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Research Article

Shrinkage estimators of shape parameter of contaminated Pareto model with insurance application

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Abstract

In this paper, a Pareto distribution in the presence of outliers is proposed as a claim size distribution. The shrinkage estimators of the shape parameter α are derived. Also, estimators of Premium are considered and compared by using simulation study. Finally, an actual example is proposed for obtaining different estimators of the Premium.

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1. Introduction

One of important topics in actuarial science is Premium. The premium (P) is the amount of money that insurer must collect from the Policy holder in order to cover the expected losses. Tsanakas and Desli [22], Heilpern [12] and Young [23] proposed various methods for determining premium. Assume that X represents the amounts of claims that have occurred in the past period, then the net premium can be obtained in the form $P = E(X)$. The Pareto distribution was introduced by [19], and was studied for distribution income over a population. Also Benktander [2] explained that one of the applications of Pareto distribution is for modeling the severity of insurance claim losses in motor insurance. The probability density function (pdf) of Pareto distribution is

$$f_X(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \qquad 0 < \theta \le x, \quad \alpha > 0, \quad \theta > 0,$$

and cumulative distribution function (cdf)

$$F_X(x) = 1 - \left(\frac{\theta}{x}\right)^\alpha.$$

Quandt [20] discussed various methods of estimation in Pareto distribution and Rytgaard [21] proposed maximum likelihood estimator (MLE) and the moment estimator for the shape parameter of the Pareto distribution, also Dixit and Jabbari Nooghabi [6] are derived the (MLE) of pdf and cdf for the Pareto distribution. Furthermore, Ebegil and Ozdemir [9] studied on shrinkage estimator classes for the shape parameter of Pareto distribution.

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On the other hand, outliers may exist in any data set. An outlier refers to some of the observations that deviates very much from the other observed data point. Many definitions of outliers as well as detecting of outliers are expressed. For examples one can refer to [1, 10, 11, 13–17]. In this regard, many statistical models in the presence of outliers are proposed, for example Dixit [4] introduced a model of outliers, for more information about this model refer to [5] and one can see the other model of outliers in [18]. Dixit and Jabbari Nooghabi [7] considered uniformly minimum variance unbiased estimator (UMVUE) of pdf and cdf for the Pareto distribution contaminated by k outliers. The article is laid out as follows. In Section 2, Pareto model contaminated by k outliers is described and the shrinkage estimators are derived in Section 3. In Section 4, the estimators of the net Premium will be proposed. In Section 5, by simulation studies, MLE, UMVUE, moment method (MM), and shrinkage estimators of Premium are compared. Finally, we have given an example to calculate the Premium in Section 6.

2. Model

Assume that n observations of the Pareto distribution such that k observations have a Pareto distribution with parameters α, $\beta\theta$, and the pdf of

$$f_2(x;\alpha,\beta,\theta) = \frac{\alpha(\beta\theta)^\alpha}{x^{\alpha+1}}, \quad 0<\beta\theta\le x, \quad \alpha>0, \quad \beta>1, \quad \theta>0, \tag{2.1}$$

and $n-k$ out of n random variables are arisen from

$$f_1(x;\alpha,\theta) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, \quad 0<\theta\le x, \quad \alpha>0, \tag{2.2}$$

where the shape parameter α is unknown. Also the scale parameter θ, contamination factor β, and the number of outliers k are known.

According to [8] and [7], the joint pdf of $(X_1, X_2, \cdots, X_n)$ contaminated by k outliers is introduced as

$$f(x_1,x_2,\cdots,x_n;\alpha,\beta,\theta) = \frac{\alpha^n\beta^{k\alpha}\theta^{n\alpha}}{c(n,k)}\left(\prod_{i=1}^{n} x_i\right)^{-(\alpha+1)} \sum_{A_1=1}^{n-k+1} \sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}\prod_{j=1}^{k} I(x_{A_j}-\beta\theta)I(x_{A_j}-\theta), \tag{2.3}$$

where $c(n,k)=\frac{n!}{k!(n-k)!}$ and $I(A)$ is proposed as

$$I(A) = \begin{cases} 1, & A>0, \\ 0, & otherwise. \end{cases} \tag{2.4}$$

Observe that the indicator function, $I(A)$ in (2.4) is defined based on (2.1) and (2.2). The marginal distribution of X_i, $i=1,2,\cdots,n$, is

$$f(x_i;\alpha,\beta,\theta) = b\frac{\alpha(\beta\theta)^\alpha}{x_i^{\alpha+1}}I(x_i-\beta\theta) + \bar{b}\frac{\alpha\theta^\alpha}{x_i^{\alpha+1}}I(x_i-\theta), \quad \alpha>0, \quad \beta>1, \quad \theta>0, \tag{2.5}$$

where, $b=\frac{k}{n}$ and $\bar{b}=1-b=\frac{n-k}{n}$.

3. Shrinkage estimator

Dixit and Jabbari Nooghabi [7] obtained the MLE, UMVUE, and moment estimator of shape parameter for the Pareto distribution contaminated by k outliers, which are defined, respectively of

$$\hat{\alpha}_{ml} = \frac{n}{\ln(T)-k\ln(\beta)-n\ln(\theta)}, \tag{3.1}$$

$$\hat{\alpha}_{mu} = \frac{n-1}{\ln(T) - k\ln(\beta) - n\ln(\theta)}, \tag{3.2}$$

and

$$\hat{\alpha}_{mm} = \frac{\mu_1'}{\mu_1' - \theta(b\beta + \bar{b})}, \tag{3.3}$$

where $T = \prod_{i=1}^{n} X_i$ and $\mu_1' = \frac{1}{n}\sum_{i=1}^{n} X_i$. In this section, different shrinkage estimators of α related to $\hat{\alpha}_{ml}$, $\hat{\alpha}_{mu}$, and $\hat{\alpha}_{mm}$ are proposed. Let

$$\hat{\alpha}_{msh} = w\hat{\alpha} + (1-w)\alpha_0, \tag{3.4}$$

where $0 \leq w \leq 1$. $\hat{\alpha}_{msh}$ is called shrinkage estimator of α and α_0 is our "guess" for parameter α.

Note: In order to find out that the α_0 is close to the true value (α) or not, we may test $H_0 : \alpha = \alpha_0$ vs. $h_1 : \alpha \neq \alpha_0$ using test statistic $V = \frac{2n\alpha_0}{\hat{\alpha}_{ml}} \sim \chi^2_{2n}$ at a predetermined significance level. If the value of test statistic v is too large or too small. then we will reject H_0. So, if we perform test with an $\eta = 0.05$ level of significance, we will reject null hypothesis if $v < \chi^2_{0.025,2n}$ or $v > \chi^2_{0.975,2n}$. One can refer to [3] for more details.

