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INVERSE PROBLEM FOR PERIODIC STURM-LIOUVILLE OPERATOR

ABSTRACT

In this paper it is studied periodic Sturm-Liouville problem.It is obtained for Hill operatör the generalized degeneracy of the fundemental integral equation for on two partially non coinciding spectra.
Keywords: Hill Equation, Spectrum, General Degeneracy, Normalized Numbers, Translation Operator

PERİYODİK STURM-LIOUVILLE OPERATORÜ İÇİN TERS PROBLEM

ÖZET

Bu makalede periyodik Sturm-Liouville problemi ele alınarak,bu problemde Hill operatörü için kısmen çakışmayan iki spektruma göre ters problemin esas integral denkleminin genel dejeneriliği gösterilmi**S**tir.
Anahtar Kelimeler: Hill Denklemi, Spektrum, Genel Dejenere, Normlaştırıcı Sayılar, Dönüşüm Operatörü

1. INTRODUCTION (GİRİŞ)

We consider periodic

$$-y'' + q(x)y = \lambda y, \qquad (1.1)$$

$$y(0) = y(\pi),\ y'(0) = y'(\pi)$$

and anti-periodic

$$-y'' + q(x)y = \lambda y, \qquad (1.2)$$

$$y(0) = y(\pi),\ y'(0) = y'(\pi)$$

Sturm-Liouville problems [10],[11],[14]. we assume that the potential q(x) is a smooth periodic function of π period We denote the spectrum of the periodic problem by $\lambda_0 < \lambda_3 \le \lambda_4 < \lambda_7 \le \lambda_9 ..$and the spectrum of the anti-periodic problem by $\lambda_1 \le \lambda_2 < \lambda_5 \le \lambda_6 < \lambda_8 \le ...$ [9], [17].

Along with the problem (1.1) and (1.2), we consider yet another problem

$$-y'' + q(x)y = \lambda y, \qquad (1.3)$$

$$y'(0) = -y'(\pi) = 0$$

we denote the spectrum of the problem (1.3) by $\eta_1 < \eta_2 < \eta_3 < ...$ It is well known that $\lambda_1 \le \eta_1 \le \lambda_2 < \lambda_3 \le \eta_2 \le \lambda_4 < ..$We denote by $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ solutions of (1.1) satisfying the initial conditions $\varphi(0,\lambda) = \psi'(0,\lambda) = -1$ and $\varphi'(0,\lambda) = \psi(0,\lambda) = 0.$ [14-22]. We call $\Delta(\lambda) = \psi'(\pi,\lambda) + \varphi(\pi,\lambda)$ a Hill (or Ljapunov) function. We have

- **Theorem 1:** If the Δ and $\tilde{\Delta}$ coincide and the spectrum (η_n) and ($\tilde{\eta}_n$) differ in a finite number of their terms, then the Gelfand-Levitan integral equation is degenerate in the extended sense.

2. RESEARCH SIGNIFICANCE (ÇALIŞMANIN ÖNEMİ)

The integral equation *K*(x.t) has given and this function be a general degenerated function. This presentation is very important, so functions , which are defined in K(x,t), are corresponding to eigenvalues of Hill equation. Hill operator has a great importance in Quantum mechanics and solid state Physics.

3. INVERSE PROBLEMS ON HILL EQUATION (HİLL DENKLEMİ ÜZERİNDE İNVERS PROBLEM)

- **Theorem 1:** For $\forall \quad n > N,\quad \tilde{\lambda}_n = \lambda_n$ and for $n = 1,2,..N$, $\tilde{\lambda}_n \ne \lambda_n$, $q(x+\pi) = q(x)$ and $\tilde{q}(x+\pi) = \tilde{q}(x)$

$$K(x,t)+F(x,t)+\int_0^{\pi} K(x,s)F(t,s)ds = 0, \ (0 \le x \le s \le \pi)$$

the integral equation of the invers problem.generalized degeneracy. So, we can show $K(x,t), \ F(x,t)$ as

$$K(x,t) = \sum_{n=1}^{N} c_n f_n(x)\tilde{g}_n(t)$$

$$F(x,t) = \sum_{n=1}^{N} c_n p_n(x)\tilde{\sigma}_n(t).$$

Here with order $f_n(x)$, $p_n(x)$ and $\tilde{g}_n(t), \ \tilde{\sigma}_n(t)$ solutions of *periodic* Sturm-Liouville equations potentials $q(x)$ and $\tilde{q}(x)$.

