RINGS WITH CHAIN CONDITIONS ON UNIFORM IDEALS

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ABSTRACT

In[4], Smith and Vedadi characterized modules which satisfy DCC (respectively, ACC) condition on non-essential submodules are uniform or Artinian (respectively, Noetherian). In this study, we will investigate rings which satisfy ACC on uniform right ideals and we will discuss some ring extensions.

Keywords: Essential Modules, Uniform Modules, Torsion-free Modules.

ÜNİFORM İDEALLER ÜZERİNDE ZİNCİR KOŞULUNU SAĞLAYAN HALKALAR

ÖZET

Smith ve Vedadi "*Modules with Chain Conditions on Non-essential Submodules*" [Commun. Algebra 32 (5), 1881-2004, (2004)] adlı makalede essential olmayan altmodüller üzerinde DCC (benzer şekilde, ACC) koşulunu sağlayan modullerin üniform veya Artinian (benzer şekilde, Noetherian) olduğunu karakterize etti. Biz bu çalışmada, üniform sağ idealler üzerinde ACC koşulunu sağlayan halkaları inceleyeceğiz, ve bazı halka genişlemelerini tartışacağız.

Anahtar kelimeler: Essential Modüller, Üniform Modüller, Torsion-free Modüller.

1.INTRODUCTION

Throughout this paper, all rings are associative with identity and all modules are unital right modules.

Let M be a right R-module. A submodules N of M is called *essential* if $N \cap K \neq 0$ for every non-zero submodule N of M . For each essential submodule N of M, M/N is Noetherian if and only if M has ACC on essential submodules. In[4], Smith and Vedadi characterized modules which satisfies on descending chain condition (for short DCC) or as cending chain condition (for short ACC) on (non-)essential submodules, and they shown that if M satisfies ACC on essential submodules then, for any submodule N of M , N and M/N are also satisfies ACC on essential submodules ([4, Corollary 1.3]). If all non-zero submodules of M are essential in M , then M is called *uniform*. Examples of such modules are, for any ring R, simple modules and indecomposable extending modules (i.e., every submodule is essential in a direct summand). Clearly uniform modules and Noetherian modules satisfy ACC on non-essential submodules.

Let R be a ring and I be a right ideal of R. I is called uniform, if any non-zero right ideal J of R with $J \subseteq I$ is essential in I. To goal of this paper, to characterize rings which satisfy ACC on uniform right ideals (i.e., for every ascending chain $I_1 \subseteq I_2 \subseteq \ldots$ of uniform ideals I_i $(i \geq 1)$ of R such that there exists a positive integer n such that $I_n = I_{n+1}$). Following [4], M is Noetherian if and only if M satisfies ACC on non-essential submodules, and a semiprime ring R satisfies ACC on non-essential right ideals if and only if R is right uniform or right Noetherian (see [4, Theorem2.9]).

We assume that a ring satisfies ACC on uniform right ideals. The class of these types rings is not closed under homomorphic images (Example 2.5). Among the other results, we will also show that ;

Theorem: Let n be any positive integer. Then;

(1) R satisfies ACC condition on uniform right ideals of R if and only if $\aleph_n(R)$ satisfies ACC condition on uniform right ideals of $\aleph_n(R)$.

(2) Let $R \subseteq R' \subseteq \aleph_n(R)$. R satisfies ACC condition on uniform right ideals of R if and only if R' satisfies ACC condition on uniform right ideals of R'.

Theorem: If $R[x]$ satisfies ACC condition on uniform right ideals of $R[x]$ then $R[x,x^{-1}]$ satisfies ACC condition on uniform right ideals of its.
General background materials can be found in [1], [2] and [5].

2.PROPERTIES AND EXTENSIONS OF RINGS WITH CHAIN CONDITIONS ON UNIFORM IDEALS

In [4, Theorem 1.8], they proved that *M* satisfies ACC on non-essential submodules if and only if every non-essential submodule of *M* is Noetherian if and only if every non-essential submodule of *M* is finitely generated. In this paper, under above result, we will consider to rings which satisfy ACC on uniform right ideals of *R*.