Now, we have to find w by using mean squares error (MSE) of $\hat{\alpha}_{msh}$ such that minimized MSE of $\hat{\alpha}_{msh}$, that is

$$\begin{aligned}\mathrm{MSE}(\hat{\alpha}_{\mathrm{msh}}) &= E(\hat{\alpha}_{msh} - \alpha)^2 \\ &= E\left(w^2(\hat{\alpha} - \alpha_0)^2\right) + (\alpha - \alpha_0)^2 - 2E(w(\hat{\alpha} - \alpha_0)(\alpha - \alpha_0)).\end{aligned}$$

Then

$$\frac{\partial \mathrm{MSE}(\hat{\alpha}_{\mathrm{msh}})}{\partial w} = 2wE(\hat{\alpha} - \alpha_0)^2 - 2(\alpha - \alpha_0)E(\hat{\alpha} - \alpha_0) = 0,$$

and the true value of w is

$$w = \frac{(\alpha - \alpha_0)E(\hat{\alpha} - \alpha_0)}{E(\hat{\alpha} - \alpha_0)^2}. \tag{3.5}$$

We know that w depends on the unknown parameter α, so we replace it by $\hat{\alpha}$.

$$\hat{w} = \frac{(\hat{\alpha} - \alpha_0)E(\hat{\alpha} - \alpha_0)}{E(\hat{\alpha} - \alpha_0)^2}. \tag{3.6}$$

To obtain the first shrinkage estimator, we use $\hat{\alpha}_{ml}$ instead of $\hat{\alpha}$.

$$\hat{\alpha}_{msh_1} = w_1\hat{\alpha}_{ml} + (1 - w_1)\alpha_0, \tag{3.7}$$

where

$$w_1 = \frac{(\hat{\alpha}_{ml} - \alpha_0)E(\hat{\alpha}_{ml} - \alpha_0)}{E(\hat{\alpha}^2_{ml}) - 2\alpha_0 E(\hat{\alpha}_{ml}) + \alpha_0^2}.$$

By substituting $E(\hat{\alpha}_{ml}) = \frac{n\alpha}{\alpha - 1}$ and $E(\hat{\alpha}^2_{ml}) = \frac{n^2\alpha^2}{(n-1)(n-2)}$, then w_1 is given as

$$w_1 = \frac{(n-2)(\hat{\alpha}_{ml} - \alpha_0)(n\alpha - n\alpha_0 + \alpha_0)}{n^2\alpha^2 - 2n(n-2)\alpha_0\alpha + \alpha_0^2(n-1)(n-2)},$$

and $\hat{w}_1$ is obtained as

$$\hat{w}_1 = \frac{(n-2)(\hat{\alpha}_{ml} - \alpha_0)(n\hat{\alpha}_{ml} - n\alpha_0 + \alpha_0)}{n^2\hat{\alpha}^2_{ml} - 2n(n-2)\alpha_0\hat{\alpha}_{ml} + \alpha_0^2(n-1)(n-2)}. \tag{3.8}$$

The second shrinkage estimator of parameter α is related to UMVUE of α, i.e.

$$\hat{\alpha}_{msh_2} = w_2\hat{\alpha}_{mu} + (1 - w_2)\alpha_0, \tag{3.9}$$

where

$$w_2 = \frac{(\hat{\alpha}_{mu} - \alpha_0)E(\hat{\alpha}_{mu} - \alpha_0)}{E(\hat{\alpha}_{mu} - \alpha_0)^2}.$$

By substituting $E(\hat{\alpha}_{mu}) = \alpha$ and $E(\hat{\alpha}^2_{mu}) = \frac{(n-1)\alpha^2}{n-2}$, w_2 is given by

$$w_2 = \frac{(n-2)(\hat{\alpha}_{mu} - \alpha_0)(\alpha - \alpha_0)}{(n-1)\alpha^2 - (n-2)\alpha_0(2\alpha - \alpha_0)},$$

and by replacing α by $\hat{\alpha}_{mu}$

$$\hat{w}_2 = \frac{(n-2)(\hat{\alpha}_{mu} - \alpha_0)^2}{(n-1)\hat{\alpha}^2_{mu} - (n-2)\alpha_0(2\hat{\alpha}_{mu} - \alpha_0)}. \tag{3.10}$$

The third shrinkage estimator is proposed by using the MM estimator ($\hat{\alpha}_{mm}$). So $\hat{\alpha}_{msh_3}$ is defined as

$$\hat{\alpha}_{msh_3} = w_3\hat{\alpha}_{mm} + (1 - w_3)\alpha_0, \tag{3.11}$$

where

$$w_3 = \frac{(\hat{\alpha}_{mm} - \alpha_0)E(\hat{\alpha}_{mm} - \alpha_0)}{E(\hat{\alpha}_{mm} - \alpha_0)^2}.$$

We know that as n increased $E(\hat{\alpha}_{mm}) \simeq \alpha$ and $E(\hat{\alpha}^2_{mm}) \simeq var(\hat{\alpha}_{mm}) + \alpha^2$, where

$$var(\hat{\alpha}_{mm}) \simeq \frac{(1 + \alpha^2 - \alpha)(b\beta^2 + \bar{b})}{n(b\beta + \bar{b})^2(\alpha - 2)}.$$

Thus, $w_3 \simeq \frac{(\hat{\alpha}_{mm}-\alpha_0)(\alpha-\alpha_0)n(b\beta+\bar{b})^2(\alpha-2)}{(1+\alpha^2-\alpha)(b\beta^2+\bar{b})+(\alpha-\alpha_0)^2 n(b\beta+\bar{b})^2(\alpha-2)}$ and for estimating w_3, we replace α by $\hat{\alpha}_{mm}$.

$$\hat{w}_3 \simeq \frac{(\hat{\alpha}_{mm} - \alpha_0)^2 n(b\beta + \bar{b})^2(\hat{\alpha}_{mm} - 2)}{(1 + \hat{\alpha}^2_{mm} - \hat{\alpha}_{mm})(b\beta^2 + \bar{b}) + (\hat{\alpha}_{mm} - \alpha_0)^2 n(b\beta + \bar{b})^2(\hat{\alpha}_{mm} - 2)}. \tag{3.12}$$

4. Estimation of net Premium

In this section, the estimators of net Premium by using the estimators of α are derived. We know that net Premium is defined as

$$P = E(X), \tag{4.1}$$

where X represents the corresponding claim amount (for more information about the methods of determining the Premium refer to [23]. By using Equation (2.5), the net Premium is derived as

$$\begin{aligned} P &= E(X) \\ &= b\int_{\beta\theta}^{\infty} x\frac{\alpha(\beta\theta)^\alpha}{x^{\alpha+1}}dx + \bar{b}\int_{\theta}^{\infty} x\frac{\alpha\theta^\alpha}{x^{\alpha+1}}dx \\ &= b\alpha(\beta\theta)^\alpha \frac{x^{-\alpha+1}}{-\alpha+1}\Big|_{\beta\theta}^{\infty} + \bar{b}\alpha\theta^\alpha\frac{x^{-\alpha+1}}{-\alpha+1}\Big|_{\theta}^{\infty} \\ &= -b\frac{\alpha\beta\theta}{-\alpha+1} - \bar{b}\frac{\alpha\theta}{-\alpha+1} \\ &= \frac{\alpha\theta(b\beta + \bar{b})}{1-\alpha}, \end{aligned}$$

and we have

$$P = \frac{\alpha\theta(b\beta + \bar{b})}{\alpha - 1}. \tag{4.2}$$

Also, another derivation of the net Premium is as follows:

$$\begin{aligned} P &= \frac{\alpha\theta(b\beta+\bar{b})}{\alpha-1} = \theta(b\beta+\bar{b}) \times \frac{1}{1-\frac{1}{\alpha}} \\ &= \theta(b\beta+\bar{b}) \sum_{j=0}^{\infty} \frac{1}{\alpha^j}. \end{aligned} \tag{4.3}$$

