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MULTIVALENT HARMONIC FUNCTIONS INVOLVING MULTIPLIER TRANSFORMATION

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Abstract. In the present investigation we study a subclass of multivalent harmonic functions involving multiplier transformation. An equivalent convolution class condition and a sufficient coefficient condition for this class is acquired. We also show that this coefficient condition is necessary for functions belonging to its subclass. As an application of coefficient condition, a necessary and sufficient hypergeometric inequality is also given. Further, results on bounds, inclusion relation, extreme points, a convolution property and a result based on the integral operator are obtained.

1. Introduction

A continuous complex-valued function $f = u + iv$ which is defined in a simply-connected domain $\mathbb{D}$ is said to be harmonic in $\mathbb{D}$ if both u and v are real-valued harmonic in $\mathbb{D}$. In any simply-connected domain $\mathbb{D} \subset \mathbb{C}$ we can write $f = h + \overline{g}$, where h and g are analytic in $\mathbb{D}$, where h is called the analytic part and g is called the co-analytic part of f. A necessary and sufficient condition for f to be locally univalent and orientation preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [6]). Let H denote a class of harmonic functions $f = h + \overline{g}$ which are harmonic, univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ so that f is normalized by $f(0) = h(0) = f_z(0) - 1 = 0$.

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It should be worthy to note that the family H reduces to the well known class S of normalized univalent functions if the co-analytic part of f is identically zero, that is if $g = 0$.

The concept of multivalent harmonic complex valued functions by using argument principle, was given by Duren et al. [8]. Using this concept, Ahuja and Jahagiri [1], [2] introduced a class $H(m)$ of m-valent harmonic and orientation preserving functions $f(z) = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are m-valent functions of the form

$$h(z) = z^m + \sum_{n=m+1}^{\infty} a_n z^n, \ g(z) = \sum_{n=m}^{\infty} b_n z^n, |b_m| < 1, m \in \mathbb{N} = \{1, 2, 3....\} \tag{1}$$

which are analytic in $\Delta = \{z : |z| < 1\}$. For $p, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ complex parameters $\alpha_i\,(i = 1, 2, ..., p)$ and $\beta_i(\neq -n, n \in \mathbb{N})$ $(i = 1, 2, .., q)$, the generalized hypergeometric function ${}_pF_q(\alpha_1, ...\alpha_p, \beta_1, ...\beta_q; z) = {}_pF_q\left((\alpha_i); (\beta_i); z\right)$ is defined by

$$ {}_pF_q\left((\alpha_i); (\beta_i); z\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n ... (\alpha_p)_n}{(\beta_1)_n ... (\beta_q)_n\, n!}\ z^n \ \ (p \leq q + 1; z \in \Delta) \tag{2}$$

where $(\lambda)_n$ represents the Pochhammer symbol defined, in terms of Gamma function, by

$$(\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \left\{ \begin{array}{c} 1, \quad n = 0, \lambda \neq 0 \\ \lambda(\lambda + 1)(\lambda + 2) ... (\lambda + n - 1), n \in \mathbb{N} \end{array} \right\}$$

The convolution of two analytic functions $\phi(z) = \sum_{n=o}^{\infty} a_n z^n$ and $\psi(z) = \sum_{n=0}^{\infty} b_n z^n$ defined on Δ is an analytic function given by

$$\phi(z) * \psi(z) = \sum_{n=0}^{\infty} a_n b_n z^n = \psi(z) * \phi(z).$$

For $\alpha_j \in \mathbb{C}$ $(j = 1, 2, \ldots, p)$ and $\beta_j \in \mathbb{C} \backslash \{0, -1, 2, \ldots\}$ $(j = 1, 2, \ldots, q)$, Dziok and Srivastava [9] introduced the following operator for an analytic function $h(z)$ of the form (1) is given by

$$\begin{aligned} H_m^{p,q}[\alpha_1]\, h(z) &= z^m\ {}_pF_q(\alpha_1, ...\alpha_p; \beta_1, ...\beta_q; z) * h(z) && (3) \\ &= z^m + \sum_{n=m+1}^{\infty} \theta_n([\alpha_1]; p, q) a_n z^n && (4) \end{aligned}$$

where

$$\theta_n([\alpha_1];p,q)=\frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{1}{(n-m)!},n\geq m. \tag{5}$$

Several results on harmonic functions by involving generalised hypergeometric functions and involving certain linear operator have recently been studied in [3–5, 11–16, 18, 19, 22]. Motivated with the operator defined by Srivastava et al. in [20], we define a mutiplier operator $\mathcal{L}_{\lambda,p,q}^{t,[\alpha_1]}$ for an analytic function $h(z)$ of the form (1) as follows:

$\mathcal{L}_{\lambda,p,q}^{0,[\alpha_1]}h(z)=h(z)$

$\mathcal{L}_{\lambda,p,q}^{1,[\alpha_1]}h(z)=\mathcal{L}_{\lambda,p,q}^{\alpha_1}h(z)=(1-\lambda)H_m^{p,q}[\alpha_1]h(z)+\frac{\lambda z}{mz'}\left(H_m^{p,q}[\alpha_1]h(z)\right)',(\lambda\geq 0)$

$\mathcal{L}_{\lambda,p,q}^{2,\alpha_1}h(z)=\mathcal{L}_{\lambda,p,q}^{\alpha_1}\left(\mathcal{L}_{\lambda,p,q}^{1,[\alpha_1]}h(z)\right)$

and in general for $t\in\mathbb{N}$,

$\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}h(z)=\mathcal{L}_{\lambda,p,q}^{\alpha_1}\left(\mathcal{L}_{\lambda,p,q}^{t-1,\alpha_1}h(z)\right).$

The series expression is given by

$$\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}h(z)=z^m+\sum_{n=m+1}^{\infty}\theta_n^t(\alpha_1;\lambda;p;q)a_nz^n, \tag{6}$$

where

$$\theta_n^t([\alpha_1];\lambda;p;q)=\left(\frac{\prod_{i=1}^{p}(\alpha_i)_{n-m}}{\prod_{i=1}^{q}(\beta_i)_{n-m}}\frac{[m+\lambda(n-m)]}{m(n-m)!}\right)^t,(n\in\mathbb{N},n\geq m,t\in\mathbb{N}_0). \tag{7}$$