We start the following fundamental lemma to this paper;

Lemma 2.1. *The following are equivalent for a ring R.*
(1) *R satisfies ACC condition on uniform right ideals of R.*
(2) *Every uniform right ideal is Noetherian.*
(3) *Every uniform right ideal is finitely generated.*

Proof. $(1) \Rightarrow (2) \Rightarrow (3)$ are clear.
$(3) \Rightarrow (1)$ Let $I_1 \subseteq I_2 \subseteq \ldots$ be any ascending chain of uniform right ideals of *R*. Let $A = \bigcup_{i=1}^{\infty} I_i$. Since *A* is also uniform ideal of *R*, *A* is finitely generated by (3).

Let $A = x_1R + x_2R + \cdots + x_nR$. Now we may choose a maximal right ideal A_{n_j} on right ideals $A_{n_1}, A_{n_2}, \ldots, A_{n_k}$ such that $x_i \in A_{n_i}$ for i =1, 2, ..., k. Hence we can obtain that $A \subseteq A_{n_j} \subseteq A$. This implies that $A = A_{n_j} = A_k$ for any $A_{n_j} \subseteq A_k$. □

Recall that the module *M* is called *extending*,or *a CS-module*, if every submodule of *M* is essential in a direct summand of *M*. More generally, *M* is called *uniform-extending* if every uniform submodule is essential in a direct summand of *M*. Some properties of uniform-extending modules can be found in [3].

Theorem 2.2. *Let R be a right uniform-extending ring with ACC on uniform right ideals. Then R is right Noetherian.*

Proof. Let *I* be an uniform right ideal of *R* and we consider *R/I*. Since *R* is right uniform-extending ring, there exists an idempotent $e \in R$ such that *I* is essential in *Re*. Then $I \oplus R(1-e)$ is essential in *R*. Since $R/(I \oplus R(1-e)) \cong Re/I$ is Noetherian by Lemma 2.1. Now $R/I \cong (Re/I) \oplus R(1-e)$. By [2,Theorem 19.4(2)], *R* is right Noetherian. □

For any element *a* of a ring *R*, we set $r(a) = \{r \in R : ar = 0\}$, i.e., *r*(*a*) is the right annihilator of *a* in *R*.

Lemma 2.3. *If R is not Noetherian but satisfies ACC on uniform right ideals, then aR is an uniform right ideal of R for* $a \in R$.

Proof. Assume that *aR* is not uniform right ideal of *R*. By hypothesis, *aR* is Noetherian. But $R/r(a) \cong aR$ is not Noetherian. This is a contradiction. Hence *aR* is an uniform ideal of *R*. □

Theorem 2.4. *Any ring which satisfies ACC on uniform right ideals either aR is Noetherian or aR is an uniform ideal of R.*

Proof. By Lemma 2.3. □

For any positive integer *n*, $\aleph_n(R)$ and $\cap_n(R)$ denote the matrix ring and the upper triangular matrix ring over *R*, respectively.

Lemma 2.5. Let n be any positive integer. If R satisfies ACC condition on uniform right ideals of R, then $\aleph_n(R)$ is an uniform Noetherian right R-module.

Proof. Since $\aleph_n(R) \cong R^{n^2}$, $\aleph_n(R)$ is an uniform Noetherian right R-module.

Theorem 2.6. *Let n be any positive integer. Then;*

(1) *R satisfies ACC condition on uniform right ideals of R if and only if* $\aleph_n(R)$ *satisfies ACC condition on uniform right ideals of* $\aleph_n(R)$.
(2) *Let* $R \subseteq R' \subseteq \aleph_n(R)$. *R satisfies ACC condition on uniform right ideals of R if and only if* R' *satisfies ACC condition on uniform right ideals of* R'.

