared the performance of the approximate Bayes estimators by using Lindley approximation under squared-error loss function with the performance of the ML estimators using Monte Carlo simulation. As a result, MSEs and risk values decrease for both estimators when (n, l) values are kept constant and (m, k) values are increased. As (n, l) values increase, it is seen that the performances of ML and the approximate Bayes estimators converge to 0. As expected, the performances of the estimates in the event of a complete sample case are better than the censoring case. In addition, it is seen that the widths of the asymptotic confidence intervals and Bootstap confidence intervals decrease and the coverage possibilities approach to 0.95 when (n, l) values are kept constant and (m, k) values are increased. Furthermore, the performance of the Bayes estimator is better than the performance of the ML estimator for a small sample size. The existence of many new discrete distributions is known in the literature. For future studies the SSR can be applied to discrete distributions. Because, there is few studies about estimation of SSR for discrete distribution.

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