According to (2.3), Dixit and Jabbari Nooghabi [7] obtained the distribution of $T = \prod_{i=1}^{n} X_i$ as $h^*(t)$

$$h^*(t) = \frac{\alpha^n \theta^{n\alpha} \beta^{k\alpha}}{\Gamma(n)} t^{-(\alpha+1)} [\ln(t) - k\ln(\beta) - n\ln(\theta)]^{n-1} I(t - \beta^k \theta^n). \tag{4.4}$$

So, assuming $S = \ln(T) - k\ln(\beta) - n\ln(\theta)$, distribution of S is $\Gamma(n, \alpha)$, i.e

$$g(s) = \frac{\alpha^n}{\Gamma(n)} e^{-\alpha s} s^{n-1}, \; s > 0. \tag{4.5}$$

According to [21],

$$E(S^j) = \frac{(n+j-1)^{(j)}}{\alpha^j}, \tag{4.6}$$

where $(n+j-1)^{(j)} = (n+j-1)\cdots(n+1)n$. Using the Rao-Blackwell theorem, a MVUE of $\frac{1}{\alpha^j}$ is $c_j S^j$ where $c_j = \frac{1}{(n+j-1)^{(j)}}$, so we have

$$E(c_j S^j) = \frac{1}{\alpha^j}.$$

Therefore, a MVUE of P is

$$\hat{P}_0 = \theta(b\beta+\bar{b}) \sum_{j=0}^{\infty} c_j S^j, \tag{4.7}$$

and

$$E(\hat{P}_0) = \frac{\alpha\theta(b\beta+\bar{b})}{\alpha-1}. \tag{4.8}$$

Now, we can find variance of $\hat{P}_0$. It is obvious that

$$\begin{aligned} \hat{P}_0^2 &= \left(\theta(b\beta+\bar{b}) \sum_{j=0}^{\infty} c_j S^j\right)^2 \\ &= \theta^2(b\beta+\bar{b})^2 \sum_{j=1}^{\infty} S^j \sum_{r=0}^{j-1} c_r c_{j-r}. \end{aligned}$$

So

$$E(\hat{P}_0^2) = \theta^2(b\beta+\bar{b})^2 \sum_{j=1}^{\infty} B_j \frac{1}{\alpha^j}, \tag{4.9}$$

where

$$B_j = \sum_{r=0}^{j-1} \frac{c_r c_{j-r}}{c_j} = \sum_{r=0}^{j-1} \frac{(n+j-1)^{(r)}}{(n+r-1)^{(r)}}, \quad j = 2, 3, \cdots.$$

In addition

$$\begin{aligned} E^2(\hat{P}_0) &= \theta^2(b\beta+\bar{b})^2\left(\sum_{j=0}^{\infty}\frac{1}{\alpha^j}\right)^2 \\ &= \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j}(j-1), \end{aligned} \tag{4.10}$$

and

$$var(\hat{P}_0) = \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}(B_j-(j-1))\frac{1}{\alpha^j},$$

or

$$var(\hat{P}_0) = \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j}\sum_{r=0}^{j-1}\frac{(n+j-1)^{(r)}-(n+r-1)^{(r)}}{(n+r-1)^{(r)}}. \tag{4.11}$$

Now, we need to calculate the Cramer-Rao bound of variance. (4.2) implies that

$$\alpha = \frac{P}{P-\theta(b\beta+\bar{b})}. \tag{4.12}$$

Now, considering the joint density of $(X_1, X_2, \cdots, X_n)$ in (2.3), we have

$$\begin{aligned} \ln(f(x_1,\cdots,x_n,P,\beta,\theta)) &= -\frac{P}{P-\theta(b\beta+\bar{b})}s-\sum_{i=1}^{n}\ln(x_i) \\ &\quad -\ln(c(n,k))+n\ln\left(\frac{P}{P-\theta(b\beta+\bar{b})}\right) \\ &\quad +\ln\left(\sum_{A_1=1}^{n-k+1}\sum_{A_2=A_1+1}^{n-k+2}\cdots\sum_{A_k=A_{k-1}+1}^{n}\prod_{j=1}^{k}I(x_{A_j}-\beta\theta)I(x_{A_j}-\theta)\right), \end{aligned}$$

where s is the value of random variable S. Then,

$$\begin{aligned} \frac{\partial\ln(f(x_1,\cdots,x_n,P,\beta,\theta))}{\partial P} &= \frac{\theta(b\beta+\bar{b})s}{(P-\theta(b\beta+\bar{b}))^2}-\frac{n\theta(b\beta+\bar{b})}{P(P-\theta(b\beta+\bar{b}))} \\ &= -\left(\frac{n}{\alpha}-s\right)\frac{(\alpha-1)^2}{\theta(b\beta+\bar{b})}. \end{aligned}$$

So,

$$\begin{aligned} E\left(\frac{\partial\ln(f(x_1,\cdots,x_n,P,\beta,\theta))}{\partial P}\right)^2 &= E\left(\left(\frac{n}{\alpha}-s\right)^2\frac{(\alpha-1)^4}{\theta^2(b\beta+\bar{b})^2}\right) \\ &= \frac{n(\alpha-1)^4}{\alpha^2\theta^2(b\beta+\bar{b})^2}, \end{aligned} \tag{4.13}$$

and the Cramer-Rao bound of variance is

$$var(\hat{P}_0) \geq \frac{\alpha^2\theta^2(b\beta+\bar{b})^2}{n(\alpha-1)^4}. \tag{4.14}$$

Now, by using the MLE of α, we can find MLE of P.

$$\hat{P}_1 = \frac{\hat{\alpha}_{ml}\theta(b\beta+\bar{b})}{\hat{\alpha}_{ml}-1} = \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{\hat{\alpha}_{ml}^j} = \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{n^j}S^j.$$

So,

$$\begin{aligned} E(\hat{P}_1) &= \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{n^j}E(S^j) \\ &= \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{(n+j-1)^{(j)}}{n^j}\frac{1}{\alpha^j} \geq \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{\alpha^j} = P. \end{aligned} \tag{4.15}$$