Proof. (1) Assume that R satisfies ACC condition on uniform right ideals of R. Let I be an uniform right ideal of $\aleph_n(R)$. Since $\aleph_n(R)$ is an uniform Noetherian right R-module by Lemma 2.5, I is a Noetherian right R-submodule of $\aleph_n(R)$. This implies that I_R is a finitely generated. Hence I is a Noetherian ideal of $\aleph_n(R)$. By Lemma 2.1, $\aleph_n(R)$ satisfies ACC condition on uniform right ideals of $\aleph_n(R)$.
Converse is clear.
(2) Assume R' satisfies ACC condition on uniform right ideals of R'. Since $R \subseteq \aleph_n(R) \subseteq \aleph_n(R')$, then R satisfies ACC condition on uniform right ideals of R.

For converse, let I be an uniform right ideal of R'. By (1), I_R is a finitely generated right R-submodule of $\aleph_n(R)$. Therefore I'_R is finitely generated R module. □

Corollary 2.7. Let n be any positive integer. Then R satisfies ACC condition on uniform right ideals of R if and only if $\cap_n(R)$ satisfies ACC condition on uniform right ideals of $\cap_n(R)$.

Given a ring R, $R[X]$ denotes the polynomial ring with X a set of commuting indeterminates over R. If $X = \{x\}$, then we use $R[x]$ in place of $R[\{x\}]$. The ring of Laurent polynomials in x, with coefficients in a ring R, consits of all formal sums $\sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and *k, n* are integers. We denote this ring by $R[x, x^{-1}]$.

Let *R* be a trivial extension of *S* by *M*, that is $R = S \times M$ as an abelian group, with multiplication given by the rule $(s,m)(s',m') = (ss', sm' + s'm)$, where $s, s' \in S$ and $m, m' \in M$. $I = 0 \times M$ is an ideal of *R*.

Example 2.8. Let *s*=*k*[*x*], where *k* is a field of characteristic zero and let *M* denote the *S*-submodule of $k[x; x^{-1}]/S$. The trivial extension $R = S \times M$ is a commutative ⊗-algebra and $I = 0 \times M$ is an ideal of *R*. Since *M* is faithful uniform *S*-module, *I* is an uniform ideal of *R*. Hence *R* satisfies ACC condition on uniform right ideals of *R* by Lemma 2.1.

Lemma 2.9. *Let S be a commutative domain, let M be a non-zero torsion-free S-module and* $R = S \times M$ *. Then I is a non-essential ideal of R if and only if* $I = 0 \times N$ *I for some non-essential submodule N of the S-module M.*

Proof. See [4,Lemma4.4] □

Lemma 2.10. *Let S be a commutative domain, let M be a non-zero torsion-free S-module and* $R = S \times M$ *. Then I is an uniform ideal of R if and only if* $I = 0 \times N$ *for some uniform submodule N of the S-module M.*

Proof. For the proof, we completely follow proof of Lemma 2.9. If *N* is an uniform submodule of *M*, then there exits non-zero submodules *A, B* of *N* such that $A \cap B \neq 0$. Note that $0 \times N$, $0 \times A$, and $0 \times B$ are non-zero ideals of *R* such that $0 \times A \cap 0 \times B \neq 0$. If we take $I = 0 \times N$, clearly *I* is an uniform ideal of *R*.

Conversely, let *I* be an uniform ideal of *R*. Suppose that *I* does not contained in $0 \times M$. Let $(a, x) \in I$ for some $a \neq 0 \in S$ and $x \in M$. By assumption, there exists non-zero submodules *N, L* of *M* such that $N \cap L = 0$. Since *M* is a torsion-free S-module, $N_a \cap L_a = 0$. On the other hand, since $0 \times N_a 0$ and $0 \times L_a$ are non-zero ideals of R, we can obtain that $0 \times N_a \subseteq I$ and $0 \times L_a \subseteq I$. This implies that $I \subseteq 0 \times M$. Let $K = \{m \in M : (0, m) \in I\}$. It is easy to check that $I = 0 \times K$ and K is an uniform submodule of M. □

Theorem 2.11. *Let S be a commutative domain, let M be a non-uniform torsion-free S-module and* $R = S \times M$ *. Then the commutative ring R satisfies ACC condition on uniform ideals of R if and only if M satisfies ACC on uniform submodules.*

Proof. By Lemmas 2.10 and 2.1. □

Note that the class of rings which satisfy ACC on uniform ideal is not closed under homomorphic images.