Similarly for the analytic function $g(z)$ given in (1),

$$\mathcal{L}_{\lambda,r,s}^{t,\gamma_1}g(z)=\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)b_nz^n \tag{8}$$

where

$$\phi_n^t([\gamma_1];\lambda;r;s)=\left(\frac{\prod_{i=1}^{r}(\gamma_i)_{n-m+1}}{\prod_{i=1}^{s}(\delta_i)_{n-m+1}}\frac{[m+\lambda(n-m)]}{m(n-m+1)!}\right)^t,(n\in\mathbb{N},n\geq m,t\in\mathbb{N}_0). \tag{9}$$

We note that when $t=1$ and $\lambda=0$ the linear operator $\mathcal{L}_{\lambda,p,q}^{t,\alpha_1}$ would reduce to the operator $H_m^{p,q}[\alpha_1]$ which includes (as its special cases) various other linear operators introduced and studied by Hohlov et al. [7], Owa [17] and Ruscheweyh [21].

Now, for $f=h+\overline{g}\in H(m)$ (where $h(z)$ and $g(z)$ are of the form (1)), in terms of the operators defined in (6) and (8) we defined a linear operator $\pounds_{\lambda,r,s}^{t,p,q}\left([\alpha_1];[\gamma_1]\right):=\mathcal{I}:H\left(m\right)\rightarrow H\left(m\right)$ by

$$\mathcal{I}f(z)=\pounds_{\lambda,p,q}^{t,\alpha_1}h(z)+\overline{\pounds_{\lambda,r,s}^{t,\gamma_1}g(z)}. \tag{10}$$

For the purpose of this paper, on applying the linear operator $\mathcal{I}f(z)$, motivated with the class defined in [10] we define a class $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ of functions $f\in H\left(m\right)$ if it satisfy the condition

$$\Re\left\{(1-\lambda)\frac{\mathcal{I}f(z)}{z^m}+\lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'}+\lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}\right\}>\frac{\beta}{m} \tag{11}$$

where $\lambda\geq 0, 0\leq k\leq 1,\ 0\leq\beta<m$ and $z=re^{i\theta}\left(r<1,\theta\in\mathbb{R}\right),\ z'=\frac{\partial}{\partial\theta}\left(z\right),z''=\frac{\partial^2}{\partial\theta^2}\left(z\right),\left(\mathcal{I}f(z)\right)'=\frac{\partial}{\partial\theta}\left(\mathcal{I}f(z)\right)$ and $\left(\mathcal{I}f(z)\right)''=\frac{\partial^2}{\partial\theta^2}\left(\mathcal{I}f(z)\right).$

Based on some particular values of λ and k, we denote following classes:

(1) for $\lambda=0$, $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;0,k\right)=A_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$
(2) for $\lambda=1$, $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;1,k\right)=B_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$
(3) for $k=0$, $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,0\right)=C_{m,}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$
(4) for $k=1$, $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,1\right)=D_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$
(5) for $\lambda=1$ and $k=0$ $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;1,0\right)=E_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta\right)$
(6) for $\lambda=1$ and $k=1$ $R_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;1,1\right)=F_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta\right)$

Let $\widetilde{H}\left(m\right)$ be a subclass of $H(m)$ whose members $f=h+\overline{g}$ are such that, h and g are of the form

$$h(z)=z^m-\sum_{n=m+1}^{\infty}\left|a_n\right|z^n,\ g(z)=\sum_{n=m}^{\infty}\left|b_n\right|z^n,\left|b_m\right|<1. \tag{12}$$

We further denote $\widetilde{R}_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)=R_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)\cap\widetilde{H}\left(m\right).$

In this paper, an equivalent convolution class condition is derived and a coefficient inequality is obtained for the functions $f=h+\overline{g}\in H(m)$ to be in the class $R_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$. It is also proved that this inequality is necessary for $f=h+\overline{g}$ to be in $\widetilde{R}_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$ class. As an application of coefficient inequality a necessary and sufficient hypergeometric inequality is also given. Further, based on the coefficient inequality, results on bounds,inclusion relations, extreme points, convolution and convex combination and on an integral operator are obtained.

Throughout in this paper, we consider that the parameters involved in the operator $\pounds_{\lambda,r,s}^{t,p,q}\left[m,[\alpha_1];[\gamma_1]\right]$ such as $\alpha_i\left(i=1,2,...,p\right),\gamma_i\left(i=1,2,...,r\right),\beta_i\left(i=1,2,..,q\right),$

δ_i $(i = 1, 2, .., s)$, are positive real and $\theta_n^t(\alpha_1; \lambda; p; q)$, $\phi_n^t(\gamma_1; \lambda; r; s)$ given by (7), (9) are bounded with $\theta_n^t(\alpha_1; \lambda; p; q) \geq \frac{n}{m}$, $\phi_n^t(\gamma_1; \lambda; r; s) \geq \frac{n}{m}$ $(n \geq m)$.

2. Coefficient Inequality

Theorem 1. *Let* $\lambda \geq 0, 0 \leq k \leq 1$, $0 < \beta \leq m, m \in \mathbb{N}$. *If the function* $f = h + \overline{g} \in H(m)$ *(where* h *and* g *are of the form (1)) satisfies*

$$\sum_{n=m+1}^{\infty} \frac{\left|m^2 + \lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t(\alpha_1; \lambda; p; q)\,|a_n| + \sum_{n=m}^{\infty} \frac{\left|m^2 + \lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t(\gamma_1; \lambda; r; s)\,|b_n| \leq 1, \tag{13}$$

then f *is sense-preserving, harmonic multivalent in* Δ *and* $f \in R_{m,k}^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda\right)$.