Also

$$E(\hat{P}_1)^2 = \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{(j-1)(n+j-1)^{(j)}}{n^j\alpha^j},$$

$$(E(\hat{P}_1))^2 = \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j n^j}\sum_{r=0}^{j-1}(n+r-1)^{(r)}\,(n+j-r-1)^{(j-r)},$$

and

$$\begin{aligned} var(\hat{P}_1) &= \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j n^j}\sum_{r=0}^{j-1}\Big((n+j-1)^{(j)}-(n+r-1)^{(r)}(n+j-r-1)^{(j-r)}\Big) \\ &= \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j n^j}\sum_{r=0}^{j-1}(n+j-1)^{(j-r)}((n+j-1)^{(r)}-(n+r-1)^{(r)}) \\ &> \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j}\sum_{r=0}^{j-1}\frac{1}{n^r}\Big((n+j-1)^{(r)}-(n+r-1)^{(r)}\Big) \\ &> \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j}\sum_{r=0}^{j-1}\frac{(n+j-1)^{(r)}-(n+r-1)^{(r)}}{(n+r-1)^{(r)}} = var(\hat{P}_0). \end{aligned}$$

In addition, we use the UMVUE of α for obtaining the second estimator of P.

$$\begin{aligned} \hat{P}_2 &= \frac{\hat{\alpha}_{mu}\theta(b\beta+\bar{b})}{\hat{\alpha}_{mu}-1} \\ &= \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{\hat{\alpha}_{mu}^j} \\ &= \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{(n-1)^j}S^j. \end{aligned} \tag{4.16}$$

So,

$$\begin{aligned} E(\hat{P}_2) &= \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{(n-1)^j}E(S^j) \\ &= \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{(n+j-1)^{(j)}}{(n-1)^j}\frac{1}{\alpha^j} > \theta(b\beta+\bar{b})\sum_{j=0}^{\infty}\frac{1}{\alpha^j} = P. \end{aligned}$$

Also, $var(\hat{P}_2) = E(\hat{P}_2^2) - (E(\hat{P}_2))^2$. Then $E(\hat{P}_2^2)$ and $\big(E(\hat{P}_2)\big)^2$ as well as $var(\hat{P}_2$ are found as follows:

$$E(\hat{P}_2^2) = \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{(j-1)(n+j-1)^{(j)}}{(n-1)^j\alpha^j},$$

$$(E(\hat{P}_2))^2 = \theta^2(b\beta+\bar{b})^2\sum_{j=1}^{\infty}\frac{1}{\alpha^j(n-1)^j}\sum_{r=0}^{j-1}(n+r-1)^{(r)}\,(n+j-r-1)^{(j-r)},$$

and

$$
\begin{aligned}
var(\hat{P}_2) &= \theta^2(b\beta \ +\bar{b})^2 \\
&\quad \times \sum_{j=1}^{\infty} \frac{1}{\alpha^j} \frac{1}{(n-1)^j} \sum_{r=0}^{j-1} \Big((n+j-1)^{(j)} - (n+r-1)^{(r)}(n+j-r-1)^{(j-r)} \Big) \\
&= \theta^2(b\beta+\bar{b})^2 \sum_{j=1}^{\infty} \frac{1}{\alpha^j} \frac{1}{(n-1)^j} \sum_{r=0}^{j-1} (n+j-1)^{(j-r)}((n+j-1)^{(r)} - (n+r-1)^{(r)}) \\
&> \theta^2(b\beta+\bar{b})^2 \sum_{j=1}^{\infty} \frac{1}{\alpha^j} \sum_{r=0}^{j-1} \frac{1}{(n-1)^r} \Big((n+j-1)^{(r)} - (n+r-1)^{(r)} \Big) \\
&> \theta^2(b\beta+\bar{b})^2 \sum_{j=1}^{\infty} \frac{1}{\alpha^j} \sum_{r=0}^{j-1} \frac{(n+j-1)^{(r)} - (n+r-1)^{(r)}}{(n+r-1)^{(r)}} = var(\hat{P}_0).
\end{aligned} \tag{4.17}
$$

Now, we estimate P by using MM estimator of α.

$$
\hat{P}_3 = \frac{\hat{\alpha}_{mm}\theta(b\beta+\bar{b})}{\hat{\alpha}_{mm}-1} = \frac{\frac{\mu_1'\theta(b\beta+\bar{b})}{\mu_1'-\theta(b\beta+\bar{b})}}{\frac{\mu_1'}{\mu_1'-\theta(b\beta+\bar{b})}-1} \tag{4.18}
$$

$$
= \frac{\frac{\mu_1'\theta(b\beta+\bar{b})}{\mu_1'-\theta(b\beta+\bar{b})}}{\frac{\theta(b\beta+\bar{b})}{\mu_1'-\theta(b\beta+\bar{b})}} = \mu_1'.
$$

We know that $\hat{P}_3$ is asymptotically unbiased for P, because

$$
\begin{aligned}
E(\hat{P}_3) &= E\left(\mu_1'\right) \\
&\simeq \frac{1}{n}\sum_{i=1}^{n} E(X_i) \\
&= \frac{\alpha\theta(b\beta+\bar{b})}{\alpha-1} = P.
\end{aligned} \tag{4.19}
$$

Then, variance of $\hat{P}_3$ is

$$
\begin{aligned}
var(\hat{P}_3) &= var(\mu_1') \\
&\simeq \frac{1}{n^2}\sum_{i=1}^{n} var(X_i) \\
&= \frac{(\alpha-1)^2\alpha\theta^2(b\beta^2+\bar{b}) - (\alpha-2)\alpha^2\theta^2(b\beta+\bar{b})^2}{n(\alpha-2)(\alpha-1)^2}.
\end{aligned} \tag{4.20}
$$

Comparing $\hat{P}_1$ and $\hat{P}_2$ shows that if we use the $\hat{\alpha}_{ml}$ or $\hat{\alpha}_{mu}$ for estimating P, then we have overestimate for the Premium. But, it has been seen that $\hat{P}_1$ is more efficient than $\hat{P}_2$, because

$$
var(\hat{P}_1) < var(\hat{P}_2). \tag{4.21}
$$

Also with substitution $\hat{\alpha}_{msh_1}$, $\hat{\alpha}_{msh_2}$, and $\hat{\alpha}_{msh3}$ in equation (4.2), $\hat{P}_4$, $\hat{P}_5$, and $\hat{P}_6$ are obtained as

$$
\hat{P}_4 = \frac{(\hat{w}_1\hat{\alpha}_{ml}+(1-\hat{w}_1)\alpha_0)\theta(b\beta+\bar{b})}{w_1\hat{\alpha}_{ml}+(1-\hat{w}_1)\alpha_0-1}, \tag{4.22}
$$

$$\hat{P}_5 = \frac{(\hat{w}_2\hat{\alpha}_{mu} + (1-\hat{w}_2)\alpha_0)\theta(b\beta + \bar{b})}{w_2\hat{\alpha}_{mu} + (1-\hat{w}_2)\alpha_0 - 1}, \tag{4.23}$$

and

$$\hat{P}_6 = \frac{(\hat{w}_3\hat{\alpha}_{mm} + (1-\hat{w}_3)\alpha_0)\theta(b\beta + \bar{b})}{w_3\hat{\alpha}_{mm} + (1-\hat{w}_3)\alpha_0 - 1}, \tag{4.24}$$

where w_1, $\hat{w}_2$, and $\hat{w}_3$ are defined in Equations (3.8), (3.10), and (3.12), respectively.

