APPLICATION OF THE DIFFERENTIAL TRANSFORM METHOD TO DIFFERENTIAL-ALGEBRAIC EQUATIONS WITH INDEX 2

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ABSTRACT

In this paper, we have used the differential transform method to solve differential-algebraic equations with index 2. Two kind of differential-algebraic equations have been considered and solved numericaly, then we compared numerical and analytical solution of the given equations. Examples were presented to show the ability of the method for differential-algebraic equations. We use MAPLE computer algebra system to solve given problems [4].

Keyword: Differential transform method, Differential-algebraic equation, Index of differential-algebraic equation, Power series, MAPLE.

2 İNDEXLİ DİFERENSİYEL-CEBİRSEL DENKLEMLERE DİFERENSİYEL DÖNÜŞÜM YÖNTEMİNİN UYGULANMASI

Özet

Bu makalede, 2 indexli diferensiyel-cebirsel denklemleri çözmek için diferensiyel dönüşüm yöntemini kullandık. İki çeşit diferensiyel-cebirsel denklem gözönüne alındı ve nümerik çözümleri bulundu. Daha sonar numeric çözümlerle analitik çözümler karşılaştırıldı. Bizim ele aldığımız örneklerde diferensiyel dönüşüm yönteminin diferensiyel-cebirsel denklemlerin çözümünde güçlü olduğunu gördük. Ele aldığımız problemleri çözmek için MAPLE bilgisayar yöntemini kullandık [4].

INTRODUCTION

The Differential transform method due to Zhou [1] has been successfully used to solve a linear and nonlinear initial value problems in electric circuit analysis. Using one-dimensional differential transform, Chen and Ho [5] proposed a method to solve eigenvalue problems. The method has been applied to the partial differential equation[6,7,10], and the system of partial differential equation[11]. Hassan applied the method to solve eigenvalues and normalized eigenfunctions for a Sturm-Liouville eigenvalue problem[8,9]. The differential transform method extended to solve differential-difference equations by Arıkoglu[14]. Chen used the Differential transform method to predict the advective-dispersive transport problems[12]. The numerical solution of the differential-algebraic equation systems has been found using Differential transform method[13,15]. We have used the differential transform method to solve differential-algebraic equation with index 2.

2. THE DIFFERENTIAL TRANSFORM METHOD

The differential transform of the k th derivate of function $y(x)$ in one dimensional is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{2.1}$$

where $y(x)$ is original function and $Y(k)$ is transformed function and the differential inverse transform of $Y(k)$ is defined as

$$y(x)=\sum_{k=0}^{\infty}(x-x_0)^k Y(k) \tag{2.2}$$

From (2.1) and (2.2) is defined

$$y(x)=\sum_{k=0}^{\infty}(x-x_0)^k \frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]_{x=x_0} \tag{2.3}$$

Equation (2.3) is obtained from Taylor series expansion at $x=x_0$. From the definitions of equations (2.1) and (2.2), it is easily proven that transformed functions comply with the basic mathematics operations shown in Table 1.

Table 1. The fundamental operation of one–dimensional differential transform method

Original function	**Transformed function**
$y(x)=u(x)\pm v(x)$	$Y(k)=U(k)\pm V(k)$
$y(x)=cw(x)$	$Y(k)=cW(k)$
$y(x)=dw/dx$	$Y(k)=(k+1)W(k+1)$
$y(x)=d^j w/dx^j$	$Y(k)=(k+1)(k+2)...(k+j)W(k+j)$
$y(x)=u(x)v(x)$	$Y(k)=\sum_{r=0}^{k}U(r)V(k-r)$
$y(x)=u_1(x)u_2(x)...u_n(x)$	$Y(k)=\sum_{r=0}^{r_1}\sum_{r=r_1}^{r_2}...\sum_{r=r_{n-1}}^{k}U_1(r)U_2(r_1-r)...U_n(k-r_{n-1})$
$y(x)=x^j$	$Y(k)=\delta(k-j)=\begin{cases}1, & k=j\\ 0, & k\neq j\end{cases}$, if $x_0=0$
$y(x)=\sin x,\quad y(x)=\cos x$	$Y(k)=\frac{1}{k!}\sin\left(\frac{\pi k}{2}\right),\quad Y(k)=\frac{1}{k!}\cos\left(\frac{\pi k}{2}\right)$