Proof. Under the given parametric constraints, we have

$$\frac{n}{m} \leq \frac{\left|m^2 + \lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n \text{ and } \frac{n}{m} \leq \frac{\left|m^2 + \lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n, n \geq m. \tag{14}$$

Thus, for $f = h + \overline{g} \in H(m)$, where h and g are of the form (1), we get

$$\begin{aligned} |h'(z)| &\geq m\,|z|^{m-1} - \sum_{n=m+1}^{\infty} n\,|a_n|\,|z|^{n-1} \geq m\,|z|^{m-1}\left[1 - \sum_{n=m+1}^{\infty} \frac{n}{m}\,|a_n|\right] \\ &\geq m\,|z|^{m-1}\left[1 - \sum_{n=m+1}^{\infty} \frac{\left|m^2 + \lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n\,|a_n|\right] \\ &\geq m\,|z|^{m-1}\left[\sum_{n=m}^{\infty} \frac{\left|m^2 + \lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n\,|b_n|\right] > \sum_{n=m}^{\infty} n\,|b_n|\,|z|^{n-1} \\ &\geq |g'(z)| \end{aligned}$$

which proves that $f(z)$ is sense preserving in Δ. Now to show that $f \in R_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, we need to show (11), that is

$$\Re\left\{(1-\lambda)\,\frac{\mathcal{I}f(z)}{z^m} + \lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'} + \lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}\right\} > \frac{\beta}{m}, z \in \Delta. \tag{15}$$

Suppose

$$A(z) = (1-\lambda)\,\frac{\mathcal{I}f(z)}{z^m} + \lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'} + \lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}.$$

It is suffices to show that

$$\left|\frac{A(z) - 1}{A(z) - \frac{2\beta}{m} + 1}\right| < 1.$$

Series expansion of $A(z)$ is given by

$$\begin{aligned} A(z) &= 1+\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left\{1+\lambda(\frac{n}{m}-1)(\frac{kn}{m}+1\right\}a_n z^{n-m}+ \\ &\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left\{1+\lambda(\frac{n}{m}+1)(\frac{kn}{m}-1\right\}b_n\overline{z}^n z^{-m} \end{aligned}$$

and we have

$$\left|A(z)-\frac{2\beta}{m}+1\right|-|A(z)-1|$$

$$\begin{aligned} &= \left|2(1-\tfrac{\beta}{m})+\textstyle\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left\{1+\lambda(\tfrac{n}{m}-1)(\tfrac{kn}{m}+1\right\}a_n z^{n-m}\right. \\ &\quad \left.+\textstyle\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left\{1+\lambda(\tfrac{n}{m}+1)(\tfrac{kn}{m}-1)\right\}b_n\overline{z}^n z^{-m}\right| \\ &\quad - \left|\textstyle\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left\{1+\lambda(\tfrac{n}{m}-1)(\tfrac{kn}{m}+1\right\}a_n z^{n-m}\right. \\ &\quad \left.+\textstyle\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left\{1+\lambda(\tfrac{n}{m}+1)(\tfrac{kn}{m}-1\right\}b_n\overline{z}^n z^{-m}\right| \\ &\geq \frac{1}{m}\left[2(m-\beta)-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|m+\lambda(n-m)(\frac{kn}{m}+1)\right||a_n|\left|z^{n-m}\right|\right. \\ &\quad -\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|m+\lambda(n+m)(\frac{kn}{m}-1)\right||b_n||\overline{z}^n|\left|z^{-m}\right|- \\ &\quad \sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|m+\lambda(n-m)(\frac{kn}{m}+1)\right||a_n|\left|z^{n-m}\right| \\ &\quad \left.-\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|m+\lambda(n+m)(\frac{kn}{m}-1)\right||b_n||\overline{z}^n|\left|z^{-m}\right|\right] \\ &= \frac{1}{m}\left[2(m-\beta)-2\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|m+\lambda(n-m)(\frac{kn}{m}+1\right||a_n|\left|z^{n-m}\right|\right. \\ &\quad \left.-2\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|m+\lambda(n+m)(\frac{kn}{m}-1)\right||b_n||\overline{z}^n|\left|z^{-m}\right|\right] \\ &\geq 0 \end{aligned}$$

by (13) when $z=r\to 1$ and this proves Theorem 1. □

In our next result we show that the above sufficient coefficient condition is also necessary for functions in the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$.

Theorem 2. *Let* $\lambda\geq 0, 0\leq k\leq 1,\ 0<\beta\leq m, m\in\mathbb{N}$.*and let the function* $f=h+\overline{g}\in\widetilde{H}(m)$ *be such that* h *and* g *are given by (12). Then* $f\in$

$\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ *if and only if (13) holds. The inequality (13) is sharp for the function given by*

$$f(z) = z^m - \sum_{n=m+1}^{\infty}\frac{m(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1];\lambda;p;q)}|x_n|\,z^n \quad (16)$$
$$+\sum_{n=m}^{\infty}\frac{m(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1];\lambda;r;s)}|y_n|\,\overline{z^n},$$

$\sum_{n=m+1}^{\infty}|x_n|+\sum_{n=m}^{\infty}|y_n|=1.$

Proof. The if part, follows from Theorem 1. To prove the "only if part" let $f=h+\overline{g}\in\widetilde{H}(m)$ be such that h and g are given by (12) and $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$, then for $z=re^{i\theta}$ in Δ we obtain

$$\Re\left\{(1-\lambda)\frac{\mathcal{I}f(z)}{z^m}+\lambda(1-k)\frac{(\mathcal{I}f(z))'}{(z^m)'}+\lambda k\frac{(\mathcal{I}f(z))''}{(z^m)''}\right\}>\frac{\beta}{m}$$

$$= \Re\left\{(1-\lambda)\frac{\pounds_{\lambda,p,q}^{t,\alpha_1}h(z)+\overline{\pounds_{\lambda,r,s}^{t,\gamma_1}g(z)}}{z^m}+\lambda(1-k)\frac{z\left(\pounds_{\lambda,p,q}^{t,\alpha_1}h(z)\right)'-z\left(\overline{\pounds_{\lambda,r,s}^{t,\gamma_1}g(z)}\right)'}{mz^m}\right\}$$
$$+ \Re\left\{\lambda k\frac{z^2\left(\pounds_{\lambda,p,q}^{t,\alpha_1}h(z)\right)''+z\left(\pounds_{\lambda,p,q}^{t,\alpha_1}h(z)\right)'+z^2\left(\overline{\pounds_{\lambda,r,s}^{t,\gamma_1}g(z)}\right)''+z\left(\overline{\pounds_{\lambda,r,s}^{t,\gamma_1}g(z)}\right)'}{m^2z^m}\right\}$$

$$\geq 1-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\left|1+\lambda(\frac{n}{m}-1)(\frac{kn}{m}+1\right||a_n|\left|z^{n-m}\right|-$$
$$\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\left|1+\lambda(\frac{n}{m}+1)(\frac{kn}{m}-1)\right|b_n\left|\overline{z}^n\right|\left|z^{-m}\right|$$
$$> \frac{\beta}{m}.$$

The above inequality must hold for all $z\in\Delta$. in particular $z=r\to 1$ yields the required condition (13). Sharpness of the result can easily be verified for the function given by
(16). □

Corollary 1. $f\in\widetilde{A}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{m}{(m-\beta)}\theta_n^t([\alpha_1];p;q)\,|a_n|+$$

$$\sum_{n=m}^{\infty}\frac{m}{(m-\beta)}\phi_n^t([\gamma_1];r;s)\,|b_n|\leq 1$$

holds.