In section 3, it has been seen that $\hat{w}_1$, $\hat{w}_2$, and $\hat{w}_3$ are between zero and one. On the other hand, shrinkage estimators of Premium are dependent on the value of α_0. So, if α_0 is considered bigger than actual value of α, then the Premium is called overestimated and if the guess of α_0 is smaller than actual value of α, then it is underestimated.

5. Comparison of the estimator of Premium

In the previous section, different estimators of the Premium from the Pareto distribution contaminated by k outliers were considered. Here, performance of all these estimates are compared numerically in terms of average values and MSE. We have generated 10000 samples of size $n = 20, 50, 100, 500, 100$ in order to have the small, medium and large samples from the Pareto distribution in the presence of k outliers with $k = 1, 2, 3, 4, 5, 10, 20, 50, 100, 200$, $\alpha = 3, 10$, $\beta = 1.5$, and $\theta = 1$. In this case, we have used **R** software.

The average values and MSEs of all estimators of P are presented for different choices of k in Tables 1 and 2. According to the results of simulation study, we conclude that the estimator of Premium based on MLE ($\hat{P}_1$) has superior performance in comparing with the other estimators for every fixed n and $k \geq 10n\%$. Also the MSEs of the shrinkage estimators of Premium based on MLE ($\hat{P}_4$) and UMVUE ($\hat{P}_5$) in the most cases for every constant n and $k < 10n\%$ are the minimum. In general, the MSEs of all estimators for every fixed k are decreased as the sample size, n, increases.

6. Real data analysis

The first data set is presented in this section for the purpose of illustration and investigate how well the proposed method works in using data with small size. This data set is related to a random sample of size 20 which is taken from the well-known Danish fire losses for claim of at least 1.5 (It is obvious that intensity less than 1.5 is not entrained to be claimed). The whole data set can be found in “evir” package of **R** software. The sample is as follows:
1.581612, 1.584488, 1.756955, 1.722223, 2.036376, 2.036378, 2.051958, 2.102489, 2.146618, 2.9238653, 3.263154, 3.367496, 4.530015, 4.856098, 5.417277, 5.563852, 6.319914, 7.320644, 9.174312, 56.225426.
Using one-sample Kolmogorov-Smirnov goodness of fit test ($D = 0.13697$ and $p = 0.7992$), Pareto distribution with $\hat{\alpha} = 1.208316$ and $\theta = 1.5$ is fitted on the data set. Histogram, Pareto $Q-Q$ plot, boxplot and empirical distribution are shown in Figure 1.

According to the graphs, it seems that there is an outlier in the observations. In this data set, $x = 56.225426$ is the largest value and it maybe is an outlier. Therefore, the contaminated factor $\left(\beta = \frac{\min(x_i)}{\theta}\right)$ is obtained as 1.05. we consider $\alpha_0 = 1.1$ and we test $H_0 : \alpha = 1.1$ vs. $H_1 : \alpha \neq 1.1$ and then the value of test statistic is obtained as $v = 38.63808$ and for a significant level of $\eta = 0.05$, the critical values are $\chi^2_{0.025,40} = 24.43304$ and $\chi^2_{0.975,40} = 59.34171$. Then, we can not reject the null hypothesis. Also, the number of outliers (k) and the shape parameter (α) are unknown. For obtaining the Premium, k must be predetermined. So, we can use the likelihood function respect to k and select each k which maximizes the likelihood. Thus, MLE, UMVUE, Moment, and shrinkage estimators of α are shown in Table 3 for different values of k.

Table 1. Average values and MSEs of the estimators of Premium for $\alpha = 3$, $\beta = 1.5$ and $\theta = 1$.