Example 2.12. Let S be a commutative domain and let M be an uniform torsion-free S-module. Assume that S does not satisfy ACC on uniform ideals. Then the trivial extension $R = S \times M$ satisfies ACC on uniform ideals but $S \cong R / I = 0 \times M$ does not satisfy ACC on uniform ideals.

Theorem 2.13. *Let R be a ring and n be a positive integer.*

(1) *If R[x] satisfies ACC condition on uniform right ideals of R[x] then R satisfies ACC condition on uniform right ideals of R.*

(2) *R satisfies ACC condition on uniform right ideals of R if and only if* $R[x]/(x^n)$ *satisfies ACC condition on uniform right ideals of its, where* (x^n) *is the ideal generated by* x^n *.*

Proof. (1.) Let I be an uniform right ideal of R. Since $I[x]$ is an uniform ideal of $R[x]$, $I[x]$ is a finitely generated ideal by Lemma 2.1. For $f_i(x) = a_{i_0} + a_{i_1} + \cdots + a_{i_{n_i}} x^{n_i} \in I[x] \quad (i = 1, 2, \ldots, n)$, we may write $I[x] = f_1(x)R[x] + f_2(x)R[x] + \cdots + f_n(x)R[x]$. By coefficients of $f_i(x)$ $(i = 1, 2, \ldots, n)$ we may assume that $a \in a_1 R + a_2 R + \cdots + a_n R$ for any $a \in I$. This implies that each a_i is in I $(i = 1, 2, \ldots, n)$. Hence $I = a_1 R + a_2 R + \cdots + a_n R$. That is I is finitely generated, and so R satisfies ACC condition on uniform right ideals of R by Lemma 2.1.

(2.) Let $y = \bar{x} \in R[x]/(x^n) = S$. Then $S[y] = R + Ry + \cdots + Ry^{n-1}$ because $y^n = 0$. Let *I* be an uniform ideal of *S*[*y*]. If we repeat prof of (1), we can see that S satisfies ACC condition on uniform right ideals of S. Converse is clear from Corollary 2.4. □

For the converse of Theorem 2.13 (1), we assume that the class of rings with satisfy ACC condition on uniform right ideals of its is closed under homomorphic images. Let I be an uniform right ideal of R[x]. We consider the nonempty set J={ the leading coefficient of $f(x): f(x) \in I$ }. Clearly, J is an uniform right ideal of R. By Lemma 2.1., J is finitely generated. By Hilbert-Basis Theorem, I is also a finitely generated ideal of R[x]. Hence R[x] satisfies ACC condition on uniform right ideals of R[x] by Lemma 2.1.

Recall that *R* is called *right (strongly) Ore* if given *a,b*∈*R* with *b* regular there exists $a_1, b_1 \in R$ with b_1 regular such that $ab_1 = ba_1$ ($ab_1 = ab$, respectively). In [4, Corollary 2.6], they shown that a right nonsingular ring satisfies ACC on non-singular right ideals if and only if *R* is Noetherian or *R* is right Ore domain.

We consider a subset *S* of *R*. *S* is called *right (strongly) Ore set* if *S* is a multiplicatively closed (i.e., $1_R \in S$ and *S* is closed under multiplication of *R*) and *S* satisfies right (strongly) Ore condition, respectively.

Theorem 2.14. (1) *Let R be a ring and S be a strongly Ore set of R. If S contains no non-zero divisors of R and R satisfies ACC condition on uniform right ideals of R, then the localization* $S^{-1}R$ *also satisfies ACC condition on uniform right ideals of its.*
(2) *Assume that M is a multiplicative monoid in R consisting of central regular elements. If R satisfies ACC condition on uniform right ideals of R, then so is* RM^{-1}.