Corollary 2. $f\in\widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];p;q)\,|a_n|+$$
$$\sum_{n=m}^{\infty}\frac{\left|m^2+(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];r;s)\,|b_n|\leq 1$$

holds.

Corollary 3. $f\in\widetilde{C}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)m\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];\lambda;p;q)\,|a_n|+$$
$$\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)m\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\leq 1$$

holds.

Corollary 4. $f\in\widetilde{D}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n^2-m^2)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];\lambda;p;q)\,|a_n|+$$
$$\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n^2-m^2)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\leq 1$$

holds.

Corollary 5. $f\in\widetilde{E}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+(n-m)m\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];p;q)\,|a_n|+$$
$$\sum_{n=m}^{\infty}\frac{\left|m^2+(n+m)m\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];r;s)\,|b_n|\leq 1$$

holds.

Corollary 6. $f\in\widetilde{F}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\right)$ *if and only if*

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+(n^2-m^2)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];p;q)\,|a_n|+$$

$$\sum_{n=m}^{\infty} \frac{\left|m^2+(n^2-m^2)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1]\,;r;s)\,|b_n| \leq 1$$

holds.

On applying coefficient inequality (13), we get a sufficient condition in the form of hypergeometric inequality for certain function $f = h + \overline{g} \in H(m)$ to be in $R_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ class and it is proved that this inequality is necessary for certain $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$.

Corollary 7. *Let $\lambda \geq 0, 0 \leq k \leq 1$, $0 < \beta \leq m, m \in \mathbb{N}$, and let the function $f = h + \overline{g} \in H(m)$ where h and g are of the form (1) be such that*

$$|a_n| \leq \frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}, n \geq m+1 \tag{17}$$

and (18)

$$|b_n| \leq \frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}, n \geq m. \tag{19}$$

If (in case $p = q+1$) $\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i > 0$ and (in case $r = s+1$) $\sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i > 0$,
the hypergeometric inequality

$$\left[\left[{}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right)-1\right]\frac{(m+\lambda(n-m))}{m}\right]^t + \left[{}_rF_s\left((\gamma_i)\,;(\delta_i)\,;1\right)\frac{(m+\lambda(n-m))}{m}\right]^t \leq 1 \tag{20}$$

holds, then $f \in R_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$. Further, if

$$\begin{aligned} f(z) &= z^m - \sum_{n=m+1}^{\infty} \frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}\; z^n \\ &\quad + \sum_{n=m}^{\infty} \frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}\overline{z^n} \\ &\in \widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right), \end{aligned} \tag{21}$$

then (20) holds.

Proof. To prove the result, we need to show by Theorem 1 the inequality:

$$S_1 \quad : \quad = \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n|$$

$$+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n|$$
$$\leq \quad 1.$$

By (17) and (19), we get by (20),

$$\begin{aligned} S_1 &\leq \sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1]\,;\lambda;p;q)+\sum_{n=m}^{\infty}\phi_n^t([\gamma_1]\,;\lambda;r;s) \\ &= \left[\left[{}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right)-1\right]\frac{(m+\lambda(n-m))}{m}\right]^t \\ &+\left[{}_rF_s\left((\gamma_i)\,;(\delta_i)\,;1\right)\frac{(m+\lambda(n-m))}{m}\right]^t\leq 1 \end{aligned}$$

where, under the given conditions

$$\begin{aligned} \sum_{n=m+1}^{\infty}\theta_n^T([\alpha_1]\,;\lambda;p;q) &= \left[\left(\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}(\alpha_i)_n}{\prod_{i=1}^{q}(\beta_i)_n}\frac{1}{n!}-1\right)\frac{(m+\lambda(n-m))}{m}\right]^t \\ &= \left[\left[{}_pF_q\left((\alpha_i)\,;(\beta_i)\,;1\right)-1\right]\frac{(m+\lambda(n-m))}{m}\right]^t. \end{aligned}$$

Similarly

$$\begin{aligned} \sum_{n=m}^{\infty}\phi_n^T([\gamma_1]\,;\lambda;r;s) &= \left[\sum_{n=0}^{\infty}\frac{\prod_{i=1}^{r}(\gamma_i)_n}{\prod_{i=1}^{s}(\delta_i)_n}\frac{1}{n!}\frac{(m+\lambda(n-m))}{m}\right]^t \\ &= \left[{}_rF_s\left((\gamma_i)\,;(\delta_i)\,;1\right)\frac{(m+\lambda(n-m))}{m}\right]^t. \end{aligned}$$

Further, (20) holds by Theorem 2, if $f(z)$ of the form (21) belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$. This proves the result. □

In particular if we take $\lambda=0, t=1$ we get the following hypergeometric inequality which is sufficient for certain function $f=h+\overline{g}\in H(m)$ to be in $R_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ class, and this inequality is necessary for certain $f\in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$.