3. APPLICATIONS

Example 1. The test problem consider the following differential-algebraic equation system,

$$\begin{aligned} y_1' &= y_1 y_2^2 z_1^2 \\ y_2' &= y_1^2 y_2^2 - 3y_2^2 z_1 \\ 0 &= y_1^2 y_2 - 1 \end{aligned} \qquad (3.1)$$

with initial condition

$y_1(0)=1,\quad y_2(0)=1,\quad z_1(0)=1.$

The exact solutions are

$y_1(t)=e^t$,

$y_2(t)=e^{-2t}$,

$z_1(t) = e^{2t}$.

Taking differential transform of equations (3.1) and by using the related relations in Table 1, we obtain

$$(k+1)Y_1(k+1)=\sum_{r=o}^{r_1}\sum_{r=r_1}^{r_2}\sum_{r=r_2}^{r_3}\sum_{r=r_3}^{r_4}Y_1(r)Y_2(r_1-r)Y_2(r_2-r_1)Z_1(r_3-r_2)Z_1(k-r_3)$$
$$(k+1)Y_2(k+1)=\sum_{r=o}^{r_1}\sum_{r=r_1}^{r_2}\sum_{r=r_2}^{k}Y_1(r)Y_1(r_1-r)Y_2(r_2-r_1)Y_2(k-r_2)-3\sum_{r=o}^{r_1}\sum_{r=r_1}^{k}Y_2(r)Y_2(r_1-r)Z_1(k-r_1) \quad (3.2)$$

If the equations (3.2) are rearranged, we obtain

$$Y_1(k+1)=\frac{1}{(k+1)}\sum_{r=o}^{r_1}\sum_{r=r_1}^{r_2}\sum_{r=r_2}^{r_3}\sum_{r=r_3}^{r_4}Y_1(r)Y_2(r_1-r)Y_2(r_2-r_1)Z_1(r_3-r_2)Z_1(k-r_3)$$
$$Y_2(k+1)=\frac{1}{(k+1)}\left[\sum_{r=o}^{r_1}\sum_{r=r_1}^{r_2}\sum_{r=r_2}^{k}Y_1(r)Y_1(r_1-r)Y_2(r_2-r_1)Y_2(k-r_2)-3\sum_{r=o}^{r_1}\sum_{r=r_1}^{k}Y_2(r)Y_2(r_1-r)Z_1(k-r_1)\right] \quad (3.3)$$
$$\delta(k)=\sum_{r=0}^{r_1}\sum_{r=r_1}^{k}Y_1(r)Y_1(r_1-r)Y_2(k-r_1)$$

By applying the initial conditions equations

$$Y_1(0)=Y_2(0)=Z_1(0)=1 \quad (3.4)$$

For $k=0$ into the equations (3.3), the initial transformation coefficients are therefore determined by

$$Y_1(1)=Y_1(0).Y_2(0).Y_2(0).Z_1(0)Z_1(0)=1$$
$$Y_2(1)=Y_1(0).Y_1(0).Y_2(0).Y_2(0)-3.Y_2(0).Y_2(0).Z_1(0)=-2 \quad (3.5)$$
$$1=Y_1(0).Y_1(0).Y_2(0)$$

