Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 51 (4) (2022), 1058 – 1071
DOI : 10.15672/hujms.967923

Research Article

Variational equations and Killing magnetic trajectories on timelike surfaces in semi-Riemannian manifolds

Kübra Şahin, Zehra Özdemir*

Department of Mathematics, Faculty of Science and Arts, Amasya University, Amasya, Turkey

Abstract

In this article, Darboux frame variations for timelike surfaces in semi-Riemannian manifolds are discussed. In addition, the Killing equations are examined by using the Darboux frame curvature variations. Then, magnetic trajectories are generated by means of the variational vector fields. Furthermore, parametric representations of all magnetic trajectories on the de Sitter space $\mathbb{S}_1^2$ are presented. Moreover, various examples of magnetic trajectories are given in order to illustrate the theoretical results.

Mathematics Subject Classification (2020). 53B50, 37C10, 14H45, 14H50, 35A15, 70E17, 53Z04

Keywords. vector fields, magnetic flows, ordinary differential equations, special curves, motion of a rigid body with a fixed point, variational methods

1. Introduction

The magnetic field is described mathematically as a divergence free vector field in three dimensional spaces. In three dimensional space, a Killing vector field is a vector field that preserves the metric. Namely, $\mathbf{B}$ is a Killing vector field if the Lie derivative with respect to $\mathbf{B}$ of the metric g vanishes, $L_{\mathbf{B}}g = 0$. Thus, Killing vector fields are divergence free vector fields in three dimensional space and they defines a magnetic field called as Killing magnetic vector field, [1].

When a charged particle enters a magnetic field, the Frenet vector fields of the particle are affected by the magnetic field Ω and release a force called the Lorentz force defined as follows

$$(\digamma(\zeta).\eta) = \Omega(\zeta, \eta), \tag{1.1}$$

where the Lorentz force $\digamma$ associated with the magnetic vector field $\mathbf{B}$ is given by

$$\digamma(\zeta) = \mathbf{B} \times \zeta. \tag{1.2}$$

Under the influence of the Lorentz force, the particle begins to follow a new trajectory called the magnetic curves satisfy the following equation

$$\digamma(d_1) = \mathbf{B} \times d_1 = \nabla_{d_1} d_1 \tag{1.3}$$

*Corresponding Author.
Email addresses: kbrshn.89@hotmail.com (K. Şahin), zehra.ozdemir@amasya.edu.tr (Z. Özdemir)
Received: 09.07.2021; Accepted: 14.02.2022

where d_1 is a unit tangent vector field of the magnetic curve and ∇ is the Levi-Civita connection.

Magnetic trajectories in Riemannian spaces are studied by Barros and Sunada, [3, 16]. Magnetic curves related to the Killing magnetic field using the variational method were determined by Barros, [2]. Then, the magnetic curves related to the Killing magnetic curves are expanded in Euclidean and Minkowski spaces, [5, 8, 9]. In Minkowski 3-space, Killing equations is computed by Gürbüz, [10]. Then, Bozkurt et al.[4] and Özdemir et al.[14] introduced the Killing B-magnetic and N-magnetic curves in $3D$ Riemannian and semi-Riemannian manifolds. Then, the null and pseudo null curve variations are computed and applied to magneticcurves by Özdemir, [13, 15].

In section 1, the theory has been introduced and some studies related to the theory have been given. In section 2, basic definitions and concepts related to the theory are introduced. In section 3, Killing equations related to Darboux elements on timelike surfaces are given. In section 4, magnetic curves on timelike surfaces are determined by means of the Killing equations. In this section, as an applications, the parametric expressions of all magnetic trajectories on the de Sitter space $\mathbb{S}^2_1$ are also determined and various examples are visualized by using the MAPLE program.

2. Fundamental backgrounds

Let (M, g) be a semi-Riemannian manifold, then for all $\zeta = (\zeta_1, \zeta_2, \zeta_3)$, $\eta = (\eta_1, \eta_2, \eta_3) \in \chi(M)$ the inner product is given by

$$(\zeta.\eta) = -\zeta_1\eta_1 + \zeta_2\eta_2 + \zeta_3\eta_3, \tag{2.1}$$

and the cross product is defined as

$$\zeta \times \eta = (-\zeta_2\eta_3 + \zeta_3\eta_2,\ \zeta_3\eta_1 - \zeta_1\eta_3,\ \text{-}\zeta_2\eta_1 + \zeta_1\eta_2). \tag{2.2}$$

For all $\zeta, \eta, \nu \in \chi(M)$, the mixed product is given by

$$(\zeta \times \eta.\nu) = dv(\zeta, \eta, \nu) \tag{2.3}$$

where dv denotes a volume on M. A non-zero vector $\zeta \in \chi(M)$ is called space-like on the condition that $(\zeta.\zeta) > 0$, time-like when $(\zeta.\zeta) < 0$ and lightlike(null) when $(\zeta.\zeta) = 0$. Any two vectors $\zeta, \eta \in \chi(M)$ are called orthogonal provided that$(\zeta.\eta) = 0$. Let ζ, η be two null vectors then they are linearly dependent on thecondition that $(\zeta.\eta) = 0$. A curve in Minkowski 3-space is called spacelike (resp. timelike, lightlike) curve, if its tangent vector is a spacelike (resp. timelike, lightlike) vector, [11]. The semi-Riemannian curvature tensor is defined by

$$\Re(\zeta, \eta)\nu = -\nabla_\zeta\nabla_\eta\nu + \nabla_\eta\nabla_\zeta\nu + \nabla_{[\zeta,\eta]}\nu. \tag{2.4}$$

Then, the sectional curvature of a non degenerated plane generated by $\{\zeta, \eta\}$ is presented by

$$K(\zeta, \eta) = \frac{(\Re(\zeta, \eta)\zeta.\eta)}{(\zeta.\zeta)(\eta.\eta) - (\zeta.\eta)^2}. \tag{2.5}$$

If the semi-Riemannian manifold M has constant sectional curvature C then it is called semi-Riemannian space form and the related curvature tensor is given by

$$\Re(\zeta, \eta)\nu = C\{(\nu.\zeta)\eta - (\nu.\eta)\zeta\}. \tag{2.6}$$

Definition 2.1. The surface S in the Minkowski 3-space $\mathbb{E}^3_1$ is a timelike surface when the induced metric on the surface is a Lorentz metric. In other words, the normal vector on the timelike surface is a spacelike vector [12].