Corollary 8. *Let $\lambda\geq 0, 0\leq k\leq 1$, $0<\beta\leq m, m\in\mathbb{N}$, and let the function $f=h+\overline{g}\in H(m)$ where h and g are of the form (1) be such that*

$$|a_n| \quad\leq\quad \frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}, n\geq m+1 \tag{22}$$

$$|b_n| \leq \frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}, n \geq m. \tag{23}$$

If (in case $p=q+1$) $\sum_{i=1}^{q}\beta_i - \sum_{i=1}^{p}\alpha_i > 0$ and (in case $r=s+1$) $\sum_{i=1}^{s}\delta_i - \sum_{i=1}^{r}\gamma_i > 0$, the hypergeometric inequality

$$ {}_pF_q\left((\alpha_i);(\beta_i);1\right) +_r F_s\left((\gamma_i);(\delta_i);1\right) \leq 2 \tag{24}$$

holds, then $f \in R_{m,k}^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda\right)$. Further, if

$$\begin{aligned} f(z) &= z^m - \sum_{n=m+1}^{\infty}\frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|}\; z^n \\ &+ \sum_{n=m}^{\infty}\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|}\overline{z^n} \\ &\in \widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right), \end{aligned} \tag{25}$$

then (20) holds.

3. Inclusion Relation

The inclusion relations between the classes $\widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$ and $\widetilde{A}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$ for different values of λ In this section inclusion relation between the classes and for different values of $\widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$ $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$

Theorem 3. *for $n \in \{1,2,3..\}$ and $0 \leq \beta < m$, we have*

(1) $\widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right) \subset \widetilde{A}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$
(2) $\widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right) \subset \widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right), 0 \leq \lambda \leq 1$
(3) $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right) \subset \widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right), \lambda \geq 1.$

Proof. (i) Let $f(z) \in \widetilde{B}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta,k\right)$. in view of corollaries 1 and 2, we have

$$\begin{aligned} &\sum_{n=m+1}^{\infty}\frac{m}{(m-\beta)}\theta_n^t([\alpha_1];p;q)\,|a_n| + \sum_{n=m}^{\infty}\frac{m}{(m-\beta)}\phi_n^t([\gamma_1];r;s)\,|b_n| \\ &\leq \sum_{n=m+1}^{\infty}\frac{\left|m^2+(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];p;q)\,|a_n| + \\ &\sum_{n=m}^{\infty}\frac{\left|m^2+(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];r;s)\,|b_n| \leq 1 \end{aligned}$$

(ii)Let $f(z) \in \widetilde{B}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta, k\right)$.For $0 \leq \lambda \leq 1$,we can write

$$\begin{aligned}
& \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; \lambda; p; q)\,|a_n| + \\
& \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; \lambda; r; s)\,|b_n| \\
\leq \quad & \sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; p; q)\,|a_n| + \\
& \sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; r; s)\,|b_n| \\
\leq \quad & 1
\end{aligned}$$

by corollary 2 and (ii) follows from Theorem 2.

(iii) By the Theorem 2, if $\lambda \geq 1$,we have

$$\begin{aligned}
& \sum_{n=m+1}^{\infty} \frac{\left|m^2+(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; p; q)\,|a_n| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; r; s)\,|b_n| \\
\leq \quad & \sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m(m-\beta)} \theta_n^t([\alpha_1]; \lambda; p; q)\,|a_n| \\
& + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m(m-\beta)} \phi_n^t([\gamma_1]; \lambda; r; s)\,|b_n| \\
\leq \quad & 1.
\end{aligned}$$

Therefore the result follows from corollary 2. □

4. Bounds

Our next theorems provide the bounds for the function in the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ which are followed by a covering result for this class.

Theorem 4. *Let $\lambda \geq 0, 0 \leq k \leq 1$, $0 < \beta \leq m, m \in \mathbb{N}$. if $f = h+\overline{g} \in \widetilde{H}(m)$, where h and g are of the form (12) belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then for $|z| = r < 1$,*

$$|\mathcal{I}f(z)| \leq (1+|b_m|)\,r^m + \frac{mr^{m+1}}{m+1}\left(1 - \frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right), \tag{26}$$

and

$$|\mathcal{I}f(z)| \geq (1-|b_m|)\,r^m - \frac{m}{m+1}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right)r^{m+1}. \qquad (27)$$

The result is sharp.

Proof. Let $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then on using (13), related to (12), by (10), we get for $|z| = r < 1$,

$$\begin{aligned}
&|\mathcal{I}f(z)| \\
\leq\ & (1+|b_m|)\,r^m + \sum_{n=m+1}^{\infty}\left(\theta^t([\alpha_1];\lambda;p;q)\,|a_n| + \phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\right)r^n \\
\leq\ & (1+|b_m|)\,r^m + r^{m+1}\sum_{n=m+1}^{\infty}\left(\theta_n^t([\alpha_1];\lambda;p;q)\,|a_n| + \phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\right) \\
\leq\ & (1+|b_m|)\,r^m + \frac{mr^{m+1}}{m+1}\left(\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];\lambda;p;q)\,|a_n| + \right. \\
& \left.\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\right) \\
\leq\ & (1+|b_m|)\,r^m + \frac{mr^{m+1}}{m+1}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right)
\end{aligned}$$

which proves the result (26). The result (27) can similarly be obtained. The bounds (26) and (27) are sharp for the function given by

$$f(z) = z^m + |b_m|\,\overline{z^m} + \frac{m}{(m+1)\,\phi_{m+1}^t([\gamma_1];\lambda;r;s)}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}\,|b_m|\right)\overline{z^{m+1}}$$

for $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m,\ |b_m| < \frac{1-\frac{\beta}{m}}{1+2\lambda(k-1)}$. □

Corollary 9. *Let* $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m, m \in \mathbb{N}$. *If* $f = h + \overline{g} \in \widetilde{H}(m)$ *with* h *and* g *are of the form (12) belongs to the class* $\widetilde{R}_m^t\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, *then*

$$\left\{\omega : |\omega| < 1 - \frac{m}{m+1} + \left(\frac{m\,(1+2\lambda(k-1))}{(m+1)\left(1-\frac{\beta}{m}\right)} - 1\right)|b_m|\right\} \subset f(\Delta).$$

Theorem 5. *Let* $\lambda \geq 0, 0 \leq k \leq 1,\ 0 < \beta \leq m, m \in \mathbb{N}$ *and let*

$$\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s) \leq \min\left(\theta_n^t([\alpha_1];\lambda;p;q), \phi_n^t([\gamma_1];\lambda;r;s)\right), n \geq m+1$$

If $f=h+\overline{g}\in\widetilde{H}(m)$, *where* h *and* g *are of the form (12), belongs to the class* $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$, *then for* $|z|=r<1$,

$$|f(z)|\le(1+|b_m|)\,r^m+\frac{m}{(m+1)\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}|b_m|\right)r^{m+1}, \tag{28}$$

and

$$|f(z)|\ge(1-|b_m|)\,r^m-\frac{m}{(m+1)\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}|b_m|\right)r^{m+1}. \tag{29}$$

The result is sharp.