Taking $k=1$ in equations (3.3), we obtain

$$Y_1(2)=\frac{1}{2}\left[Y_1(0).Y_2(0).Y_2(0).Z_1(0)Z_1(1)+Y_1(0).Y_2(0).Y_2(0).Z_1(1)Z_1(0)+Y_1(0).Y_2(0).Y_2(1).Z_1(0)Z_1(0)\right.$$
$$\left.+Y_1(0).Y_2(1).Y_2(0).Z_1(0)Z_1(0)+Y_1(1).Y_2(0).Y_2(0).Z_1(0)Z_1(0)\right]$$
$$Y_2(2)=\frac{1}{2}\left[Y_1(0).Y_1(0).Y_2(0).Y_2(1)+Y_1(0).Y_1(0).Y_2(1).Y_2(0)+Y_1(0).Y_1(1).Y_2(0).Y_2(0)\right.$$
$$\left.+Y_1(1).Y_1(0).Y_2(0).Y_2(0)-3\left(Y_2(0).Y_2(0)Z_1(1)+Y_2(0).Y_2(1)Z_1(0)+Y_2(1).Y_2(0)Z_1(0)\right)\right]$$
$$0=Y_1(0).Y_1(0).Y_2(1)+Y_1(0).Y_1(1).Y_2(1)+Y_1(1).Y_1(0).Y_2(0)$$

If we continue in same manner, we obtain

$$Y_1(0)=1,\ Y_1(1)=1,\ Y_1(2)=\frac{1}{2},\ Y_1(3)=\frac{1}{6},\ Y_1(4)=\frac{1}{24},,$$
$$Y_1(5)=\frac{1}{120},\ Y_1(6)=\frac{1}{720},\ Y_1(7)=\frac{1}{5040},\ Y_1(8)=\frac{1}{40320},....$$
$$Y_2(0)=1,\ Y_2(1)=-2,\ Y_2(2)=2,\ Y_2(3)=-\frac{4}{3},\ Y_2(4)=\frac{2}{3},,$$
$$Y_2(5)=-\frac{4}{15},\ Y_2(6)=\frac{4}{45},\ Y_2(7)=-\frac{8}{315},\ Y_2(8)=\frac{2}{315},....$$
$$Z_1(0)=1,\ Z_1(1)=2,\ Z_1(2)=2,\ Z_1(3)=\frac{4}{3},\ Z_1(4)=\frac{2}{3},,$$
$$Z_1(5)=\frac{4}{15},\ Z_1(6)=\frac{4}{45},\ Z_1(7)=\frac{8}{315},\ Z_1(8)=\frac{2}{315},.... \quad (3.6)$$

Substituting all above coefficients into equation (2.2), we have solutions as

$$y_1(t)=1+t+\frac{1}{2}t^2+\frac{1}{6}t^3+\frac{1}{24}t^4+\frac{1}{120}t^5+\frac{1}{720}t^6+\frac{1}{5040}t^7+\frac{1}{40320}t^8+O(t^9)$$
$$y_2(t)=1-2t+2t^2-\frac{4}{3}t^3+\frac{2}{3}t^4-\frac{4}{15}t^5+\frac{4}{45}t^6-\frac{3}{315}t^7+\frac{2}{315}t^8+O(t^9)$$
$$z_1(t)=1+2t+2t^2+\frac{4}{3}t^3+\frac{2}{3}t^4+\frac{4}{15}t^5+\frac{4}{45}t^6+\frac{3}{315}t^7+\frac{2}{315}t^8+O(t^9)$$

Table 2. Compared of the numerical and exact solution of the first test problem, where y_2 is exact solution, $\tilde{y}_2$ is numerical solution.

t	y_2	$\tilde{y}_2$	$\lvert y_2-\tilde{y}_2\rvert$
0.1	0.8187307531	0.8187307532	$0.1\ 10^{-9}$
0.2	0.6703200460	0.6703200461	$0.1\ 10^{-9}$
0.3	0.5488116361	0.5488116345	$0.16\ 10^{-8}$
0.4	0.4493289641	0.4493289365	$0.276\ 10^{-7}$
0.5	0.3678794412	0.3678791888	$0.2524\ 10^{-6}$
0.6	0.3011942119	0.3011926747	$0.15372\ 10^{-5}$
0.7	0.24655969639	0.2465899006	$0.70633\ 10^{-5}$
0.8	0.2018965180	0.2018701030	0.0000264150
0.9	0.1652988882	0.1652144772	0.0000844110
1.0	0.1353352832	0.1350970021	0.0002382811