A sphere of center p_0 and radius r is defined by

$$\mathbb{S}^2_1(r; p_0) = \{p \in \mathbb{E}^3_1 : (p - p_0.p - p_0) = r^2\}.$$

The tangent plane at p is $T_pM = Span\{p - p_0\}^{\perp}$ and $N(p) = \frac{1}{r}(p - p0)$. This vector is a spacelike vector and, therefore, the surface is accepted as a timelike surface. When p_0 is the origin and $r = 1$, the surface is also called *de Sitter space* and denoted by

$$\mathbb{S}_1^2 = \{(\zeta_1, \zeta_2, \zeta_3) \in \mathbb{E}_1^3 : -\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 1\},$$

[11]. De Sitter space $\mathbb{S}_1^2$ is a surface analog embedded in Minkowski space $\mathbb{R}_1^3$ being one of the Euclidean spheres in mathematics and physics. It could be noted that this manifold analog is symmetric at a maximum level and includes positive constant curvature. The de Sitter space is named after Willem de Sitter (1872–1934) at Leiden University [6, 7]. When $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}_1^3$, $\varphi(U) = S$ is a timelike embedding surface and $\alpha : I \subset \mathbb{R} \to U$ is a regular curve on S. A curve γ on the surface S defined by $\gamma(s) = \varphi(\alpha(s))$ is found and since φ is a timelike embedding, we could conclude a unit spacelike normal vector field d_3 along the surface S. Then, we have found a orthonormal frame $\{d_1, d_2, d_3\}$ which is called the Darboux frame along the non-null curve γ where $d_2(s) = d_3(s) \wedge d_1(s)$ is a unit spacelike vector. Then, the Darboux frame equations of γ is given by

$$\begin{aligned} d_1' &= k_g d_2 - \varepsilon k_n d_3, \\ d_2' &= k_g d_1 + \varepsilon \tau_r d_3, \\ d_3' &= k_n d_1 + \tau_r d_2, \end{aligned} \tag{2.7}$$

where $k_g = -\varepsilon(d_1'.d_2)$, $k_n = -\varepsilon(d_1'.d_3)$ and $\tau_g = \varepsilon(d_3'.d_2)$, $\varepsilon = \pm 1$ are *geodesic curvature*, the *asymptotic curvature*, and the *principal curvature* of γ on the surface S in $\mathbb{E}_1^3$, respectively. Moreover, s is the arc-length parameter of γ. In particular, the following equations are presented:

$$(d_1.d_1) = -(d_2.d_2) = \varepsilon, \;\; (d_3.d_3) = 1, (d_1.d_2) = (d_2.d_3) = (d_1.d_3) = 0,$$

[17]. Moreover, it is found as the following

$$d_2 \wedge d_1 = \varepsilon d_3, \;\; d_3 \wedge d_2 = d_1, \;\; d_3 \wedge d_1 = d_2, \tag{2.8}$$

where $\varepsilon = \pm 1$.

3. Killing equations of Darboux frame for timelike surfaces in 3D semi-Rimannian spaces

Lemma 3.1. *Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}_1^3$, $\varphi(U) = S$ be a timelike embedding surface and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on S in 3D semi Riemannan space form $(M(C), g)$. The variation of γ defined by $\Gamma : I \times (-\varepsilon, \varepsilon) \to M(C)$ with $\gamma(s, 0)$ the initial curve satisfy $\Gamma(s, 0) = \gamma(s)$. The related variational vector field is given by $\mathbf{B}(s) = \frac{\partial \Gamma(s,t)}{\partial t}|_{t=0}$ and the speed function is defined as $W(s, t) = \frac{\partial \Gamma(s,t)}{\partial t} = v(s, t)T(s, t)$. Then, we find the following functions of the non-null curve $\gamma_t(s)$:*
1. Speed function $v(s, t) = \left\| \frac{\partial \Gamma(s,t)}{\partial t} \right\|$,
2. Geodesic curvature functions $k_g(s, t)$,
3. Normal curvature functions $k_n(s, t)$,
4. Torsion curvature functions $\tau_g(s, t)$,
The variations of these functions, at $t = 0$, are calculated as follows:

$$\mathbf{B}(v) = \left(\frac{\partial v}{\partial t}(s, t)\right)\Big|_{t=0} = -v\rho \tag{3.1}$$

$$\mathbf{B}(k_g) = (\frac{\partial k_g}{\partial t}(s,t))\Big|_{t=0} = \varepsilon(\Re(\mathbf{B},d_1)d_1.d_2) - \varepsilon(\nabla^2_{d_1}\mathbf{B}.d_2) + \frac{k_n}{k_g}((-\Re(\mathbf{B},d_1)d_1.d_3) + (\nabla^2_{d_1}\mathbf{B}.d_3)) + 2\rho k_g + 2\rho\frac{k_n^2}{k_g} \quad (3.2)$$

$$\mathbf{B}(k_n) = (\frac{\partial k_n}{\partial t}(s,t))\Big|_{t=0} = \varepsilon(\Re(\mathbf{B},d_1)d_1.d_3) - \varepsilon(\nabla^2_{d_1}\mathbf{B}.d_3) + \frac{k_g}{k_n}((-\Re(\mathbf{B},d_1)d_1.d_2) + (\nabla^2_{d_1}\mathbf{B}.d_2)) - 2\varepsilon\rho\frac{k_g^2}{k_n} + 2\rho k_n, \quad (3.3)$$

$$\mathbf{B}(\tau_g) = (\frac{\partial \tau_g}{\partial t}(s,t))\Big|_{t=0} = -\varepsilon k_n(\nabla_{d_1}\mathbf{B}.d_2) - \varepsilon\frac{k_n}{k_g}((-\Re(\mathbf{B},d_1)d_1.d_1) + (\nabla^2_{d_1}\mathbf{B}.d_1)) + \frac{k_n^2}{k_g}(\nabla_{d_1}\mathbf{B}.d_3) - \varepsilon\frac{\rho'}{k_g}. \quad (3.4)$$

where $\rho = (\nabla_{d_1}\mathbf{B}.d_1)$ and $\Re$ stands for the curvature tensor of M.