Proof. Let $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$, then on using (13), from (12), we get for $|z|=r<1$,

$$\begin{aligned}
|f(z)| &\le (1+|b_m|)\,r^m+\sum_{n=m+1}^{\infty}(|a_n|+|b_n|)\,r^n\\
&\le (1+|b_m|)\,r^m+r^{m+1}\sum_{n=m+1}^{\infty}(|a_n|+|b_n|)\\
&\le (1+|b_m|)\,r^m+\frac{r^{m+1}}{\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)}\\
&\times \sum_{n=m+1}^{\infty}\left(\theta_n^t([\alpha_1];\lambda;p;q)\,|a_n|+\phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\right)\\
&\le (1+|b_m|)\,r^m+\frac{mr^{m+1}}{(m+1)\,\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)}\\
&\quad\left(\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];\lambda;p;q)\,|a_n|+\right.\\
&\quad\left.\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];\lambda;r;s)\,|b_n|\right)\\
&\le (1+|b_m|)\,r^m\\
&+ \frac{mr^{m+1}}{(m+1)\,\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}|b_m|\right)r^{m+1},
\end{aligned}$$

which proves (28). The result (29) can similarly be obtained. The bounds (28) and (29) are sharp for the function given by

$$f(z)=z^m+|b_m|\,\overline{z^m}+\frac{mr^{m+1}}{(m+1)\,\delta_{m+1}^t([\alpha_1];[\gamma_1];\lambda;p;q;r;s)}\left(1-\frac{1+2\lambda(k-1)}{1-\frac{\beta}{m}}|b_m|\right)\overline{z^{m+1}}$$

for $|b_m| < \frac{1-\frac{\beta}{m}}{1+2\lambda(k-1)}$. □

Corollary 10. *Let $\lambda \geq 0, 0 \leq k \leq 1$, $0 < \beta \leq m, m \in \mathbb{N}$ and let $\delta^t_{m+1}([\alpha_1];[\gamma_1];\lambda;p;q;r;s) \leq \min\left(\theta^t_n([\alpha_1];\lambda;p;q), \phi^t_n([\gamma_1];\lambda;r;s)\right)$, $n \geq m+1$. If $f = h + \overline{g} \in \widetilde{H}(m)$ with h and g are of the form (12) belongs to the class $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$, then*

$$\left\{\begin{array}{c}\omega : |\omega| < 1 - \frac{m}{(m+1)\delta^t_{m+1}([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} + \\ \left(\frac{m(1+2\lambda(k-1))}{(m+1)\left(1-\frac{\beta}{m}\right)\delta^t_{m+1}([\alpha_1];[\gamma_1];\lambda;p;q;r;s)} - 1\right)|b_m|\end{array}\right\} \subset f(\Delta).$$

5. Extreme Points

In this section, we determine the extreme points for the class $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$.

Theorem 6. *let $f = h + \overline{g} \in \widetilde{H}(m)$ and*

$$\begin{aligned} h_m(z) &= z^m, \\ h_n(z) &= z^m - \frac{m(m-\beta)}{|m^2 + \lambda(n-m)(kn+m)|\,\theta^t_n([\alpha_1];\lambda;p;q)} z^n \ \ (n \geq m+1), \\ g_n(z) &= z^m + \frac{m(m-\beta)}{|m^2 + \lambda(n+m)(kn-m)|\,\phi^t_n([\gamma_1];\lambda;r;s)} \overline{z^n} \ \ (n \geq m), \end{aligned}$$

then the function $f \in \widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ if and only if it can be expressed as $f(z) = \sum_{n=m}^{\infty}(x_n h_n(z) + y_n g_n(z))$ where $x_n \geq 0, y_n \geq 0$ and $\sum_{n=m}^{\infty}(x_n + y_n) = 1$. In particular, the extreme points of $\widetilde{R}^t_m\left([\alpha_1]_{p,q}, [\gamma_1]_{r,s}; \beta; \lambda, k\right)$ are $\{h_n\}$ and $\{g_n\}$.

Proof. Suppose that

$$f(z) = \sum_{n=m}^{\infty}(x_n h_n(z) + y_n g_n(z))$$

Then,

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty}(x_n + y_n) z^m - \sum_{n=m+1}^{\infty} \frac{m(m-\beta)}{|m^2 + \lambda(n-m)(kn+m)|\,\theta^t_n([\alpha_1];\lambda;p;q)} x_n z^n \\ &\quad + \sum_{n=m}^{\infty} \frac{m(m-\beta)}{|m^2 + \lambda(n+m)(kn-m)|\,\phi^t_n([\gamma_1];\lambda;r;s)} y_n \overline{z^n} \\ &= z^m - \sum_{n=m+1}^{\infty} \frac{m(m-\beta)}{|m^2 + \lambda(n-m)(kn+m)|\,\theta^t_n([\alpha_1];\lambda;p;q)} x_n z^n \end{aligned}$$

$$+\sum_{n=m}^{\infty}\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}y_n\overline{z^n}$$
$$\in\quad \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$$

by Theorem 2, since,

$$\begin{aligned}
&\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\\
&\times\left(\frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1]\,;\lambda;p;q)}x_n\right)\\
&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\\
&\times\left(\frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}y_n\right)\\
=\;&\sum_{n=m+1}^{\infty}x_n+\sum_{n=m}^{\infty}y_n=1-x_m\le 1.
\end{aligned}$$