n	k	$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$	$\hat{P}_5$	$\hat{P}_6$
20	1	1.55642749	1.60229435	1.53524700	1.66416200	1.69708542	1.87780071
		(0.03735982)	(0.04787402)	(0.04249945)	(0.04922419)	(0.06265609)	(0.11521449)
	2	1.59592948	1.64316270	1.57425726	1.70532774	1.73930003	1.92494161
		(0.04048907)	(0.05580576)	(0.04146690)	(0.05985986)	(0.07685504)	(0.14623444)
	3	1.63386828	1.68212386	1.61105923	1.74711585	1.78198941	1.97162513
		(0.04381345)	(0.06289389)	(0.04394502)	(0.07396727)	(0.09427820)	(0.18202006)
	4	1.65078642	1.72396472	1.67426297	1.79062962	1.82655534	2.01904343
		(0.05144540)	(0.07738014)	(0.05296227)	(0.09339569)	(0.11780126)	(0.22280091)
	5	1.71250063	1.76337194	1.69166215	1.83075212	1.86724665	2.06428660
		(0.06440288)	(0.09357250)	(0.08656033)	(0.11470327)	(0.14249255)	(0.26765058)
50	1	1.52320772	1.53943438	1.51584535	1.58146143	1.59040927	1.88791038
		(0.01514761)	(0.01484179)	(0.01846070)	(0.01764386)	(0.01958129)	(0.10695981)
	2	1.53848220	1.55487889	1.53114508	1.59734454	1.60640321	1.90625519
		(0.01410725)	(0.01431743)	(0.01796014)	(0.01849745)	(0.02076507)	(0.11917898)
	3	1.55258619	1.56909466	1.54486727	1.61181804	1.62088284	1.92445025
		(0.01314803)	(0.01380723)	(0.01664909)	(0.01919757)	(0.02171684)	(0.13146123)
	4	1.56976370	1.56653266	1.56153593	1.62983107	1.63907112	1.94453943
		(0.01349651)	(0.01481056)	(0.01623605)	(0.02187956)	(0.02483504)	(0.14607376)
	5	1.58349778	1.60037252	1.57467418	1.64379360	1.65307116	1.96130746
		(0.01387726)	(0.01568882)	(0.01619607)	(0.02391920)	(0.02717750)	(0.15932423)
	10	1.65799611	1.67561792	1.64965233	1.72125996	1.73089950	2.05539015
		(0.02109785)	(0.02559950)	(0.02361489)	(0.04154408)	(0.04645208)	(0.24133068)
100	1	1.51187428	1.51967689	1.50776788	1.54236645	1.54469440	1.89271335
		(0.00989749)	(0.00922283)	(0.01167126)	(0.00865590)	(0.00880215)	(0.10551694)
	2	1.51876929	1.52659758	1.51498620	1.54933849	1.55166830	1.90177770
		(0.00910391)	(0.00853553)	(0.01107035)	(0.00829244)	(0.00847367)	(0.11134638)
	3	1.52434417	1.53217059	1.52021368	1.55480043	1.55709418	1.90888881
		(0.00847986)	(0.00799662)	(0.01066358)	(0.00797774)	(0.00817166)	(0.11631617)
	4	1.53442577	1.54234447	1.53020487	1.56538158	1.56774483	1.92061226
		(0.00772859)	(0.00740492)	(0.00952330)	(0.00789939)	(0.00815775)	(0.12420305)
	5	1.54087795	1.54881362	1.53748584	1.57183020	1.57418147	1.92957800
		(0.00730977)	(0.00709049)	(0.00990445)	(0.00788866)	(0.00818034)	(0.13057285)
	10	1.57974665	1.58790363	1.57535806	1.61164962	1.61409356	1.97683755
		(0.00661703)	(0.00703962)	(0.00826047)	(0.00981105)	(0.01032341)	(0.16659789)
	20	1.65388304	1.66240503	1.65013302	1.68715354	1.68968909	2.07049703
		(0.01334168)	(0.01505941)	(0.01499548)	(0.02172088)	(0.02264910)	(0.25114913)
500	1	1.50253691	1.50404941	1.50163932	1.50858214	1.50866379	1.89811997
		(0.00640544)	(0.00619787)	(0.00686885)	(0.00563203)	(0.00562301)	(0.10539636)
	2	1.50353338	1.50504542	1.50277864	1.50957201	1.50965343	1.89957473
		(0.00628704)	(0.00608275)	(0.00675646)	(0.00552774)	(0.00551894)	(0.10637450)
	3	1.50531081	1.50682537	1.50444244	1.51135997	1.51144149	1.90171757
		(0.00599115)	(0.00579152)	(0.00648332)	(0.00524917)	(0.00524056)	(0.10774385)
	4	1.50678973	1.50830572	1.50594709	1.51284478	1.51292640	1.90372263
		(0.00579489)	(0.00559961)	(0.00623711)	(0.00507066)	(0.00506230)	(0.10903842)
	5	1.50678973	1.50830572	1.50594709	1.51284478	1.51292640	1.90372263
		(0.00579489)	(0.00559961)	(0.00623711)	(0.00507066)	(0.00506230)	(0.10903842)
	10	1.51567614	1.51720074	1.51508310	1.52176480	1.52184687	1.91517054
		(0.00468527)	(0.00451613)	(0.00509443)	(0.00406651)	(0.00405960)	(0.11671604)
	20	1.53092129	1.53246189	1.53002954	1.53707513	1.53715808	1.93406424
		(0.00310650)	(0.00298246)	(0.00355735)	(0.00266816)	(0.00266369)	(0.12996091)
	50	1.57584204	1.57742755	1.57487848	1.58217464	1.58225999	1.99081675
		(0.00124603)	(0.00126127)	(0.00163963)	(0.00136771)	(0.00137087)	(0.17399193)
	100	1.65126623	1.65292880	1.65060354	1.65791202	1.65800179	2.08613125
		(0.00719235)	(0.00745985)	(0.00758250)	(0.00832932)	(0.00834639)	(0.26244742)
1000	1	1.50129213	1.50204529	1.50064520	1.50430414	1.50432410	1.89887737
		(0.00600758)	(0.00589943)	(0.00627541)	(0.00558890)	(0.00558629)	(0.10540275)
	2	1.50129213	1.50204529	1.50064520	1.50430414	1.50432410	1.89887737
		(0.00600758)	(0.00589943)	(0.00627541)	(0.00558890)	(0.00558629)	(0.10540275)
	3	1.50235297	1.50310599	1.50211165	1.50536138	1.50538126	1.90075613
		(0.00583767)	(0.00573108)	(0.00606261)	(0.00542544)	(0.00542287)	(0.10661717)
	4	1.50324982	1.50400352	1.50289110	1.50626218	1.50628211	1.90175628
		(0.00572032)	(0.00561503)	(0.00593381)	(0.00531334)	(0.00531081)	(0.10726734)
	5	1.50401113	1.50476523	1.50366626	1.50702497	1.50704490	1.90267026
		(0.00560682)	(0.00550260)	(0.00584771)	(0.00520407)	(0.00520157)	(0.10787600)
	10	1.50813304	1.50888975	1.50780405	1.51115935	1.51117940	1.90780830
		(0.00503459)	(0.00493623)	(0.00525363)	(0.00465502)	(0.00465267)	(0.11125394)
	20	1.51534859	1.51610850	1.51474375	1.51838623	1.51840633	1.91679878
		(0.00413778)	(0.00405002)	(0.00438545)	(0.00380098)	(0.00379892)	(0.11733816)
	50	1.53799548	1.53876696	1.53750873	1.54107996	1.54110038	1.94551627
		(0.00195775)	(0.00190361)	(0.00216981)	(0.00175562)	(0.00175445)	(0.13780492)
	100	1.57534453	1.57613449	1.57502991	1.57850186	1.57852275	1.99292526
		(0.00061440)	(0.00061803)	(0.00083100)	(0.00064388)	(0.00064426)	(0.17521868)
	200	1.65042990	1.65125764	1.65028537	1.65373929	1.65376120	2.08793694
		(0.00638890)	(0.00651727)	(0.00662189)	(0.00691890)	(0.00692261)	(0.26374275)

Table 2. Average values and MSEs of the estimators of Premium for $\alpha = 10$, $\beta = 1.5$ and $\theta = 1$.