Proof: We assume that the potential $q(x)$ is a smooth periodic function of period π.

$$-y'' + q(x)y = \lambda y, \tag{3.1}$$

$$y(0) = y(\pi), \ y'(0) = y'(\pi) \tag{3.2}$$

and

$$-y'' + q(x)y = \lambda y,$$

$$y(0) = -y(\pi), \ y'(0) = -y'(\pi) \tag{3.3}$$

we consider the two Sturm-Liouville problems. We call the first problem a *periodic* Sturm-Liouville problem, and the second an *anti-periodic* Sturm-Liouville problem. We denote the spectrum of the first problem by

$$\lambda_0 < \lambda_3 \le \lambda_4 < \lambda_7 \le \lambda_9 ..$$

and the spectrum of the second by

$$\lambda_1 \le \lambda_2 < \lambda_5 \le \lambda_6 < \lambda_8 \le ...$$

Both spectra can be arranged in a single chain of inequalities, namely,

$$\lambda_0 < \lambda_1 \le \lambda_2 < \lambda_3 \le \lambda_4 < ..$$

Along with the problems (3.1)-(3.2) and (3.1)-(3.3), we consider another problem (3.4)

$$-y'' + q(x)y = \lambda y,$$

$$y'(0) = -y'(\pi) = 0 \tag{3.4}$$

We denote the spectrum of the problem (3.4) by

$$\eta_1 < \eta_2 < \eta_3 < ...$$

It is well known that

$$\lambda_1 \le \eta_1 \le \lambda_2 < \lambda_3 \le \eta_2 \le \lambda_4 < ..$$

We denote by $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ solution of (3.1)-(3.2) and (3.1)-(3.3) satisfying the initial conditions $\varphi(0,\lambda)=\psi^{'}(0,\lambda)=-1$ and $\varphi^{'}(0,\lambda)=\psi(0,\lambda)=0.$ Further, we put

$$\Delta(\lambda)=\psi^{'}(\pi,\lambda)+\varphi(\pi,\lambda) \tag{3.5}$$

We call $\Delta(\lambda)$ a *Ljapunov function*. The eigenvalues of the periodic problem are the roots of the equation $\Delta(\lambda)=2$; the eigenvalues of the anti-periyodik problem are the roots of the equation $\Delta(\lambda)=-2$. Solutions of (3.1) are π periodic if and only if $\varphi^{'}(\pi,\lambda)=\psi(\pi,\lambda)=0$. It follovs from the constancy of the Wronskiyan that

$$W\{\psi,\varphi\}=\begin{vmatrix}\varphi & \varphi^{'}\\ \psi & \psi^{'}\end{vmatrix}=\varphi(\pi,\lambda)\psi^{'}(\pi,\lambda)-\varphi^{'}(\pi,\lambda)\psi(\pi,\lambda)=1 \tag{3.6}$$

Puting $\lambda=\lambda_n$ in (3.5) and (3.6), we obtain the system of equations

$$\psi^{'}(\pi,\lambda_n)+\varphi(\pi,\lambda_n)=\Delta(\lambda_n)$$
$$\psi^{'}(\pi,\lambda_n)\varphi(\pi,\lambda_n)=1$$

from which it follows that

$$\varphi(\pi,\lambda_n)=\Delta(\lambda_n)\mp\sqrt{\Delta^2(\lambda_n)-1} \tag{3.7}$$

$$\psi^{'}(\pi,\lambda_n)=\Delta(\lambda_n)\pm\sqrt{\Delta^2(\lambda_n)-1} \tag{3.8}$$

using $\Delta^2(\lambda_n)=1$, and for this reason from (3.7) and (3.8) and from asimptotik formulas, we obtain for $\varphi(x,\lambda)$ and $\psi^{'}(x,\lambda)$