Conversely, let $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ and let

$$|a_n|=\frac{m\,(m-\beta)\,x_n}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1]\,;\lambda;p;q)}$$

and

$$|b_n|=\frac{m\,(m-\beta)\,y_n}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}$$

and

$$x_m=1-\sum_{n=m+1}^{\infty}x_n-\sum_{n=m}^{\infty}y_n,$$

then, we get

$$\begin{aligned}
f\,(z)\;&=\;z^m-\sum_{n=m+1}^{\infty}|a_n|\,z^n+\sum_{n=m}^{\infty}|b_n|\,\overline{z}^n\\
&=\;h_m(z)-\sum_{n=m+1}^{\infty}\frac{m\,(m-\beta)\,x_n}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1]\,;\lambda;p;q)}x_nz^n\\
&\quad+\sum_{n=m}^{\infty}\frac{m\,(m-\beta)\,y_n}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1]\,;\lambda;r;s)}y_n\overline{z^n}\\
&=\;h_m(z)+\sum_{n=m+1}^{\infty}\left(h_n(z)-h_m(z)\right)x_n+\sum_{n=m}^{\infty}\left(g_n(z)-h_m(z)\right)y_n
\end{aligned}$$

$$= h_m(z)\left(1-\sum_{n=m+1}^{\infty}x_n-\sum_{n=m}^{\infty}y_n\right)+\sum_{n=m+1}^{\infty}h_n(z)x_n+\sum_{n=m}^{\infty}g_n(z)y_n$$
$$= \sum_{n=m}^{\infty}\left(x_nh_n(z)+y_ng_n(z)\right).$$

This proves the Theorem 6. □

6. Convolution and Convex Combinations

In this section, we show that the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ is invariant under convolution and convex combinations of its members.

Let the function $f=h+\overline{g}\in\widetilde{H}(m)$ where h and g are of the form (12) and

$$F(z)=z^m-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\,|A_n|\,z^n+\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\,|B_n|\,\overline{z^n}\in\widetilde{H}(m).\tag{30}$$

The convolution between the functions of the class $\widetilde{H}(m)$ is defined by

$(f*F)(z)=f(z)*F(z)=z^m-\sum_{n=m+1}^{\infty}\theta_n^t([\alpha_1];\lambda;p;q)\,|a_nA_n|\,z^n+\sum_{n=m}^{\infty}\phi_n^t([\gamma_1];\lambda;r;s)\,|b_nB_n|\,\overline{z^n}$

Theorem 7. *Let* $\lambda\geq 0, 0\leq k\leq 1$, $0<\beta\leq m, m\in\mathbb{N}$, *if* $f\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ *and* $F\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$, *then* $f*F\in\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$.

Proof. Let $f=h+\overline{g}\in\widetilde{H}(m)$,where h and g are of the form (12) and $F\in\widetilde{H}(m)$ of the form (30) be in $\widetilde{R}_m^t\left([\alpha_1]_{p,q},[\gamma_1]_{r,s};\beta;\lambda,k\right)$ class. Then by theorem (2), we have

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];\lambda;p;q)\,|A_n|$$
$$+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1];\lambda;r;s)\,|B_n|$$
$$\leq 1$$

which in view of (14), yields

$$|A_n| \leq \frac{m\,(m-\beta)}{|m^2+\lambda(n-m)(kn+m)|\,\theta_n^t([\alpha_1];\lambda;p;q)}\leq\frac{m}{n}\leq 1, n\geq m+1$$
$$|B_n| \leq \frac{m\,(m-\beta)}{|m^2+\lambda(n+m)(kn-m)|\,\phi_n^t([\gamma_1];\lambda;r;s)}\leq\frac{m}{n}\leq 1, n\geq m.$$

Hence, by Theorem 2,

$$\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1];\lambda;p;q)\,|a_nA_n|$$

$$
\begin{aligned}
&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_nB_n| \\
\le\ &\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n| \\
&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n| \\
\le\ & 1
\end{aligned}
$$

which proves that $f*F\in\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$. □

We prove next that the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ is closed under convex combination.

Theorem 8. *Let $\lambda\ge 0, 0\le k\le 1,\ 0<\beta\le m, m\in\mathbb{N}$, the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ is closed under convex combination.*

Proof. Let $f_j\in\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$, $j\in\mathbb{N}$ be of the form

$$
f_j\,(z)=z^m-\sum_{n=m+1}^{\infty}|A_{j,n}|\,z^n+\sum_{n=m}^{\infty}|B_{j,n}|\,\overline{z^n}, j\in\mathbb{N}.
$$

Then by Theorem 2, we have for $j\in\mathbb{N}$,

$$
\begin{aligned}
&\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^T([\alpha_1]\,;\lambda;p;q)\,|A_{j,n}| \qquad (31)\\
&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^T([\gamma_1]\,;\lambda;r;s)\,|B_{j,n}| \\
\le\ & 1.
\end{aligned}
$$

For some $0\le t_j\le 1$, let $\sum_{j=1}^{\infty}t_j=1$, the convex combination of $f_j\,(z)$ may be written as

$$
\sum_{j=1}^{\infty}t_j\ f_j\,(z)=z^m-\sum_{n=m+1}^{\infty}\sum_{j=1}^{\infty}t_j\,|A_{j,n}|\,z^n+\sum_{n=m}^{\infty}\sum_{j=1}^{\infty}t_j\,|B_{j,n}|\,\overline{z^n}
$$

Now by (31),

$$
\begin{aligned}
&\sum_{n=m+1}^{\infty}\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\sum_{j=1}^{\infty}t_j\,|A_{j,n}| \\
&+\sum_{n=m}^{\infty}\frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\sum_{j=1}^{\infty}t_j\,|B_{j,n}|
\end{aligned}
$$

$$= \sum_{j=1}^{\infty} t_j \left[\frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)} \theta_n^t([\alpha_1]\,;\lambda;p;q)\,|A_{j,n}| + \sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)} \phi_n^t([\gamma_1]\,;\lambda;r;s)\,|B_{j,n}| \right] \le \sum_{j=1}^{\infty} t_j = 1$$

and so again by Theorem 2, we get $\sum_{j=1}^{\infty} t_j\ f_j\,(z) \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$. This proves the result. □

7. Integral Operator

In this section, we study a closure property of the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$ involving the generalized Bernardi Libera-Livingston Integral operator $L_{m,c}$ which is given for $f = h + \overline{g} \in \widetilde{H}\,(m)$ by

$$L_{m,c}(f) = \frac{c+m}{z^c}\int_0^z t^{c-1}h(t)dt + \overline{\frac{c+m}{z^c}\int_0^z t^{c-1}g(t)dt}, \quad c > -m, z \in \Delta. \tag{32}$$

Theorem 9. *Let* $\lambda \ge 0, 0 \le k \le 1, 0 < \beta \le m, m \in \mathbb{N}$, *if* $f \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$, *then* $L_{m,c}(f) \in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$.