Proof. Let $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = S$ be a timelike embedding surface and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on S in $3D$ semi-Riemannian space form $M(C)$ and $\mathbf{B}$ be a vector field along the curve γ. If we compute the covariant derivative $\nabla_{\mathbf{B}}$ of $v(s,t)$, then we calculate

$$\begin{aligned} \mathbf{B}(v) &= (\frac{\partial v}{\partial t}(s,t))\Big|_{t=0} = (\frac{\partial^2\Gamma(s,t)}{\partial t\partial s}.vd_1) \\ &= v(\nabla_{d_1}\mathbf{B}.d_1) \\ &= -v\rho. \end{aligned} \quad (3.5)$$

Using the Darboux frame equations, we obtain,

$$k_g = -\varepsilon(\nabla_{d_1}d_1.d_2). \quad (3.6)$$

The covariant derivative $\nabla_{\mathbf{B}}$ of the geodesic curvature $k_g(s,t)$ is calculated as

$$\mathbf{B}(k_g) = (\frac{\partial k_g}{\partial t}(s,t))\Big|_{t=0} = -\varepsilon(\nabla_{\mathbf{B}}\nabla_{d_1}d_1.d_2) - \varepsilon(\nabla_{d_1}d_1.\nabla_{\mathbf{B}}d_2). \quad (3.7)$$

On the other hand, we compute

$$\nabla_{\mathbf{B}}\nabla_{d_1}d_1 = -\Re(\mathbf{B},d_1)d_1 + \nabla_{d_1}\nabla_{\mathbf{B}}d_1 + \nabla_{[V,d_1]}d_1 \quad (3.8)$$

and

$$\begin{aligned} [\mathbf{B},W] &= \nabla_{\mathbf{B}}W - \nabla_W\mathbf{B} \\ &= \frac{\partial^2\Gamma(s,t)}{\partial t\partial s} - \frac{\partial^2\Gamma(s,t)}{\partial s\partial t} \\ &= 0. \end{aligned} \quad (3.9)$$

Thus, if we take the equation $W = vT$ into account, we reach

$$[\mathbf{B},vd_1] = \mathbf{B}(v)d_1 - v\nabla_{d_1}\mathbf{B} + v[\mathbf{B},d_1] = 0. \quad (3.10)$$

This gives

$$\begin{aligned} [\mathbf{B},d_1] &= -\frac{1}{v}\mathbf{B}(v)d_1, \;\; \rho = -\frac{1}{v}\mathbf{B}(v) \\ &= \rho d_1. \end{aligned} \quad (3.11)$$

Using the eq.(3.10) and eq.(3.11) with the equation $\nabla_{\mathbf{B}}d_1 = [\mathbf{B},d_1] + \nabla_{d_1}\mathbf{B}$, we find

$$\nabla_{d_1}\nabla_{\mathbf{B}}d_1 = \rho' d_1 + \rho\nabla_{d_1}d_1 + \nabla^2_{d_1}\mathbf{B}. \quad (3.12)$$

From the Darboux frame formulae, we obtain

$$d_2 = \frac{1}{k_g}(\nabla_{d_1} d_1 + \varepsilon k_n d_3) \tag{3.13}$$

If we take the covariant derivative $\nabla_{\mathbf{B}}$ of eq.(3.13), we get

$$\nabla_{\mathbf{B}} d_2 = \frac{-k_g'}{k_g^2}\frac{\partial k_g}{\partial t}(\nabla_{d_1} d_1 + \varepsilon k_n d_3) + \frac{1}{k_g}(\nabla_{\mathbf{B}}\nabla_{d_1} d_1 + \varepsilon k_n \nabla_{\mathbf{B}} d_3);$$

Since, $\frac{\partial k_g}{\partial t} = 0$, we calculate

$$\nabla_{\mathbf{B}} d_2 = \frac{1}{k_g}(\nabla_{\mathbf{B}}\nabla_{d_1} d_1 + \varepsilon k_n \nabla_{\mathbf{B}} d_3). \tag{3.14}$$

Using the eqs.(3.7)-(3.12) the expression for $\mathbf{B}(k_g)$ becomes

$$\begin{aligned}\mathbf{B}(k_g) = (\frac{\partial k_g}{\partial t}(s,t))\Big|_{t=0} &= \varepsilon(\Re(\mathbf{B}, d_1)d_1.d_2) - \varepsilon(\nabla^2_{d_1}\mathbf{B}.d_2) + \frac{k_n}{k_g}((-\Re(\mathbf{B}, d_1)d_1.d_3) \\ &+(\nabla^2_{d_1}\mathbf{B}.d_3)) + 2\rho k_g + 2\rho\frac{k_n^2}{k_g}\end{aligned}$$

Similar calculations above we compute the following equations

$$\begin{aligned}\mathbf{B}(k_n) = (\frac{\partial k_n}{\partial t}(s,t))\Big|_{t=0} &= \varepsilon(\Re(\mathbf{B}, d_1)d_1.d_3) - \varepsilon(\nabla^2_{d_1}\mathbf{B}.d_3) + \frac{k_g}{k_n}((-\Re(\mathbf{B}, d_1)d_1.d_2) \\ &+(\nabla^2_{d_1}\mathbf{B}.d_2)) - 2\varepsilon\rho\frac{k_g^2}{k_n} + 2\rho k_n,\end{aligned}$$

$$\begin{aligned}\mathbf{B}(\tau_g) = (\frac{\partial \tau_g}{\partial t}(s,t))\Big|_{t=0} &= -\varepsilon k_n(\nabla_{d_1}\mathbf{B}.d_2) - \varepsilon\frac{k_n}{k_g}((-\Re(\mathbf{B}, d_1)d_1.d_1) + (\nabla^2_{d_1}\mathbf{B}.d_1)) \\ &+\frac{k_n^2}{k_g}(\nabla_{d_1}\mathbf{B}.d_3) - \varepsilon\frac{\rho'}{k_g}.\end{aligned}$$

□

Proposition 3.2. *Assume that* $\mathbf{B}(s)$ *be the restriction to* $\gamma(s)$ *of a Killing vector field* $\mathbf{B}$ *of* S*, then the variations of the Darboux curvature functions and speed function of* γ *satisfy:*

$$\mathbf{B}(v) = \mathbf{B}(k_g) = \mathbf{B}(k_n) = \mathbf{B}(\tau_g) = 0. \tag{3.15}$$