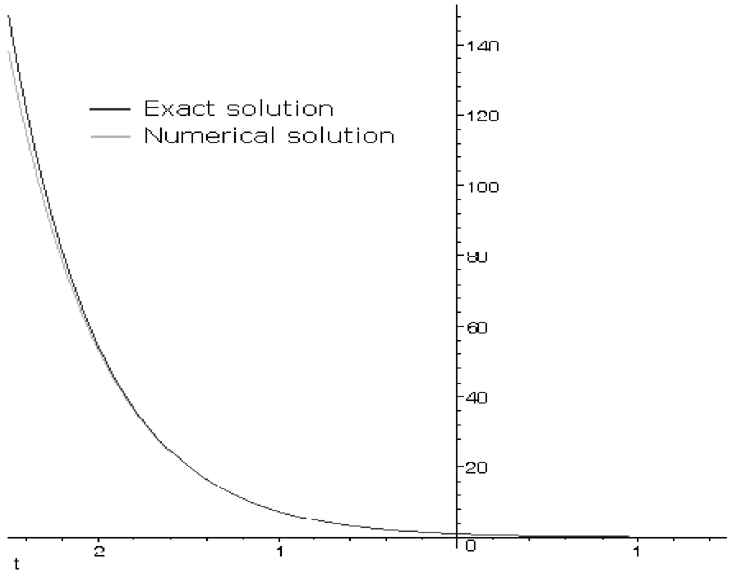

Figure 3.1. Graph of the functions y_2 and $\tilde{y}_2$ in the first test problem.

Example 2. We consider the differential-algebraic equation as a second example

$$\begin{aligned} y' &= y^2 + z + \cos t - 1, \\ v' &= y^2 + v^2 - \sin t - 1, \\ 0 &= y^2 + v^2 - 1. \end{aligned} \tag{3.7}$$

The initial values are

$$y(0) = 0,$$
$$v(0) = 1,$$
$$z(0) = 1.$$

The analytical solutions are

$$y(t) = \sin t,$$
$$v(t) = \cos t,$$
$$z(t) = \cos^2 t.$$

Taking the transform of the given differential algebraic equations. It is obtained that

$$
\begin{aligned}
(k+1)Y(k+1) &= \sum_{r=0}^{k} Y(r)Y(k-r) + Z(k) + \frac{1}{k!}\cos\left(\frac{\pi k}{2}\right) - \delta(k),\\
(k+1)V(k+1) &= \sum_{r=0}^{k} Y(r)Y(k-r) + \sum_{r=0}^{k} V(r)V(k-r) - \frac{1}{k!}\sin\left(\frac{\pi k}{2}\right) - \delta(k),\\
0 &= \sum_{r=0}^{k} Y(r)Y(k-r) + \sum_{r=0}^{k} V(r)V(k-r) - \delta(k).
\end{aligned}
\tag{3.8}
$$

Taking $k=0$ in equation(3.8), we obtain,

$$
\begin{aligned}
Y(1) &= Y(0)\times Y(0) + Z(0),\\
V(1) &= Y(0)\times Y(0) + V(0)\times V(0) - 1,\\
1 &= Y(0)\times Y(0) + V(0)\times V(0).
\end{aligned}
\tag{3.9}
$$

From (3.9), we have $Y(1)=1, V(1)=0$. Taking $k=1$ in equation(3.8), we obtain

$$
\begin{aligned}
2Y(2) &= \left[Y(0).Y(1) + Y(1).Y(0) + Z(1)\right],\\
2V(2) &= \left[Y(0).Y(1) + Y(1).Y(0) + V(0).V(1) + V(1).V(0) - 1\right].
\end{aligned}
\tag{3.10}
$$