Proof. (1) Let *J* be a right ideal of $S^{-1}R$. Under the homomorphism $f : R \rightarrow S^{-1}R$, defined by $r \rightarrow r/I_R$, $f^{-1}(J)$ is a right ideal of *R*. Let $I = f^{-1}(J)$. Suppose that *I* is not uniform ideal of *R*. That is there exists non-zero right ideals *A, B* of *R* with *A, B* $\subseteq I$ such that $A \cap B = 0$. Note that $S^{-1}I = J$, and so $S^{-1}A$ and $S^{-1}B$ are non-zero right ideals of $S^{-1}R$. Since *S* contains no non-zero divisors of *R*, we have $S^{-1}(A \cap B) = 0$. This implies that $S^{-1}A \cap S^{-1}B = 0$. This contradiction gives to us that *I* is an uniform ideal of *R*. Hence *I* is a finitely generated. Let $I = x_1R + x_2R + \cdots + x_nR$, where $x_1, x_2, \ldots, x_n \in I$. For $a/s \in S^{-1}I$ with $a \in I$ and $s \in S$, we can write

$$\begin{aligned} a/s &= (x_1/1_R)(r_1 s/st_1) + (x_2/1_R)(r_2 s/st_2) + \cdots + (x_n/1_R)(r_n s/st_n) \\ &\in (x_1/1_R)S^{-1}R + (x_2/1_R)S^{-1}R + \cdots + (x_n/1_R)S^{-1}R \end{aligned}$$

Where some elements $r_1, r_2, \ldots, r_n \in R$ and $s, t_1, t_2, \ldots, t_n \in S$ with $a = x_1r_1 + x_2r_2 + \ldots + x_nr_n$ and $sx_nr_ns = x_nr_nst_n$. This implies that

$$S^{-1}I = (x_1/1_R)S^{-1}R + (x_2/1_R)S^{-1}R + \cdots + (x_n/1_R)S^{-1}R.$$

Thus $S^{-1}I = J$ is finitely generated. By Lemma 2.1, $S^{-1}R$ also satisfies ACC condition on uniform right ideals of its.

(2) Firstly, we claim that $I \to IM^{-1}$ is one to one correspondence between the set of all ideals in *R* and the set of all ideals in RM^{-1}. Let *J* be an ideal of RM^{-1} and $I = \{r \in R : rm^{-1} \in J\}$. Clearly, $I \subset J$ and *I* is an ideal of *R*. This implies that $IM^{-1} = J$. Conversely, let *I* be an ideal of *R*. Take $a_1m_1^{-1}, a_2m_2^{-1} \in IM^{-1}$. Since *I* is an ideal of *R*, we have $a_1m_2 - a_2 - m_1 \in I$ and $a_1a_2 \in I$. Then

$$(a_1m_2 - a_2m_1)(m_1m_2)^{-1} = a_1m_1^{-1} - a_2m_2^{-1} \in IM^{-1}$$

and

$$(a_1a_2)(m_1m_2)^{-1} = a_1m_1^{-1}a_2m_2^{-1} \in IM^{-1}$$

That is IM^{-1} is an ideal of RM^{-1}. Now, the proof is similar to case (1). □

Corollary 2.15. *Let R be a ring. If* $R[x]$ *satisfies ACC condition on uniform right ideals of* $R[x]$ *then* $R[x, x^{-1}]$ *satisfies ACC condition on uniform right ideals of its.*

Proof. Assume that $R[x]$ satisfies ACC condition on uniform right ideals of $R[x]$. We consider $S=\{1,x,x^2,\ldots\}$. Since $S=\{1,x,x^2,\ldots\}$ is a strongly Ore set of $R[X]$ and $S^{-1}R[x]=R[x,x^{-1}]$, we have $R[x,x^{-1}]$ satisfies ACC condition on uniform right ideals of its by Theorem 2.14.(1).

On the other hand, let $M=\{1,x,x^2,\ldots\}$ then M is multiplicative monoid in $R[x]$ consisting of central regular elements. Since $M^{-1}R[x]=R[x,x^{-1}]$, we also obtain the corollary from Theorem2.14.(2). □

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