n	k	$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$	$\hat{P}_5$	$\hat{P}_6$
20	1	1.13985080	1.14665750	1.13913430	1.15788592	1.16215379	1.19858569
		(0.00153149)	(0.00132358)	(0.00156646)	(0.00110197)	(0.00118458)	(0.00166881)
	2	1.16673851	1.17365113	1.16599240	1.18505345	1.18936458	1.22701828
		(0.00084096)	(0.00100442)	(0.00084956)	(0.00139209)	(0.00171434)	(0.00434226)
	3	1.19513743	1.20225655	1.20159252	1.21392520	1.21836721	1.25658408
		(0.00172247)	(0.00230305)	(0.00228061)	(0.00336830)	(0.00396132)	(0.00883075)
	4	1.22208025	1.23014480	1.22286462	1.24207713	1.24661120	1.28575807
		(0.00402806)	(0.00511120)	(0.00410992)	(0.00686967)	(0.00773350)	(0.01495498)
	5	1.25081904	1.25827540	1.25195748	1.27048816	1.27514507	1.31499483
		(0.00808047)	(0.00952745)	(0.00894599)	(0.01202021)	(0.01317669)	(0.02280911)
50	1	1.12263751	1.12520483	1.12236780	1.13104370	1.13195425	1.18567406
		(0.00224725)	(0.00204352)	(0.00227271)	(0.00166080)	(0.00162008)	(0.00061461)
	2	1.13335888	1.13594196	1.13297763	1.14180067	1.14271276	1.19683529
		(0.00143467)	(0.00128587)	(0.00146187)	(0.00103039)	(0.00101032)	(0.00117901)
	3	1.14505495	1.14767798	1.14473381	1.15365508	1.15459355	1.20917698
		(0.00079683)	(0.00070717)	(0.00081405)	(0.00058809)	(0.00058979)	(0.00207645)
	4	1.15604226	1.15868753	1.15575177	1.16470431	1.16564695	1.22076955
		(0.00045063)	(0.00041867)	(0.00046521)	(0.00043173)	(0.00045449)	(0.00320299)
	5	1.16710890	1.16977828	1.16683597	1.17585390	1.17680468	1.23252804
		(0.00033947)	(0.00036628)	(0.00034738)	(0.00051541)	(0.00055972)	(0.00461794)
	10	1.22264408	1.22543958	1.22511661	1.22902974	1.22912152	1.28162473
		(0.00350878)	(0.00384872)	(0.00382495)	(0.00431384)	(0.00432677)	(0.01471710)
100	1	1.11661427	1.11786988	1.11648733	1.12089838	1.12111955	1.18118519
		(0.00265977)	(0.00253951)	(0.00267396)	(0.00227336)	(0.00225628)	(0.00033795)
	2	1.12250096	1.12376692	1.12233565	1.12683076	1.12705582	1.18732410
		(0.00210708)	(0.00200077)	(0.00212318)	(0.00176784)	(0.00175312)	(0.00055314)
	3	1.12799798	1.12926948	1.12788880	1.13234601	1.13257214	1.19316934
		(0.00165508)	(0.00156235)	(0.00166610)	(0.00136276)	(0.00135055)	(0.00083276)
	4	1.13357279	1.13485076	1.13337676	1.13794212	1.13816899	1.19900901
		(0.00125471)	(0.00117574)	(0.00126975)	(0.00100950)	(0.00099973)	(0.00117754)
	5	1.13928957	1.14057578	1.13916209	1.14369141	1.14392048	1.20508086
		(0.00090986)	(0.00084510)	(0.00092084)	(0.00071321)	(0.00070595)	(0.00160803)
	10	1.16692193	1.16823760	1.16679004	1.17142103	1.17165485	1.23429830
		(0.00016979)	(0.00017643)	(0.00017568)	(0.00021888)	(0.00022460)	(0.00471499)
	20	1.22216647	1.22373652	1.22235976	1.22706564	1.22730985	1.29283752
		(0.00327896)	(0.00345256)	(0.00329255)	(0.00386849)	(0.00390183)	(0.01608118)
500	1	1.11229961	1.11254758	1.11227750	1.11316469	1.11317302	1.17815302
		(0.00298635)	(0.00295960)	(0.00298926)	(0.00289402)	(0.00289315)	(0.00015683)
	2	1.11342866	1.11367692	1.11339589	1.11429494	1.11430329	1.17933359
		(0.00286530)	(0.00283908)	(0.00286915)	(0.00277481)	(0.00277395)	(0.00018559)
	3	1.11441075	1.11465893	1.11439345	1.11527571	1.11528402	1.18041658
		(0.00276091)	(0.00273518)	(0.00276324)	(0.00267223)	(0.00267139)	(0.00021385)
	4	1.11552200	1.11577043	1.11548720	1.11638804	1.11639636	1.18156498
		(0.00264727)	(0.00262207)	(0.00265124)	(0.00256044)	(0.00255963)	(0.00024754)
	5	1.11666849	1.11691725	1.11664607	1.11753575	1.11754409	1.18279917
		(0.00253020)	(0.00250553)	(0.00253284)	(0.00244520)	(0.00244440)	(0.00028499)
	10	1.12218747	1.12243738	1.12215643	1.12305859	1.12306696	1.18863263
		(0.00200988)	(0.00198786)	(0.00201311)	(0.00193415)	(0.00193345)	(0.00050828)
	20	1.13329638	1.13354876	1.13326561	1.13417603	1.13418449	1.20040130
		(0.00114529)	(0.00112867)	(0.00114812)	(0.00108838)	(0.00108785)	(0.00116418)
	50	1.16670027	1.16696025	1.16665396	1.16760701	1.16761574	1.23574650
		(0.00003417)	(0.00003443)	(0.00003546)	(0.00003616)	(0.00003620)	(0.00480059)
	100	1.22231408	1.22263293	1.22236032	1.22331182	1.22332100	1.29468890
		(0.00313535)	(0.00316950)	(0.00313888)	(0.00324696)	(0.00324803)	(0.01642052)
1000	1	1.11171417	1.11183790	1.11170279	1.11214659	1.11214865	1.17773658
		(0.00303511)	(0.00302157)	(0.00303651)	(0.00298802)	(0.00298780)	(0.00013493)
	2	1.11222596	1.11234969	1.11221463	1.11265825	1.11266031	1.17828438
		(0.00297926)	(0.00296584)	(0.00298063)	(0.00293263)	(0.00293241)	(0.00014753)
	3	1.11280556	1.11292939	1.11279996	1.11323824	1.11324030	1.17890355
		(0.00291610)	(0.00290281)	(0.00291693)	(0.00286992)	(0.00286970)	(0.00016208)
	4	1.11330400	1.11342782	1.11329512	1.11373641	1.11373847	1.17944078
		(0.00286247)	(0.00284931)	(0.00286361)	(0.00281674)	(0.00281653)	(0.00017535)
	5	1.11384270	1.11396656	1.11382752	1.11427523	1.11427729	1.18000346
		(0.00280572)	(0.00279268)	(0.00280749)	(0.00276045)	(0.00276024)	(0.00019038)
	10	1.11667901	1.11680326	1.11666353	1.11711309	1.11711516	1.18299828
		(0.00251415)	(0.00250178)	(0.00251593)	(0.00247119)	(0.00247098)	(0.00027928)
	20	1.12224833	1.12237321	1.12223526	1.12268468	1.12268676	1.18889678
		(0.00198858)	(0.00197754)	(0.00199005)	(0.00195026)	(0.00195008)	(0.00050697)
	50	1.13888944	1.13901614	1.13887447	1.13933204	1.13933414	1.20653162
		(0.00078735)	(0.00078036)	(0.00078855)	(0.00076321)	(0.00076309)	(0.00160225)
	100	1.16665926	1.16678904	1.16664694	1.16711262	1.16711478	1.23594922
		(0.00001691)	(0.00001696)	(0.00001742)	(0.00001738)	(0.00001738)	(0.00481410)
	200	1.22222432	1.22236030	1.22221802	1.22269937	1.22270164	1.29481144
		(0.00310550)	(0.00312068)	(0.00310573)	(0.00315882)	(0.00315907)	(0.01643698)