From (3.10), we have $Y(2)=0,\quad V(2)=-\frac{1}{2}$. If we continue in same manner, we obtain

$$
Y(3)=-\frac{1}{6},\ Y(4)=0,\ Y(5)=\frac{1}{120},\ Y(6)=0,\ Y(7)=\frac{1}{5040},\ Y(8)=0,\ldots
$$

$$
V(3)=0,\ V(4)=\frac{1}{24},\ V(5)=0,\ V(6)=-\frac{1}{720},\ V(7)=0,\ V(8)=\frac{1}{40320},\ldots
$$

$$
Z(3)=0,\ Z(4)=\frac{1}{3},\ Z(5)=0,\ Z(6)=-\frac{2}{45},\ Z(7)=0,\ Z(8)=\frac{1}{315},\ldots
$$

If we substitute above values in the equations (2.2), we have

$$
\begin{aligned}
y(t) &= t - \frac{1}{6}t^3 + \frac{1}{120}t^5 - \frac{1}{5040}t^7 + O(t^9)\\
v(t) &= 1 - \frac{1}{2}t^2 + \frac{1}{24}t^4 - \frac{1}{720}t^6 + \frac{1}{40320}t^8 + O(t^9)\\
z(t) &= 1 - t^2 + \frac{1}{3}t^4 - \frac{2}{45}t^6 + \frac{1}{315}t^8 + O(t^9)
\end{aligned}
$$

Table 3. Compared of the numerical and exact solution of the second application , where $v(t)$ is exact solution, $v(t)^*$ is numerical solution.

x	$v(t)$	$v(t)^*$	$\left\|v(t)-v(t)^*\right\|$
0.1	0.9950041653	0.9950041653	0
0.2	0.9800665778	0.9800665779	-0.1 10^{-9}
0.3	0.9553364891	0.9553364891	0
0.4	0.9210609940	0.9210609941	-0.1 10^{-9}
0.5	0.8775825619	0.8775825622	-0.3 10^{-9}
0.6	0.8253356149	0.8253356166	-0.17 10^{-8}
0.7	0.7648421873	0.7648421951	-0.78 10^{-8}
0.8	0.6967067093	0.6967067388	-0.295 10^{-7}
0.9	0.6216099683	0.6216100638	-0.955 10^{-7}
1.0	0.5403023059	0.5403025794	-0.2735 10^{-6}

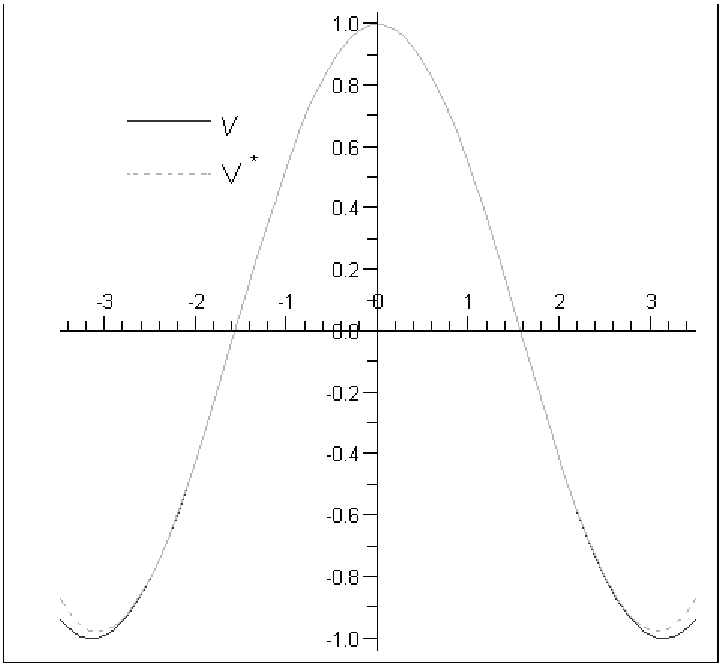

Figure 3.2. Graph of the functions V and V^* in the second test problem.

4. CONCLUSION

The method has been applied to the solution of differential-algrebraic equations. We have obtained approximant analytical solution of the given problem. If the numerical solution of the given problems are compared with their analytical solutions, the differential transform method is very effective and convergence are quite close.

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