Proof. Let $f = h + \overline{g} \in \widetilde{H}\,(m)$,where h and g are of the form (12), belongs to the class $\widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right)$. Then, it follows from (32) that

$$\begin{aligned} L_{m,c}(f) &= z^m - \sum_{n=m+1}^{\infty}\left(\frac{c+m}{c+n}\right)|a_n|\,z^n + \sum_{n=m}^{\infty}\left(\frac{c+m}{c+n}\right)|b_n|\,\overline{z^n} \\ &\in \widetilde{R}_m^t\left([\alpha_1]_{p,q}\,,[\gamma_1]_{r,s}\,;\beta;\lambda,k\right) \end{aligned}$$

by (13), since,

$$\begin{aligned} &\sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\left(\frac{c+m}{c+n}\right)\theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n| + \\ &\sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\left(\frac{c+m}{c+n}\right)\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n| \\ \le\ &\sum_{n=m+1}^{\infty} \frac{\left|m^2+\lambda(n-m)(kn+m)\right|}{m\,(m-\beta)}\theta_n^t([\alpha_1]\,;\lambda;p;q)\,|a_n| \\ &+\sum_{n=m}^{\infty} \frac{\left|m^2+\lambda(n+m)(kn-m)\right|}{m\,(m-\beta)}\phi_n^t([\gamma_1]\,;\lambda;r;s)\,|b_n| \\ \le\ & 1. \end{aligned}$$

This proves the result. □

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References

[1] Ahuja, O. P., Jahangiri, J. M., Multivalent harmonic starlike functions, *Ann. Univ. Mariac Curie-Sklodowska Section A*, 55(1) (2001), 1–13.

[2] Ahuja, O. P., Jahangiri, J. M., Errata to Multivalent harmonic starlike function, *Ann. Univ. Mariac Curie-Sklodowska Section A* 56(1) (2002), 105.

[3] Ahuja, O. P., Aghalary, R., Joshi, S. B., Harmonic univalent functions associated with k-uniformly starlike functions, *Math. Sci. Res. J.,* 9(1) (2005), 9–17.

[4] Ahuja, O. P., Güney, H. Ö., Sakar, F. M., Certain classes of harmonic multivalent functions based on Hadamard product, *J. Inequal. Appl.,* 2010 (2009), Art. ID 759251, 12pp. https://doi.org/10.1155/2009/759251

[5] Güney, H. Ö., Ahuja, O. P., Inequalities involving multipliers for multivalent harmonic functions, *J. Inequal. Pure Appl. Math.,* 7(5) Art. 190 (2006), 1–9.

[6] Clunie, J., Sheil-Small, T., Harmonic univalent functions, *Ann. Acad. Aci. Fenn. Ser. A Math.,* 9 (1984), 3–25.

[7] Carlson, B. C., Shaffer, D. B., Starlike and prestarlike hypergeometric functions, *SIAM, J. Math. Anal.*, 15 (1984), 737–745.

[8] Duren, P., Hengartner, W., Laugesen, R. S., The argument principle for harmonic functions, *Amer. Math. Monthly,* 103 (1996), 411–415.

[9] Dziok, J., Srivastava, H. M., Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.,* 103 (1999), 1–13. https://doi.org/10.1016/S0096-3003(98)10042-5

[10] Ebadian, A., Tehranchi, A., On certain classes of harmonic p- valent functions by applying the Ruscheweyh derivatives, *Filomat,* 23(1) (2009), 91–101.

[11] Murugusundaramoorthy, G., Raina, R. K., On a subclass of harmonic functions associated with Wright's generalized hypergeometric functions, *Hacettepe J. Math. Stat.,* 38(2) (2009), 129–136.

[12] Murugusundaramoorthy, G., Vijaya, K., Starlike harmonic functions in parabolic region associated with a convolution structure, *Acta Univ. Sapientiae Math.*, 2(2) (2010), 168–183.

[13] Murugusundaramoorthy, G., Vijaya, K., Raina, R. K., A subclass of harmonic univalent functions with varying arguments defined by Dziok-Srivastava operator, *Archivun Mathematicum (BRNO) Tomus*, 45 (2009), 37-46.

[14] Murugusundaramoorthy, G., Harmonic starlike functions of complex order involving hypergeometric functions, *Math. Vesnik,* 64(4) (2012), 316–325.

[15] Murugusundaramoorthy, G., Uma, K., Harmonic univalent functions associated with generalized hypergeometric functions, *Bull. Math. Anal. Appl.,* 2(2) (2010), 69–76.

[16] Omar, R., Halim, S. A., Multivalent Harmonic Functions defined by Dziok-Srivastava operator, *Bull. Malays. Math. Sci. Soc.,* 35(3)(2) (2012), 601–610.

[17] Owa, S., On the distortion theorems - I, *Kyungpook. Math. J.*, 18 (1978) 53–59.

[18] Porwal, S., Some properties of a subclass of harmonic univalent functions defined by the multiplier transformations, *Indian J. Pure Appl. Math.,* 46(3) (2015), 309–335. https://doi.org/10.1007/s13226-015-0132-9

[19] Porwal, S., On a New Subclass of Harmonic Univalent Functions Defined by Multiplier Transformation, *Mathematica Moravica,* 19(2) (2015), 75–87. http://dx.doi.org/10.5937/MatMor1502075P

[20] Srivastava, H. M., Li, Shu-Hai, Tang, H., Certain classes of k-uniformly close-to-convex functions and other related functions defined by using the Dziok-Srivastava operator, *Bull. Math. Anal. Appl.*, 1(3) (2009), 49–63.

[21] Ruscheweyh, S., New criteria for univalent functions, *Proc. Amer. Math. Soc.,* 49 (1975), 109–115. https://doi.org/10.1090/S0002-9939-1975-0367176-1

[22] Atshan, W. G., Kulkarni, S. R., Raina, R. K., A class of multivalent harmonic functions involving a generalized Ruscheweyh type operator, *Math. Vesnik*, 60 (2008), 207–213.