$$\psi^{'}(\pi,\lambda_n)=\varphi(\pi,\lambda_n)=(-1)^n. \tag{3.9}$$

Lemma: If $\varphi(x,\lambda_n)$ solution function we obtain

$$-\psi^{''}(x,\lambda_n)+q(x)\psi(x,\lambda_n)=\lambda_n\psi(x,\lambda_n),$$
$$-\psi^{''}(x,\eta_n)+q(x)\psi(x,\eta_n)=\eta_n\psi(x,\eta_n).$$

first equation multiplicate with $\psi(x,\eta_n)$ and second equation multiplicate with $\psi(x,\lambda_n)$ and we subtraction second equation from first equation we obtain

$$-\psi^{''}(x,\lambda_n)\psi(x,\eta_n)+\psi^{''}(x,\eta_n)\psi(x,\lambda_n)=(\lambda_n-\eta_n)\psi(x,\lambda_n)\psi(x,\eta_n)$$

Last equation integrate from 0 to π,

$$(\lambda_n-\eta_n)\int_0^{\pi}\psi(x,\lambda_n)\psi(x,\nu_n)dx=\int_0^{\pi}\left(-\psi^{''}(x,\lambda_n)\psi(x,\eta_n)+\psi^{''}(x,\eta_n)\psi(x,\lambda_n)\right)dx,$$

$$\int_0^{\pi}\psi(x,\lambda_n)\psi(x,\nu_n)dx=\frac{\psi^{'}(x,\eta_n)\psi(x,\lambda_n)-\psi^{'}(x,\lambda_n)\psi(x,\eta_n)}{(\lambda_n-\eta_n)} \tag{3.10}$$

For $n\to\infty,$ because of $\lambda_n\to\eta_n$ right side give $\frac{0}{0}$ indefiniteness From L'hospital rule, derived the dot indicates differentiation with respect to $\lambda,$

we obtain

$$\int_0^\pi \psi(x,\lambda_n)\psi(x,\nu_n)dx = \psi'(x,\eta_n)\dot{\psi}(x,\lambda_n) - \psi'(x,\lambda_n)\psi(x,\eta_n).$$

Taking with $\eta_n = \lambda_n$, we obtain

$$\int_0^\pi \psi^2(x,\lambda_n)dx = \psi'(x,\lambda_n)\dot{\psi}(x,\lambda_n) - \psi'(x,\lambda_n)\psi(x,\lambda_n).$$

If we take $x=\pi$ and using $\psi(\pi,\lambda)=0,$ we find normalized numbers

$$c_n = \int_0^\pi \psi^2(x,\lambda_n)dx = \psi'(\pi,\lambda_n)\dot{\psi}(\pi,\lambda_n). \tag{3.11}$$

Assume now that $q(x)$ and $\tilde{q}(x)$ are two potentials with the same Ljapunov function. [23].We denote by $\psi(x,\lambda)$ solution of (a) which is satisfying the initial conditions $\psi(0,\lambda)=0$, $\psi'(0,\lambda)=-1.$

$$-y'' + q(x)y = \lambda y \tag{a}$$

We denote by $\tilde{\varphi}(x,\lambda)$ solution of (b) which is satisfying the initial conditions $\tilde{\psi}(0,\lambda)=0$, $\tilde{\psi}'(0,\lambda)=-1.$

$$-y'' + \tilde{q}(x)y = \lambda y \tag{b}$$

and by $\{\eta_n\},$ the roots of the equation $\psi(\pi,\lambda)=0.$ by $\{\tilde{\eta}_n\},$ the roots of the equation $\tilde{\psi}(\pi,\lambda)=0.$It is well known that

$$X[\psi(x,\lambda)] = \tilde{\psi}(x,\lambda) = \psi(x,\lambda) + \int_0^x K(x,t)\psi(t,\lambda)dt \tag{3.12}$$

The function $K(x,t)$ satisfies the equation

$$\frac{\partial^2 K}{\partial x^2} - \tilde{q}(x)K = \frac{\partial^2 K}{\partial t^2} - q(t)K$$

and the conditions

$$K(x,0)=0$$

$$K(x,x) = \frac{1}{2}\int_0^x [\tilde{q}(z) - q(z)]dz$$

[1-3]. If the potentials $q(x)$ and $\tilde{q}(x)$ have one and the same *N*-zoned Ljapunov function, then,

$$\tilde{\lambda}_n = \lambda_n, \quad \text{for } \; n > N, \tag{3.13}$$

$$\tilde{\psi}'(\pi,\lambda_n) = \psi'(\pi,\lambda_n) = (-1)^n, \;\; \text{for } \; n > N. \tag{3.14}$$