Proof. Any local flow $\{\digamma_t\}$ generated by the Killing vector field $\mathbf{B}$ is composed of local isometries of S. Since the variations $\mathbf{B}(v)$, $\mathbf{B}(k_g)$, $\mathbf{B}(k_n)$ and $\mathbf{B}(\tau_g)$ do not depend on the variation Γ but only on $\mathbf{B}(s)$, we can variate $\gamma(s)$ in the direction of $\mathbf{B}(s)$ as follows:

$$\gamma_t(s) = \Gamma(s,t) =: \digamma_t(\gamma(s)). \tag{3.16}$$

The isometric function $\digamma_t$ gives that the functions, $v(s,t)$, $k_g(s,t)$, $k_n(s,t)$ and $\tau_g(s,t)$, do not depend on t and thus we have $\mathbf{B}(v) = \mathbf{B}(k_g) = \mathbf{B}(k_n) = \mathbf{B}(\tau_g) = 0$. □

Corollary 3.3. *If* γ *is a curve in* de Sitter space $\mathbb{S}^2_1$ *and* $\mathbf{B}$ *is a Killing vector field along the curve* γ *then we have the following equations*

i. $(\nabla_{d_1}\mathbf{B}.d_1) = 0$,

ii. $-\varepsilon k_g((-\Re(\mathbf{B}, d_1)d_1.d_2) + (\nabla^2_{d_1}\mathbf{B}.d_2)) - (\Re(\mathbf{B}, d_1)d_1.d_3) + (\nabla^2_{d_1}\mathbf{B}.d_3) = 0$,

iii. $-\varepsilon(\nabla_{d_1}\mathbf{B}.d_2) + \frac{1}{k_g}(\nabla_{d_1}\mathbf{B}.d_3) - \frac{\varepsilon}{k_g}((-\Re(\mathbf{B}, d_1)d_1.d_1) + (\nabla^2_{d_1}\mathbf{B}.d_1)) = 0$.

Proof. It clearly stated in Proposition 3.2. □

4. Killing magnetic curves on the timelike surfaces

Proposition 4.1. *Provided that $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = S$ is a timelike embedding surface and $\gamma : I \subset \mathbb{R} \to U$ is a regular curve on S. Therefore, the Lorentz force equations through the Darboux frame $\{d_1, d_2, d_3\}$ are computed as follows*

$$\begin{bmatrix} \digamma(d_1) \\ \digamma(d_2) \\ \digamma(d_3) \end{bmatrix} = \begin{bmatrix} 0 & k_g & -\varepsilon k_n \\ k_g & 0 & \varpi \\ k_n & \varepsilon\varpi & 0 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \tag{4.1}$$

where k_g, k_n and τ_g are geodesic curvature, asymptotic curvature, principal curvature of γ and $\varpi(s)$ is a function on the surface S in $\mathbb{E}^3_1$, respectively.

Proof. Suppose that $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = S$ be a spacelike embedding surface and $\gamma : I \subset \mathbb{R} \to U$ be a regular curve on S. Then the definition of the magnetic curves gives

$$\digamma(d_1) = \nabla_{d_1} d_1 = k_g d_2 - \varepsilon k_n d_3. \tag{4.2}$$

Then we can write

$$\digamma(d_2) = \lambda d_1 + \mu d_2 + \xi d_3 \tag{4.3}$$

The following equation could be found:

$$\begin{aligned} \lambda &= \varepsilon(\digamma(d_2).d_1) = -\varepsilon(\digamma(d_1).d_2) = k_g, \\ \mu &= \varepsilon(\digamma(d_2).d_2) = 0, \\ \xi &= (\digamma(d_2).d_3) = \varpi. \end{aligned} \tag{4.4}$$

These equations imply

$$\digamma(d_2) = k_g d_1 + \varpi d_3. \tag{4.5}$$

Using the similar computations, we obtain

$$\digamma(d_3) = k_n d_1 + \varepsilon\varpi d_2. \tag{4.6}$$

□

Proposition 4.2. *Provided that $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = S$ is a timelike embedded surface, $\gamma : I \subset \mathbb{R} \to U$ is a regular curve on S and $\mathbf{B}$ is a Killing vector field along the curve γ. Thenγ is a magnetic curve of a magnetic field $\mathbf{B}$ if and only if $\mathbf{B}$ can be written along γ as follows*

$$\mathbf{B}(s) = -\varepsilon\varpi d_1 - k_n d_2 + k_g d_3 \tag{4.7}$$

where $\varepsilon = \pm 1$, $\{k_g, k_n\}$ Darboux curvatures and $\varpi(s)$ is a smooth function associated with each magnetic curve of $\mathbf{B}$.

Proof. We express the equation of

$$\mathbf{B}(s) = \mu d_1 + \varsigma d_2 + \delta d_3 \tag{4.8}$$

where η, ς and δ are certain functions along a trajectory of $\mathbf{B}$. If the eqs.(1.2)-(4.1) are used, we calculate that $\mu = -\varepsilon\varpi, \varsigma = -k_n$ and $\delta = k_g$. Conversely, if γ satisfies the eq.(4.8), then it satisfies $\mathbf{B} \times d_1 = \digamma(d_1) = \nabla_{d_1} d_1$. Therefore, we can say that γ is a magnetic curve of $\mathbf{B}$. □

Theorem 4.3. *If $\varphi : U \subset \mathbb{E}^2 \to \mathbb{E}^3_1$, $\varphi(U) = S$ is a timelike embedding surface, $\gamma : I \subset \mathbb{R} \to U$ is a regular curve on S and $\mathbf{B}$ is a Killing vector field along the curve γ. If γ is a magnetic curve of $\mathbf{B}$, then the Darboux curvatures of γ satisfy the following differential equations*

$$\begin{aligned} & k_n k_g'' + \varpi k_n k_n' - 2\varepsilon k_n k_n' \tau_r - \varepsilon k_n^2 \tau_r' - \tau_r \varpi k_n k_g + \varepsilon k_n \tau_r^2 k_g + \varepsilon k_g k_n'' + k_g k_g' \varpi - 2\varepsilon k_g k_g' \tau_r \\ & -\varepsilon k_g^2 \tau_r' - \varepsilon k_n k_g \tau_r \varpi + k_g \tau_r^2 k_n = 0, \end{aligned}$$

where ϖ is constant along γ.