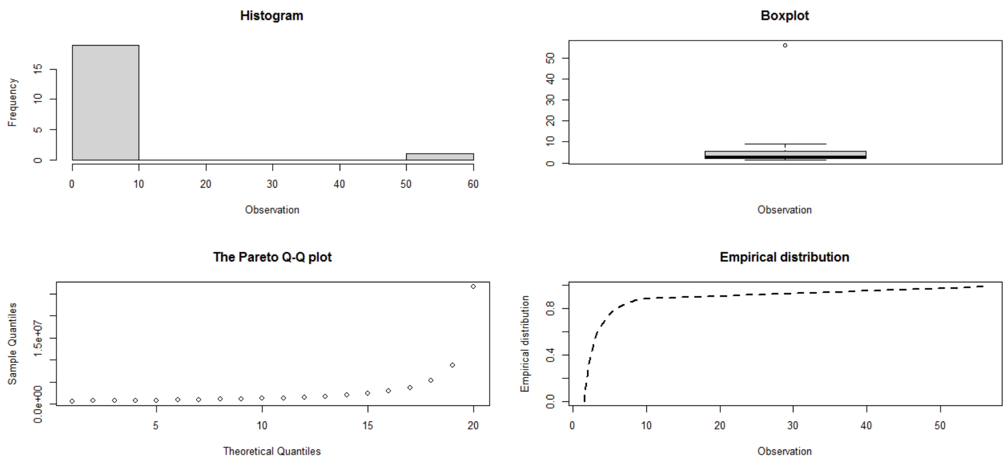

Figure 1. The goodness of fit graphs for the real data.

Table 3. The estimators of α for β=1.05, α_0=1.1 and θ=1.5.

k	$\hat{\alpha}_{ml}$	$\hat{\alpha}_{mu}$	$\hat{\alpha}_{mm}$	$\hat{\alpha}_{msh_1}$	$\hat{\alpha}_{msh_2}$	$\hat{\alpha}_{msh_3}$
1	1.138773	1.081834	1.313588	1.101657	1.099908	1.106005
2	1.141945	1.084848	1.314616	1.101980	1.099947	1.106083
3	1.145136	1.087879	1.315646	1.102338	1.099973	1.106162
4	1.148344	1.090926	1.316677	1.102733	1.099989	1.106241
5	1.151570	1.093991	1.317710	1.103166	1.099997	1.106322

Also, the values of likelihood function corresponding to k, $L(\underline{X};\hat{\alpha}_{ml})$, $L(\underline{X};\hat{\alpha}_{mu})$, $L(\underline{X};\hat{\alpha}_{mm})$, $L(\underline{X};\hat{\alpha}_{msh_1})$, $L(\underline{X};\hat{\alpha}_{msh_2})$, and $L(\underline{X};\hat{\alpha}_{msh_3})$ for $k = 1, 2, 3, 4, 5$ are shown in Table 4.

Table 4. The likelihood functions corresponding to k.

k	$L(\underline{X};\hat{\alpha}_{ml})$	$L(\underline{X};\hat{\alpha}_{mu})$	$L(\underline{X};\hat{\alpha}_{mm})$	$L(\underline{X};\hat{\alpha}_{msh_1})$	$L(\underline{X};\hat{\alpha}_{msh_2})$	$L(\underline{X};\hat{\alpha}_{msh_3})$
1	$9.362670e-21$	$9.123602e-21$	$7.558827e-21$	$9.261543e-21$	$9.251733e-21$	$9.283960e-21$
2	$1.041932e-21$	$1.015327e-21$	$8.463313e-22$	$1.028946e-21$	$1.027583e-21$	$1.031489e-21$
3	$1.836196e-22$	$5.789310e-22$	$1.500498e-22$	$1.810079e-22$	$1.807095e-22$	$1.814560e-22$
4	$4.569079e-23$	$4.452411e-23$	$3.756034e-23$	$4.495630e-23$	$4.486472e-23$	$4.506549e-23$
5	$1.510238e-23$	$1.471675e-23$	$1.248819e-23$	$1.483033e-23$	$1.479332e-23$	$1.486481e-23$

According to Table 4, the likelihood function is maximized for k=1 in all cases. So in this example, $\hat{\alpha}_{ml} = 1.138773$, $\hat{\alpha}_{mu} = 1.081834$, $\hat{\alpha}_{mm} = 1.313588$, $\hat{\alpha}_{msh_1} = 1.101657$, $\hat{\alpha}_{msh_2} = 1.099908$, and $\hat{\alpha}_{msh_3} = 1.106005$. Therefore, the risk Premiums with respect to the estimators are calculated and shown in Table 5.

Table 5. The net Premium corresponding to the estimators.

$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$	$\hat{P}_5$	$\hat{P}_6$
12.339783	19.879262	6.299058	16.296182	16.555056	15.689419

Further, according to the values of Table 4 the highest likelihood values for $k = 1$ are related to $\hat{\alpha}_{mu}$ and $\hat{\alpha}_{ml}$, so the estimator of net Premium based on MLE and UMVUE of the unknown parameter α have a better performance than the other estimators.

The second considered data is related to first example and the data contains entire data of Danish fire losses. The original data are 2167 observations, but here we considered the losses above 1.5. Therefore, the remaining data set includes 1386 observations. By using one-sample Kolmogorov-Smirnov goodness of fit test ($D = 0.017252$ and $p = 0.8039$), Pareto distribution with $\hat{\alpha} = 1.408057$ and $\theta = 1.5$ is fitted on the data set and the contaminated factor is obtained as $\beta = 1.0011$. We consider $\alpha_0 = 1.4$ and test $H_0 : \alpha = 1.4$ vs. $H_1 : \alpha \neq 1.4$, then value of test statistic is $v = 2760.449$. At the level of $\eta = 0.05$, the critical values are $\chi^2_{0.025,2772} = 2627.968$ and $\chi^2_{0.975,2772} = 2919.82$. So, the null hypothesis can not be rejected. According to the previous example, k must be determined to estimate shape parameter and the Premium. Therefore, by calculating the likelihood function for each k and comparing these values, the number of outliers is related to which has the largest likelihood value. In this example, the value of the likelihood function is maximum for $k = 1$ in all cases. Also, the estimates of α for $k = 1$ are as $\hat{\alpha}_{ml} = 1.405858$, $\hat{\alpha}_{mu} = 1.404843$, $\hat{\alpha}_{mm} = 1.484142$, $\hat{\alpha}_{msh_1} = 1.400160$, $\hat{\alpha}_{msh_2} = 1.400078$, and $\hat{\alpha}_{msh_3} = 1.400271$ and the risk Premiums respect to the estimators are shown in Table 6.

Table 6. The net Premium corresponding to the estimators.

$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$	$\hat{P}_5$	$\hat{P}_6$
5.195880	5.205140	4.598270	5.248508	5.249269	5.247468

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