We now use (3.13) and (3.14) to find the structure of the functions $K(x,t).$Putting $x=\pi$ and $\lambda=\lambda_n$ in (a), we deduce that

$$\tilde{\psi}(\pi,\lambda_n)=\psi(\pi,\lambda_n)+\int_0^{\pi}K(\pi,t)\psi(t,\lambda_n)dt.$$

Here because of $\tilde{\psi}(\pi,\lambda_n)=\psi(\pi,\lambda_n)$, we deduce

$$\int_0^{\pi}K(\pi,t)\psi(t,\lambda_n)dt=0 \qquad (\text{for} n>N)$$

$$\int_0^{\pi}K(\pi,t)\psi(t,\lambda_n)dt=\tilde{\psi}(\pi,\lambda_n) \qquad (\text{for} n\le N)$$

(for $n\le N$ because of $\tilde{\lambda}_n\ne\lambda_n$ we obtain $\tilde{\psi}(\pi,\tilde{\lambda}_n)\ne$ $\tilde{\psi}(\pi,\lambda_n)$. Also, because of $\tilde{\psi}(\pi,\tilde{\lambda}_n)=0,$ we obtain $\tilde{\psi}(\pi,\lambda_n)\ne 0$) So,we deduce

$$K(\pi,t)=\sum_{n=1}^{N}\frac{\tilde{\psi}(\pi,\lambda_n)}{c_n}\psi(t,\lambda_n). \tag{3.15}$$

Differentiating (a) and then putting $x=\pi$ and $\lambda=\lambda_n$, we obtain

$$\tilde{\psi}'(\pi,\lambda_n)=\psi'(\pi,\lambda_n)+K(\pi,\pi)\psi(\pi,\lambda_n)+\int_0^{\pi}\frac{\partial K}{\partial x}\psi(t,\lambda_n)dt$$

$$\int_0^{\pi}\frac{\partial K}{\partial x}\psi(t,\lambda_n)dt=\tilde{\psi}'(\pi,\lambda_n)-\psi'(\pi,\lambda_n) \qquad (\text{for} n>N\ )$$

Consequently,

$$\frac{\partial K}{\partial x}=\sum_{n=1}^{N}\frac{\tilde{\psi}'(\pi,\lambda_n)-\psi'(\pi,\lambda_n)}{c_n}\psi(t,\lambda_n) \tag{3.16}$$

From (3.15) and (3.16), we obtain (in the triangle $(0\le x\le s\le\pi)$)

$$-y''+\tilde{q}(x)y=\lambda y$$

$$K(x,t)=\sum_{n=1}^{N}\left\{\tilde{\psi}(\pi,\lambda_n)\tilde{\theta}(x,\lambda_n)+\left[\tilde{\psi}'(\pi,\lambda_n)-\psi'(\pi,\lambda_n)\right]\tilde{\psi}(x,\lambda_n)\right\}\frac{\psi(t,\lambda_n)}{\left\|\psi(t,\lambda_n)\right\|^2} \tag{3.17}$$

where $\tilde{\theta}(x,\lambda)$, $\tilde{\phi}(x,\lambda)$ is the solution of the equation

$$-y''+\tilde{q}(x)y=\lambda y$$

satisfying the

$$\tilde{\theta}(\pi,\lambda)=\tilde{\phi}'(\pi,\lambda)=-1,\tilde{\theta}'(\pi,\lambda)=\tilde{\phi}(\pi,\lambda)=0.$$

Relying on this representation of $K(x,t)$, we can prove the generalized of the kernel of the integral equation.

5. CONCLISIONS (SONUÇLAR)

The function *K*(x.t) has a presentation as

$$K(x,t)=\sum_{n=1}^{N} c_n f_n(x)\tilde{g}_n(t)$$

in integral equation.This function is a degenerated function. This presentation is very important,so functions $f_n(x)$,$g_n(t)$,…functions are corresponding to eigenvalues of Hill equation. Hill operator has a great importance in Quantum mechanics and solid state Physics.

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