Proof. If γ is a magnetic curve of $\mathbf{B}$, then $\mathbf{B}$ satisfies the eq.(4.7). Then the derivative of the eq.(4.7) gives

$$\nabla_{d_1}\mathbf{B} = -\varepsilon\varpi' d_1 + (-k_n' - \varepsilon\varpi k_g + k_g\tau_r)d_2 + (k_g' + \varpi k_n - \varepsilon k_n\tau_r)d_3. \tag{4.9}$$

Firstly, if we use the equation $\mathbf{B}(v) = 0$, in Lemma 3.3, we calculate ϖ is a constant. Using the derivative of eq.(4.7), we compute

$$\begin{aligned}\nabla^2_{d_1}\mathbf{B} = (k_g(-k_n' - \varepsilon\varpi k_g + k_g\tau_r) + k_n(k_g' + \varpi k_n - \varepsilon k_n\tau_r))d_1 \\ +((-k_n' - \varepsilon\varpi k_g + k_g\tau_r)' + \tau_r(k_g' + \varpi k_n - \varepsilon k_n\tau_r))d_2 \\ +((k_g' + \varpi k_n - \varepsilon k_n\tau_r)' + \varepsilon\tau_r(-k_n' - \varepsilon\varpi k_g + k_g\tau_r))d_3 = 0.\end{aligned} \tag{4.10}$$

Also, the Riemannian curvature tensor satisfy

$$\Re(\mathbf{B}, d_1)d_1 = C((d_1.\mathbf{B})d_1 - (d_1.d_1)\mathbf{B}). \tag{4.11}$$

Then, we obtain

$$\Re(\mathbf{B}, d_1)d_1 = \varepsilon C(k_n d_2 - k_g d_3). \tag{4.12}$$

If we use the eq.(4.10) and eq.(4.12), in the second equation in Lemma 3.3 and using the Proposition 3.2, we compute

$$\begin{aligned}&(k_g' + \varpi k_n - \varepsilon k_n\tau_r)' + \varepsilon\tau_r(-k_n' - \varepsilon\varpi k_g + k_g\tau_r) - \varepsilon\tfrac{k_g}{k_n}(-k_n' - \varepsilon\varpi k_g + k_g\tau_r)' \\ &-\varepsilon\tfrac{k_g\tau_r}{k_n}(k_g' + \varpi k_n - \varepsilon k_n\tau_r) = 0,\end{aligned} \tag{4.13}$$

Finally, the last equation in Lemma 3.3 is found automatically. □

Next we give the parametric representations of the magnetic curves in *de Sitter space* $\mathbb{S}^2_1$ by using the MAPLE Program.

Theorem 4.4. *Let γ be a spacelike curve in* de Sitter space $\mathbb{S}^2_1 \subset M$ *then the magnetic γ has one of the following parametric representations:*
i.

$$\gamma(s) = k_1\cos s + k_2\sin s,\ k_g = 0, \tag{4.14}$$

where $k_1, k_2 \in \mathbb{E}^3_1$.
ii.

$$\gamma(s) = h_1 + h_2 s + h_3 s^2,\ k_g = c = 1 \tag{4.15}$$

where $h_1, h_2, h_3 \in \mathbb{E}^3_1$.
iii.

$$\gamma(s) = g_1 + g_2\frac{1}{\sqrt{c^2-1}}e^{\sqrt{c^2-1}s} + g_3\frac{1}{\sqrt{c^2-1}}e^{-\sqrt{c^2-1}s},\ k_g = c \neq 1, \tag{4.16}$$

where $g_1, g_2, g_3 \in \mathbb{E}_1^3$.

iv. $k_g = \sqrt{\frac{2c_1}{\varpi}}\tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})$,

$$\gamma(s) = \begin{pmatrix} d_1 hypergeom([\frac{1}{2}, \frac{\varpi+1}{\varpi}, \frac{\varpi-1}{\varpi}], [\frac{2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{2c_1\varpi}, \\ \frac{-2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{2c_1\varpi}], \frac{1}{\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^2}) \tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2}), \\ +d_2\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^{\frac{\sqrt{2c_1\varpi-2}-\sqrt{c_1\varpi})}{\sqrt{c_1\varpi}}} \sinh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2}) \\ hypergeom([-\frac{-c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{2c_1\varpi}, -\frac{-2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}-2c_1}{2c_1\varpi}, \\ -\frac{-2c_1\varpi\sqrt{2}+\sqrt{2c_1\varpi}\sqrt{c_1\varpi-}+2c_1}{2c_1\varpi}], [-\frac{-c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{c_1\varpi}, \\ -\frac{-2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{2c_1\varpi}], \frac{1}{\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^2}) \\ +d_3\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^{-\frac{\sqrt{2c_1\varpi-2}-\sqrt{c_1\varpi})}{\sqrt{c_1\varpi}}} \sinh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2}) \\ hypergeom([\frac{-c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{2c_1\varpi}, -\frac{2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}+2c_1}{2c_1\varpi} \\ , \frac{2c_1\varpi\sqrt{2}+\sqrt{2c_1\varpi}\sqrt{c_1\varpi-}-2c_1}{2c_1\varpi}], [\frac{c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{c_1\varpi} \\ , \frac{2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1-\varpi}}{2c_1\varpi}], \frac{1}{\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^2}) \end{pmatrix}. \tag{4.17}$$

Proof. Since γ is a curve on the de Sitter space $\mathbb{S}_1^2$ the curve has the following curvatures

$$k_g,\ k_n = 1,\ \tau_g = 0.$$

The equations (4.13) reduce

$$k_g'' + \varpi k_g {k_g}' = 0. \tag{4.18}$$

If we solve the differential equation, we reach

$$k_g = c \text{ or } k_g = \sqrt{\frac{2c_1}{\varpi}}\tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})$$

where $\varpi = const$. By using the Darboux frame equation, we obtain the following differential equation

$$k_g\gamma''' - k_g'\gamma'' + (-k_g^3 + k_g)\gamma' + k_g'\gamma = 0.$$

If we solve the differential equation, we obtain following four cases:

i.

$$\gamma(s) = k_1\cos s + k_2 \sin s,\ k_g = 0, \tag{4.19}$$

where $k_1, k_2 \in \mathbb{E}_1^3$.

ii.

$$\gamma(s) = h_1 + h_2 s + h_3 s^2,\ k_g = c = 1 \tag{4.20}$$

where $h_1, h_2, h_3 \in \mathbb{E}_1^3$.

iii.

$$\gamma(s) = g_1 + g_2\frac{1}{\sqrt{c^2-1}}e^{\sqrt{c^2-1}s} + g_3\frac{1}{\sqrt{c^2-1}}e^{-\sqrt{c^2-1}s},\ k_g = c \neq 1, \tag{4.21}$$

where $g_1, g_2, g_3 \in \mathbb{E}_1^3$.

iv. $k_g = \sqrt{\frac{2c_1}{\varpi}}\tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})$, The solution of the differential equation give us the magnetic curve parameterized as eq.(4.17). □

Theorem 4.5. *If* γ *is a timelike curve in* de Sitter space $\mathbb{S}_1^2 \subset M$ *then the curve* γ *has one of the following representations:*

i.

$$\gamma(s) = a_1 e^s + a_2 e^{-s},\ k_g = 0. \tag{4.22}$$

where $a_1, a_2 \in \mathbb{E}_1^3$.

ii.

$$\gamma(s) = b_1 + b_2\frac{1}{\sqrt{c^2+1}}e^{\sqrt{c^2+1}s} + b_3\frac{1}{\sqrt{c^2+1}}e^{-\sqrt{c^2+1}s},\ k_g = c, \tag{4.23}$$

where $b_1, b_2, b_3 \in \mathbb{E}_1^3$ *and* $c \neq 0 \in \mathbb{R}$.

iii. $k_g = \sqrt{\frac{2c_1}{\varpi}} \tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})$

$$\gamma(s) = \begin{pmatrix} d_1 hypergeom([\frac{1}{2}, \frac{\varpi+1}{\varpi}, \frac{\varpi-1}{\varpi}], [\frac{2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{2c_1\varpi}, \\ \frac{-2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{2c_1\varpi}], \frac{1}{\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^2}) \tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2}), \\ +d_2 \cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^{\frac{\sqrt{2c_1\varpi+2}-\sqrt{c_1\varpi})}{\sqrt{c_1\varpi}}} \sinh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2}) \\ hypergeom([-\frac{-c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{2c_1\varpi}, -\frac{-2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}-2c_1}{2c_1\varpi}, \\ -\frac{-2c_1\varpi\sqrt{2}+\sqrt{2c_1\varpi}\sqrt{c_1\varpi+}+2c_1}{2c_1\varpi}], [-\frac{-c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{c_1\varpi}, \\ -\frac{-2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{2c_1\varpi}], \frac{1}{\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^2}) \\ +d_3 \cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^{-\frac{\sqrt{2c_1\varpi+2}-\sqrt{c_1\varpi})}{\sqrt{c_1\varpi}}} \sinh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2}) \\ hypergeom([\frac{-c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{2c_1\varpi}, -\frac{2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}+2c_1}{2c_1\varpi} \\ , \frac{2c_1\varpi\sqrt{2}+\sqrt{2c_1\varpi}\sqrt{c_1\varpi+}-2c_1}{2c_1\varpi}], [\frac{c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{c_1\varpi} \\ , \frac{2c_1\varpi+\sqrt{2c_1\varpi}+\sqrt{2c_1+\varpi}}{2c_1\varpi}], \frac{1}{\cosh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})^2}) \end{pmatrix}. \tag{4.24}$$

Proof. Since γ is a curve on the de Sitter space $\mathbb{S}_1^2$ the curve has the following curvatures

$$k_g, \ k_n = 1, \ \tau_g = 0.$$

The equations (4.13) reduce

$$k_g'' + \varpi k_g k_g{}' = 0.$$

If we solve the differential equation, we get

$$k_g = c \text{ or } k_g = \sqrt{\frac{2c_1}{\varpi}} \tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})$$

where $\varpi = const$. From the Darboux frame equation, we get:

$$k_g\gamma''' - k_g'\gamma'' + (-k_g^3 - k_g)\gamma' - k_g'\gamma = 0.$$

If we solve the differential equation, we obtain the following three cases

i.

$$\gamma(s) = a_1 e^s + a_2 e^{-s}, \ k_g = 0. \tag{4.25}$$

where $a_1, a_2 \in \mathbb{E}_1^3$.

ii.

$$\gamma(s) = b_1 + b_2 \frac{1}{\sqrt{c^2+1}} e^{\sqrt{c^2+1}s} + b_3 \frac{1}{\sqrt{c^2+1}} e^{-\sqrt{c^2+1}s}, \ k_g = c, \tag{4.26}$$

where $b_1, b_2, b_3 \in \mathbb{E}_1^3$ and $c \neq 0 \in \mathbb{R}$.

iii. $k_g = \sqrt{\frac{2c_1}{\varpi}} \tanh(\frac{\sqrt{2c_1\varpi}(c_2+s)}{2})$, the solution of the differential equation give us the magnetic curves parameterized as eq.(4.24). □

5. Examples

Example 5.1. In order to illustrate, we have considered a spacelike curve on the *de sitter space* $\mathbb{S}_1^2$ defined by

$$\gamma_1(s) = (0, \cos s, \sin s).$$

Therefore, the curve γ will have the following Darboux curvatures:

$$k_g(s) = 0, \ k_n(s) = 1, \ \tau_g = 0.$$

We could express that γ_1 is a magnetic curve. Therefore, the Killing magnetic vector field is calculated as

$$\mathbf{B}(s) = (1, \varpi \sin s, \varpi \cos s).$$

The magnetic curve is visualized in Figure 1.

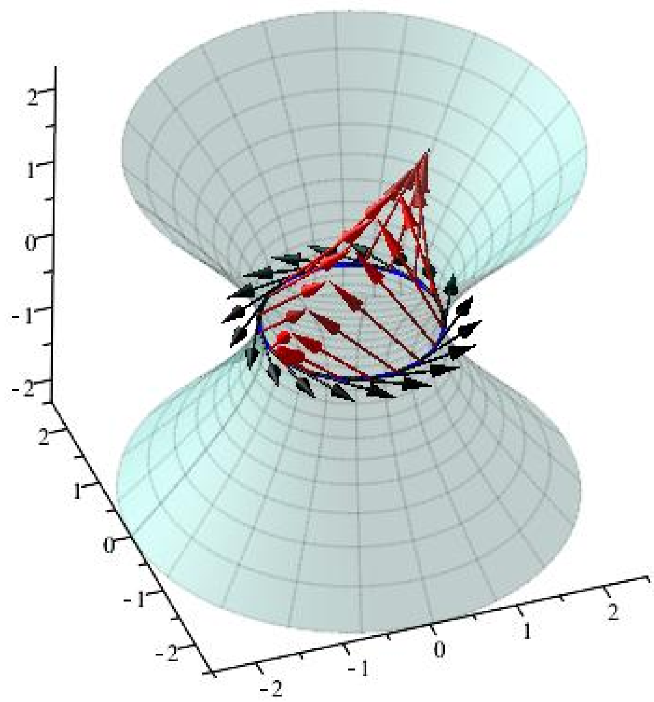

Figure 1. A charged particle motion (black) along a spacelike magnetic trajectory γ_1 (blue) in the magnetic field $\mathbf{B}$ (red) on the de Sitter space $\mathbb{S}^2_1$.

Example 5.2. If we choose $a_1 = (0, \frac{1}{2}, -\frac{1}{2})$ and $a_2 = (0, \frac{1}{2}, \frac{1}{2})$ we will have the following timelike magnetic curve on the de Sitter space $\mathbb{S}^2_1$ parameterized by

$$\gamma_2(s) = (\frac{1}{2}e^s - \frac{1}{2}e^{-s}, \frac{1}{2}e^s + \frac{1}{2}e^{-s}, 0).$$

Therefore, the Darboux frame equations would be as follows:

$$k_g(s) = 0,\ k_n(s) = 1,\ \tau_g = 0.$$

We could express that γ_2 is a magnetic curve. Therefore, the Killing magnetic vector field is calculated as

$$\mathbf{B}(s) = (\varpi\frac{1}{2}e^s + \varpi\frac{1}{2}e^{-s}, \varpi\frac{1}{2}e^s - \varpi\frac{1}{2}e^{-s}, 1).$$

The magnetic curve is visualized in Figure 2.

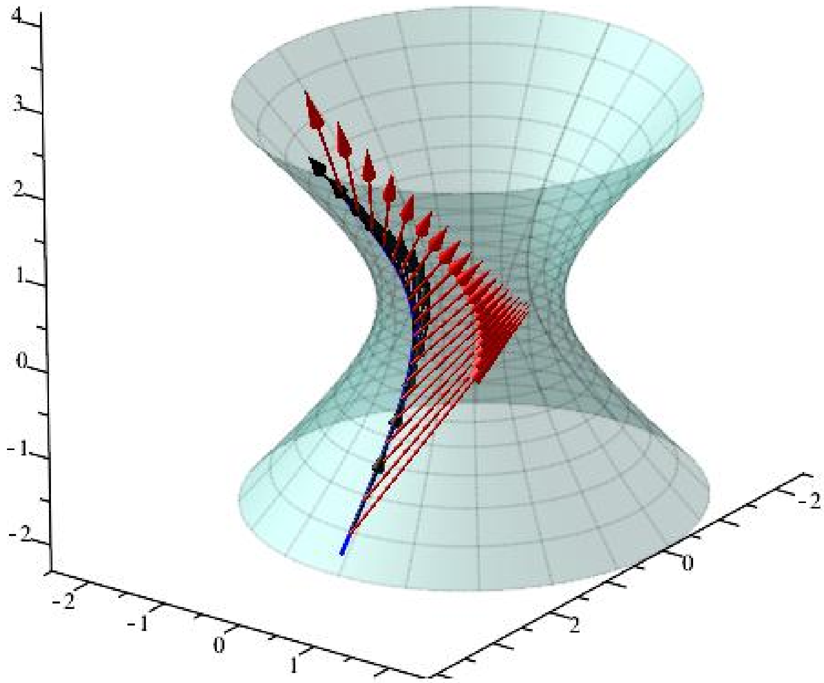

Figure 2. A charged particle motion (black) along a timelike magnetic trajectory γ_2 (blue) in the magnetic field $\mathbf{B}$ (red) on the de Sitter space $\mathbb{S}_1^2$.

Example 5.3. If we choose $h_1 = (0, -1, 0)$, $h_2 = (0, 0, 1)$ and $h_3 = (\frac{1}{2}, \frac{1}{2}, 0)$ we will have the following spacelike magnetic curve on the de Sitter space $\mathbb{S}_1^2$ parameterized by

$$\gamma_3 = (\frac{s^2}{2}, \frac{s^2}{2} - 1, s)$$

The curve will have the following Darboux curvatures:

$$k_g(s) = 1, \ k_n(s) = 1, \ \tau_g = 0.$$

We calculate that $\varpi = 0$ and therefore γ is a magnetic curve. Then, the Killing magnetic vector field calculated as

$$\mathbf{B}(s) = (-\varpi s - 1, -\varpi s - 1, -\varpi s).$$

The magnetic curve is visualized in Figure 3.

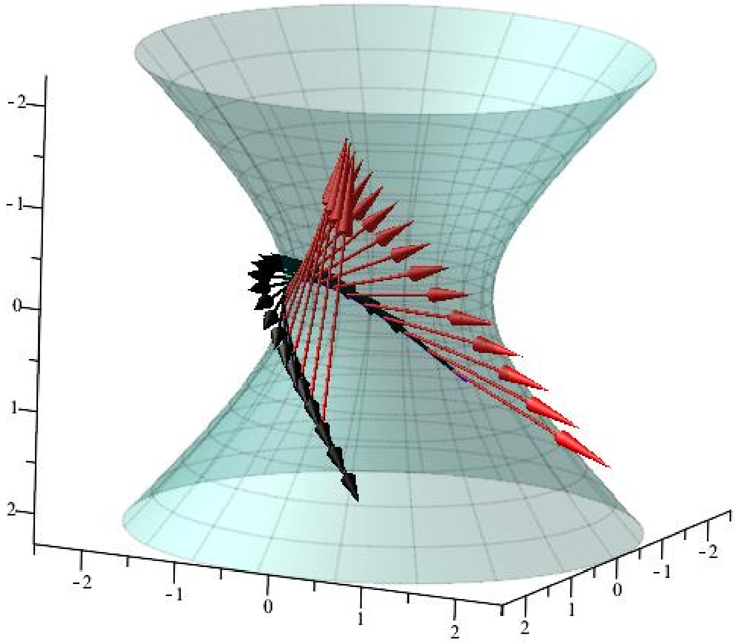

Figure 3. A charged particle motion (black) along a spacelike magnetic trajectory γ_3 (blue) in the magnetic field $\mathbf{B}$ (red) on the de Sitter space $\mathbb{S}_1^2$.

Example 5.4. If we choose the following timelike curve on the de Sitter space $\mathbb{S}_1^2$ parameterized by

$$\gamma_3 = \begin{pmatrix} \frac{-\cos s}{\sqrt{0.21}}, \\ \frac{-1.1\cos s \sin(\sqrt{0.21}/1.1)s}{\sqrt{0.21}} + \sin s \cos(\sqrt{0.21}/1.1)s, \\ \frac{1.1\cos s \cos(\sqrt{0.21}/1.1)s}{\sqrt{0.21}} + \sin s \sin(\sqrt{0.21}/1.1)s \end{pmatrix}.$$

The curve will have the following Darboux curvatures:

$$k_g(s) = \cot s, \ k_n(s) = 1, \ \tau_g = 0.$$

Then from the eq.(4.18), we calculate that $\varpi = 2$ and therefore γ is a magnetic curve in the Killing magnetic vector field calculated as

$$\mathbf{B}(s) = 2d_1 + \cot s d_2 - d_3.$$

The magnetic curve is visualized in Figure 3.

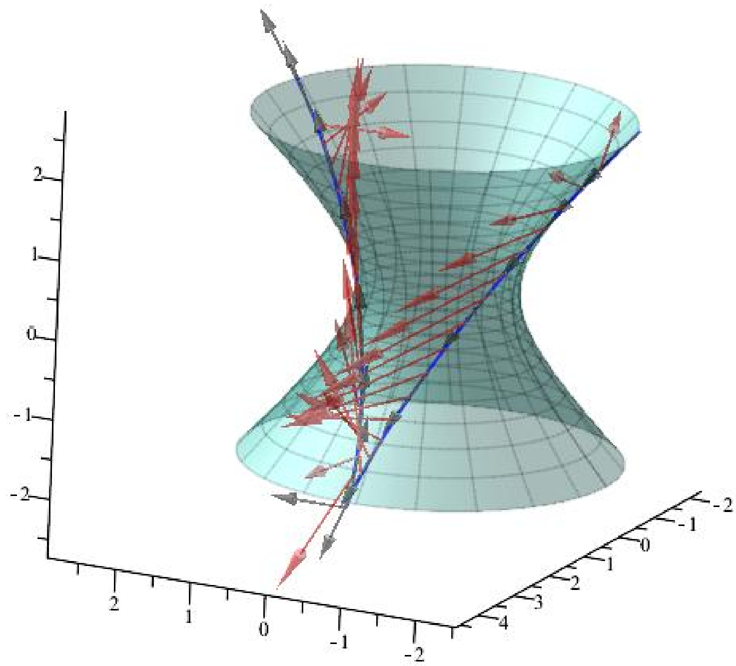

Figure 4. A charged particle motion (black) along a spacelike magnetic trajectory γ_4 (blue) in the magnetic field **B** (red) on the de Sitter space $\mathbb{S}^2_1$.

References

[1] M. Barros, J.L. Cabrerizo, M. Fernández and A. Romero, *Magnetic vortex flament flows,* J. Math. Phys. **48**, 1-27, 2007.

[2] M. Barros, A. Ferrandez, P. Lucas and M.A. Merono, *General helices in the 3-dimensional Lorentzian space forms,* Rocky. Mt. J. Math. **32**, 373-388, 2001.

[3] M. Barros, A. Romero, J.L. Cabrerizo and M. Fernández, *The Gauss Landau Hall problem on Riemannian surfaces.* J. Math. Phys.**46**, 112905, 2005.

[4] Z. Bozkurt, I. Gök, Y. Yaylı and F.N. Ekmekci, *A new approach for magnetic curves in 3D Riemannian manifolds,* J. Math. Phys. **55**, 053501, 2014.

[5] J.L. Cabrerizo, *Magnetic fields in 2D and 3D sphere,* J. Nonlinear. Math. Phys. **20**, 440-450, 2013.

[6] H.S.M. Coxeter, *A geometrical background for de Sitter's world,* Am. Math. Mon. (Math. Assoc. Am.) **50** (4), 217-228 (JSTOR 2303924), 1943.

[7] W. de Sitter, *On the relativity of inertia remarks concerning Einstein's latesthypothesis,* Proc. Kon. Ned. Acad. Wet. **19**, 1217-1225, 1917.

[8] S.L. Druta-Romaniuc and M.I. Munteanu, *Magnetic Curves corresponding to Killing magnetic fields in* E^3*,* J. Math. Phys. **52**, 113506, 2011.

[9] S.L. Druta-Romaniuc and M.I. Munteanu, *Killing magnetic curves in a Minkowski 3-space,* Nonlinear. Anal. Real. World. Appl. **14**, 383-396, 2013.

[10] N. Gurbuz, *p-Elastica in the 3-Dimensional Lorentzian Space Forms,* Turkish J. Math. **30**, 33-41, 2006.

[11] R. Lopez, *Differential Geometry of Curves and Surfaces in Lorentz-Minkowski Space.* Int. Electronic. J. Geom. **7** (1), 44-107, 2014.

[12] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Academic press, New York, 1983.

[13] Z. Özdemir, *Pseudo Null Curve Variations for Bishop Frame in 3D semi-Riemannian Manifold*, Int. J. Geom. Methods and Modern Phys. **16** (3), 1950043, 2019.

[14] Z. Özdemir, I. Gok, Y. Yaylı and F.N. Ekmekci, *Notes on Magnetic Curves in 3D semi-Riemannian Manifolds.* Turkish J. Math. **39**, 412-426, 2015.

[15] Z. Özdemir, *Null Cartan curve variations in 3D semi-Riemannian manifold*, Hacettepe J. Math. Stat. **50** (2), 351-360, 2021.

[16] T. Sunada, *Magnetic flows on a Riemann surface*, In Proceedings of the KAIST Mathematics Workshop:Analysis and Geometry, Taejeon, Korea, 3-6, 1993.

[17] H.H. Ugurlu, H. Kocayigit, *The Frenet and Darboux Instantaneous Rotain Vectors of Curves on Time-like Surfaces*, Math. Comput. Appl. **1** (2), 133-141, 1996.